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BAUSTATIK II – KOLLOQUIUM 1, Lösung

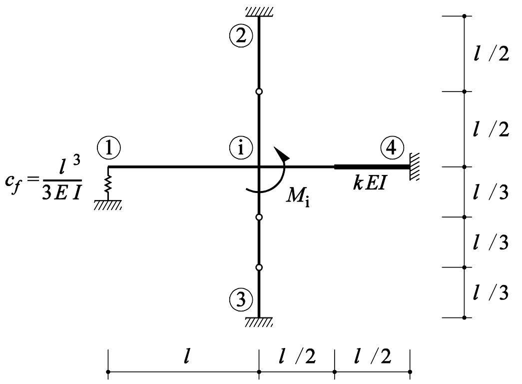
(101-0114)

Thema: Verformungsmethode

Aufgabe 1, Lösung

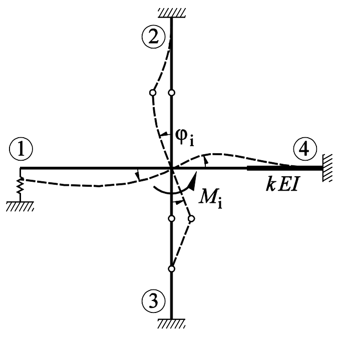
Gegeben: System (l, EI, k) und Einwirkung M_i

Gesucht.: - Stab- und Kreuzsteifigkeiten allgemein
- Schnittkraftlinien infolge der Einwirkung M_i für $k=1$

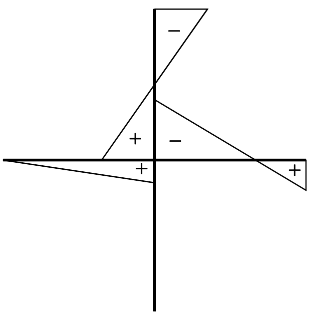


$n = 4$

Verformungen qualitativ infolge M_i :

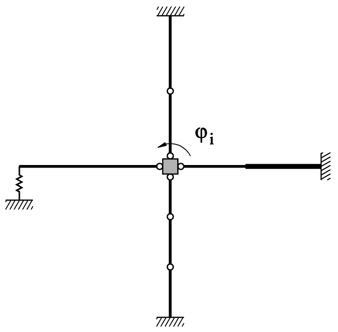


Momente qualitativ infolge M_i :



Das System ist 4-fach statisch unbestimmt. Mit der Kraftmethode müssten somit 4 ÜG eingeführt werden.

Da das System unverschieblich ist (es braucht **keine** Festhaltekraft, um eine Verschiebung zu verhindern), ergibt sich mit der Verformungsmethode nur ein einziger unbekannter Knotendrehwinkel φ_i



System unverschieblich → Stabdrehwinkel $\psi = 0$

→ Unbekannte: Knotendrehwinkel φ_i

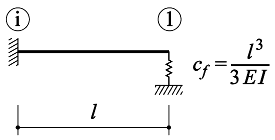
1. Festeinspannmomente

Da auf die Stäbe keine Lasten wirken und auch keine Verformungen aufgezwungen werden, gibt es keine Festeinspannmomente.

2. Stab- und Kreuzsteifigkeiten

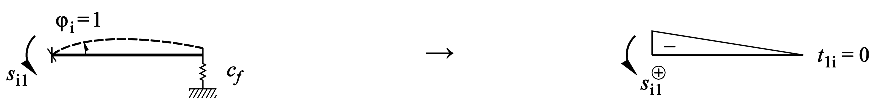
Stabsteifigkeiten $s_{ik} = M_{ik}(\varphi_i = 1)$
Kreuzsteifigkeiten $t_{ik} = M_{ik}(\varphi_k = 1)$

<u>Stab i – 1</u>:



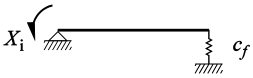
$\boxed{n = 1}$

- Stabsteifigkeit s_{i1}:



Lösung mit Kraftmethode:

GS und ÜG:

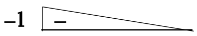


$$\boxed{\varphi_i = X_i \cdot \varphi_{ii} = 1} \qquad (\varphi_{i0} = 0)$$

$$\varphi_{ii} = \frac{1}{3} \cdot 1 \cdot \frac{1}{EI} \cdot l + \frac{1}{l} \cdot \frac{1}{l} \cdot c_f$$

$$= \frac{l}{3EI} + \frac{c_f}{l^2} = \frac{2l}{3EI}$$

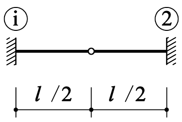
$M_i (X_i = 1)$:



$$X_i = \frac{1}{\varphi_{ii}} = \frac{1}{\frac{L}{3EI} + \frac{c_f}{l^2}} = \underline{\underline{\frac{3EI}{2l} = s_{i1}}}$$

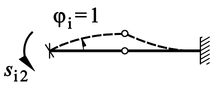
$$\underline{\underline{s_{1i} = t_{i1} = t_{1i} = 0}}$$

Stab i – 2:

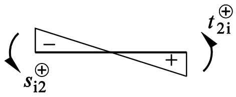


$\boxed{n = 1}$

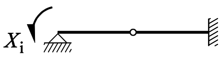
- Stabsteifigkeit s_{i2}, Kreuzsteifigkeit t_{2i}:



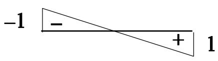
→



GS und ÜG:



$M_i (X_i = 1)$:

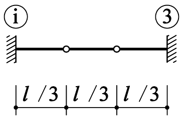


$$\boxed{\varphi_i = X_i \cdot \varphi_{ii} = 1}$$

$$\varphi_{ii} = 2 \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{EI} \cdot \frac{l}{2} = \frac{l}{3EI}$$

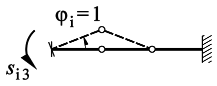
$$X_i = \frac{1}{\varphi_{ii}} = \underline{\underline{\frac{3EI}{l} = s_{i2} = t_{2i} = t_{i2} = s_{2i}}}$$

Stab i – 3:



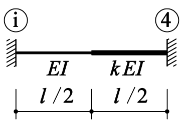
$\boxed{n=0}$ → statisch bestimmt

- Stabsteifigkeit s_{i3}:



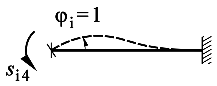
$\boxed{s_{i3}=0}$

Stab i – 4:



$\boxed{n=2}$

- Stabsteifigkeit s_{i4}, Kreuzsteifigkeit t_{4i}:



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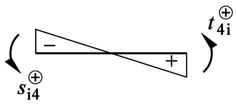
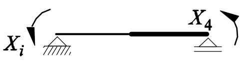
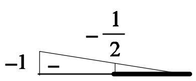


GS und ÜG:



$$\boxed{\begin{aligned}\varphi_i &= X_i\cdot\varphi_{ii} + X_4\cdot\varphi_{i4} = 1\\ \varphi_4 &= X_i\cdot\varphi_{4i} + X_4\cdot\varphi_{44} = 0\end{aligned}}$$

$M_i(X_i=1)$:



$$\varphi_{ii} = \frac{1}{6EI}\cdot\left[1\cdot\left(2\cdot 1+\frac{1}{2}\right)+\frac{1}{2}\cdot\left(1+2\cdot\frac{1}{2}\right)\right]\cdot\frac{l}{2}+\frac{1}{3kEI}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{l}{2}$$

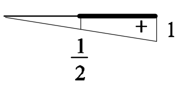
$$= \frac{7l}{24EI}+\frac{l}{24kEI}$$

$$\varphi_{44} = \frac{l}{24EI}+\frac{7l}{24kEI}$$

$$\varphi_{i4} = \frac{1}{6EI}\cdot\left[\frac{1}{2}\cdot\left(-1-2\cdot\frac{1}{2}\right)\right]\cdot\frac{l}{2}+\frac{1}{6kEI}\cdot\left[-\frac{1}{2}\cdot\left(2\cdot\frac{1}{2}+1\right)\right]\cdot\frac{l}{2}$$

$$= -\left(\frac{l}{12EI}+\frac{l}{12kEI}\right)$$

$M_4(X_4=1)$:

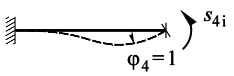


$$\left.\begin{aligned}\left(\frac{7l}{24EI}+\frac{l}{24kEI}\right)\cdot X_i-\left(\frac{7l}{12EI}+\frac{l}{12kEI}\right)\cdot X_4 &= 1\\ -\left(\frac{7l}{12EI}+\frac{l}{12kEI}\right)\cdot X_i+\left(\frac{l}{24EI}+\frac{7l}{24kEI}\right)\cdot X_4 &= 0\end{aligned}\right|$$

$$X_4 = t_{4i} = t_{i4} = \frac{16EI}{l} \cdot \frac{k \cdot (k+1)}{k^2 + 14k + 1}$$

$$X_i = s_{i4} = \frac{8EI}{l} \cdot \frac{k \cdot (k+7)}{k^2 + 14k + 1}$$

- Stabsteifigkeit s_{4i}, Kreuzsteifigkeit t_{i4}:



$$\begin{aligned} \varphi_i &= X_i \cdot \varphi_{ii} + X_4 \cdot \varphi_{i4} = 0 \\ \varphi_4 &= X_i \cdot \varphi_{4i} + X_4 \cdot \varphi_{44} = 1 \end{aligned}$$

$$\rightarrow \quad s_{4i} = \frac{8EI}{l} \cdot \frac{k \cdot (7k+1)}{k^2 + 14k + 1}$$

$$t_{i4} = t_{4i} = \frac{16EI}{l} \cdot \frac{k \cdot (k+1)}{k^2 + 14k + 1}$$

Stab- und Kreuzsteifigkeiten in Abhängigkeit von k:

		$k=0$	$k=0.5$	$k=1$	$k=2$	$k=10$	$k=\infty$
$s_{i4}\left[\frac{EI}{l}\right]$	$\frac{8k \cdot (k+7)}{k^2+14k+1}$	0	3.64	4	4.36	5.64	8
$s_{4i}\left[\frac{EI}{l}\right]$	$\frac{8k \cdot (7k+1)}{k^2+14k+1}$	0	2.18	4	7.27	23.57	56
$t_{i4} = t_{4i}\left[\frac{EI}{l}\right]$	$\frac{16k \cdot (k+1)}{k^2+14k+1}$	0	1.45	2	2.91	7.30	16

3. Stabendmomente

$$M_{ik} = M_{ik}^0 + s_{ik} \cdot \varphi_i + t_{ik} \cdot \varphi_k - (s_{ik} + t_{ik}) \cdot \psi_{ik}$$

$$M_{ki} = M_{ki}^0 + s_{ki} \cdot \varphi_k + t_{ki} \cdot \varphi_i - (s_{ki} + t_{ki}) \cdot \psi_{ki}$$

$$M_{i1} = s_{i1} \cdot \varphi_i = \frac{3EI}{2l} \cdot \varphi_i$$

$$M_{i2} = s_{i2} \cdot \varphi_i = \frac{3EI}{l} \cdot \varphi_i$$

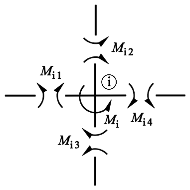
$$M_{2i} = t_{2i} \cdot \varphi_i = \frac{3EI}{l} \cdot \varphi_i$$

$$M_{i3} = M_{3i} = 0$$

$$M_{i4} = s_{i4} \cdot \varphi_i = \frac{4EI}{l} \cdot \varphi_i \ \text{(für } k=1\text{)}$$

$$M_{4i} = t_{4i} \cdot \varphi_i = \frac{2EI}{l} \cdot \varphi_i \ \text{(für } k=1\text{)}$$

4. Knotengleichgewicht



$$M_i - \sum_{k=1}^{4} M_{ik} = 0 \quad \rightarrow \quad M_{i1} + M_{i2} + M_{i3} + M_{i4} = M_i$$

$$M_i = \sum_{k=1}^{4} M_{ik} = \frac{3EI}{2l} \cdot \varphi_i + \frac{3EI}{l} \cdot \varphi_i + \frac{4EI}{l} \cdot \varphi_i = \frac{17EI}{2l} \cdot \varphi_i$$

$$\rightarrow \underline{\underline{\varphi_i = \frac{2l}{17EI} \cdot M_i}}$$

5. Endgültige Stabendmomente

$$M_{i1} = \frac{3EI}{2l} \cdot \frac{2l}{17EI} \cdot M_i = \underline{\underline{\frac{3}{17} \cdot M_i}}$$

$$M_{i2} = \frac{3EI}{l} \cdot \frac{2l}{17EI} \cdot M_i = \underline{\underline{\frac{6}{17} \cdot M_i}}$$

$$M_{2i} = \frac{3EI}{l} \cdot \frac{2l}{17EI} \cdot M_i = \underline{\underline{\frac{6}{17} \cdot M_i}}$$

$$M_{i3} = M_{3i} = \underline{\underline{0}}$$

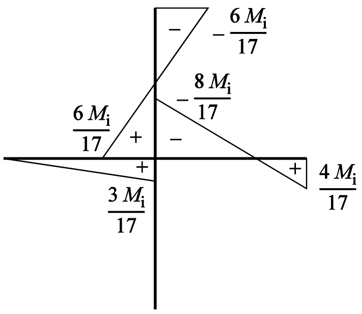
$$M_{i4} = \frac{4EI}{l} \cdot \frac{2l}{17EI} \cdot M_i = \underline{\underline{\frac{8}{17} \cdot M_i}}$$

$$M_{4i} = \frac{2EI}{l} \cdot \frac{2l}{17EI} \cdot M_i = \underline{\underline{\frac{4}{17} \cdot M_i}}$$

Kontrolle: $\Sigma M_{ik} = M_i$ i.O.

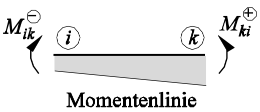
6. Schnittkraftlinien

M:



Merke:
Momentenpfeil des Stabendmomentes (Vorzeichen der Verformungsmethode beachten!) dreht ums Stabende und „greift“ in der *M*-„Fläche“ an

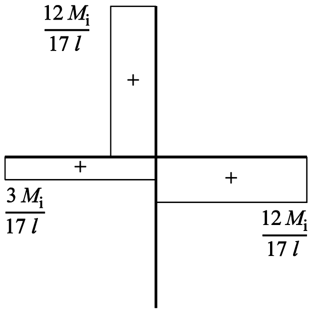
Beispiel:



Beim Bezeichnen der *M*-Flächen allgemeine Konvention verwenden.

V:

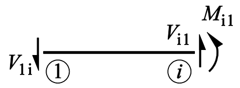
Zeichnung nach
allg. Konvention: $V^+ \uparrow \overline{\quad} \downarrow$



Berechnung nach
Konvention Verformungsmethode: $V^+ \downarrow \overline{\quad} \uparrow$

$$V_{i1} \cdot l + M_{i1} = 0$$

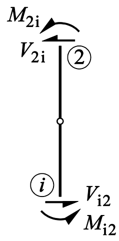
$$V_{i1} = -\frac{M_{i1}}{l} = -\underline{\underline{\frac{3M_i}{17l}}}$$



$$V_{1i} = V_{i1} = -\underline{\underline{\frac{3M_i}{17l}}}$$

$$V_{i2} \cdot l + M_{i2} + M_{2i} = 0$$

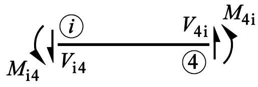
$$V_{i2} = -\left(\frac{M_{i2} + M_{2i}}{l}\right) = -\underline{\underline{\frac{12M_i}{17l}}}$$



$$V_{2i} = V_{i2} = -\underline{\underline{\frac{12M_i}{17l}}}$$

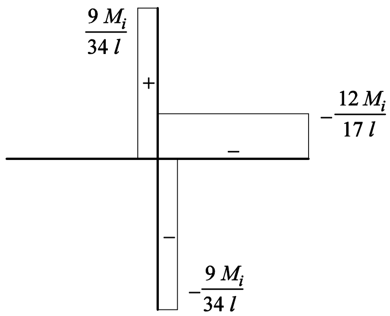
$$V_{i4} \cdot l + M_{i4} + M_{4i} = 0$$

$$V_{i4} = -\frac{(M_{i4} + M_{4i})}{l} = -\underline{\underline{\frac{12M_i}{17l}}}$$

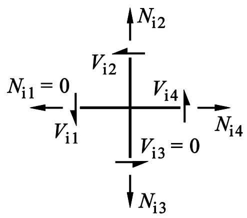


$$V_{4i} = V_{i4} = -\underline{\underline{\frac{12M_i}{17l}}}$$

N:



Berechnung von N mit der Konvention
Verformungsmethode für $V^+ \downarrow \overline{\quad} \uparrow$



$$N_{i1} = 0 \;;\; V_{i3} = 0$$

$$N_{i4} - V_{i2} = 0 \;\rightarrow\; N_{i4} = V_{i2} = -\underline{\underline{\frac{12M_i}{17l}}}$$

$$N_{i2} - N_{i3} - V_{i1} + V_{i4} = 0$$

$$N_{i2} - N_{i3} + \frac{3M_i}{17l} - \frac{12M_i}{17l} = 0$$

$$N_{i2} - N_{i3} = \frac{9M_i}{17l} \;\rightarrow\; \text{statisch unbestimmt}$$

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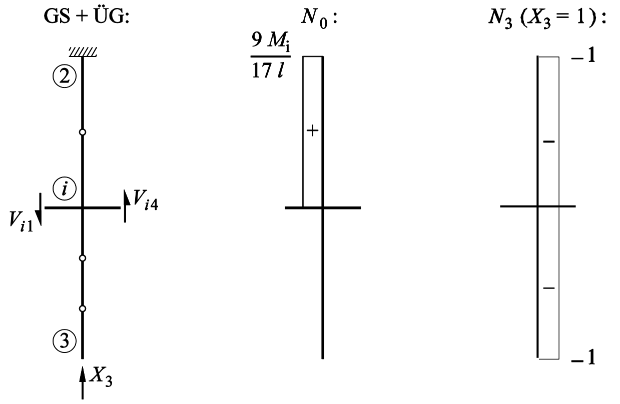
Lösung mit Kraftmethode (EA = konstant):



$$\delta_{30} = (-1) \cdot \frac{9M_\mathrm{i}}{17l} \cdot \frac{l}{EA} = -\frac{9M_\mathrm{i}}{17EA}$$

$$\delta_{33} = (-1) \cdot (-1) \cdot \frac{2l}{EA} = \frac{2l}{EA}$$

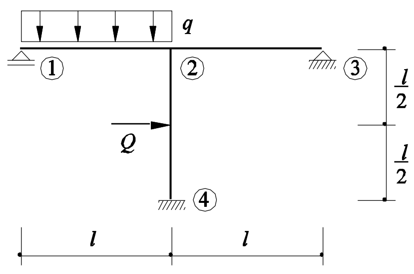
$$\delta_3 = \delta_{30} + X_3 \cdot \delta_{33} = 0 \rightarrow X_3 = -\frac{\delta_{30}}{\delta_{33}} = \frac{9M_\mathrm{i}}{34l}$$

$$\rightarrow N_{\mathrm{i}3} = -\frac{9M_\mathrm{i}}{34l} \;;\; N_{\mathrm{i}2} = \frac{9M_\mathrm{i}}{34l}$$

Aufgabe 2, Lösung

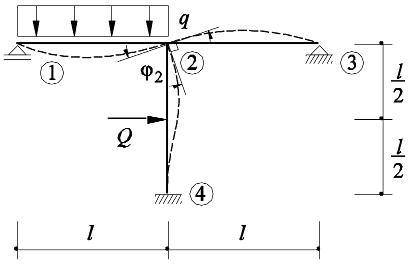
Gegeben: System $(l, EI = \text{konstant})$ und Einwirkungen q und $Q = ql$

Gesucht: Schnittkraftlinien

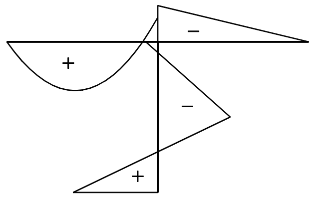


$\boxed{n = 3}$

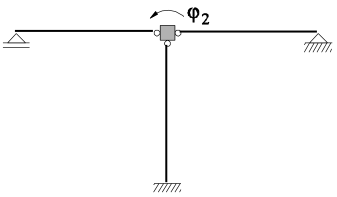
Verformungen qualitativ infolge q und Q:



Momente qualitativ infolge q und Q:



Das System ist 3-fach statisch unbestimmt. Da es unverschieblich ist (es braucht **keine** Festhaltekraft, um eine Verschiebung zu verhindern), ergibt sich mit der Verformungsmethode nur ein einziger unbekannter Knotendrehwinkel φ_2

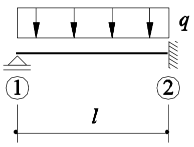


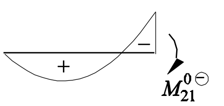
System unverschieblich → Stabdrehwinkel $\psi = 0$

→ Unbekannte: Knotendrehwinkel φ_2

1. Festeinspannmomente

Stab 1-2:



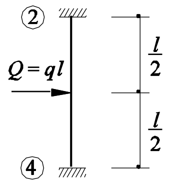


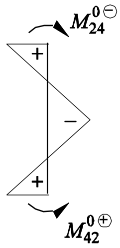
$$M_{12}^0 = 0$$

$$M_{21}^0 = -\frac{ql^2}{8}$$

Stab 2-3: keine Einwirkung → $M_{23}^0 = M_{32}^0 = 0$

Stab 2-4:



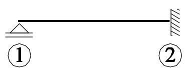


$$M_{24}^{0} = -\frac{Ql}{8} = -\frac{ql^2}{8}$$

$$M_{42}^{0} = \frac{Ql}{8} = \frac{ql^2}{8}$$

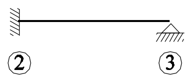
2. Stab- und Kreuzsteifigkeiten

Stab 1-2:



$$s_{21} = \frac{3EI}{l} \qquad t_{12} = t_{21} = s_{12} = 0$$

Stab 2-3:



$$s_{23} = \frac{3EI}{l} \qquad t_{23} = t_{32} = s_{32} = 0$$

Stab 2-4:

$$s_{24} = \frac{4EI}{l} \qquad t_{24} = \frac{2EI}{l}$$

$$s_{42} = \frac{4EI}{l} \qquad t_{42} = \frac{2EI}{l}$$

3. Stabendmomente

$$M_{ik} = M_{ik}^{0} + s_{ik} \cdot \varphi_i + t_{ik} \cdot \varphi_k - \left(s_{ik} + t_{ik}\right) \cdot \psi_{ik}$$

$$M_{ki} = M_{ki}^{0} + s_{ki} \cdot \varphi_k + t_{ki} \cdot \varphi_i - \left(s_{ki} + t_{ki}\right) \cdot \psi_{ki}$$

$M_{12} = 0$

$M_{21} = -\frac{ql^2}{8} + \frac{3EI}{l} \cdot \varphi_2$

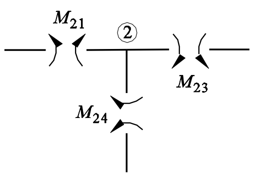
$M_{23} = 0 + \frac{3EI}{l} \cdot \varphi_2$

$M_{32} = 0$

$M_{24} = -\frac{ql^2}{8} + \frac{4EI}{l} \cdot \varphi_2$

$M_{42} = \frac{ql^2}{8} + \frac{2EI}{l} \cdot \varphi_2$

4. Knotengleichgewicht



$\Sigma M_{2k} = 0 \quad \rightarrow \quad M_{21} + M_{23} + M_{24} = 0$

$\Sigma M_{2k} = -\frac{ql^2}{8} + \frac{3EI}{l} \cdot \varphi_2 + \frac{3EI}{l} \cdot \varphi_2 - \frac{ql^2}{8} + \frac{4EI}{l} \cdot \varphi_2 = 0$

$\rightarrow -\frac{ql^2}{4} + \frac{10EI}{l} \cdot \varphi_2 = 0$

$\rightarrow \varphi_2 = \frac{ql^3}{40EI}$

5. Endgültige Stabendmomente

$M_{12} = 0$

$M_{21} = -\frac{ql^2}{8} + \frac{3ql^2}{40} = -\frac{2ql^2}{40}$

$M_{23} = \frac{3ql^2}{40}$

$M_{32} = 0$

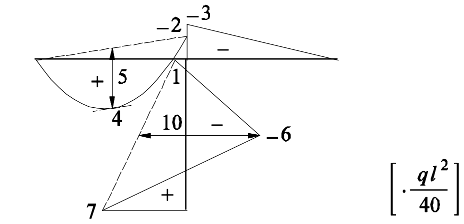
$M_{24} = -\frac{ql^2}{8} + \frac{4ql^2}{40} = -\frac{ql^2}{40}$

$M_{42} = \frac{ql^2}{8} + \frac{2ql^2}{40} = \frac{7ql^2}{40}$

Kontrolle: $\Sigma M_{2k} = 0$ i.O.

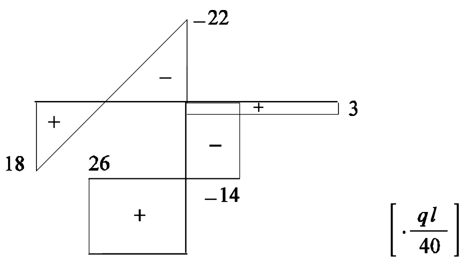
6. Schnittkraftlinien

M:



V:

Zeichnung nach
allg. Konvention: $V^+ \uparrow$ —— $\downarrow$

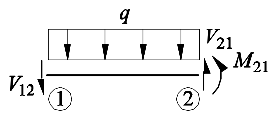


Berechnung nach
Konvention Verformungsmethode: $V^+ \downarrow$ —— $\uparrow$

$$V_{21}\cdot l-\frac{ql^2}{2}+M_{21}=0$$

$$V_{21}=\frac{ql}{2}-\frac{M_{21}}{L}=\underline{\underline{\frac{22ql}{40}}}$$

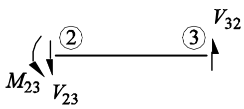
$$V_{12}=V_{21}-ql=-\underline{\underline{\frac{18ql}{40}}}$$



$$V_{23}\cdot l+M_{23}=0$$

$$V_{23}=-\frac{M_{23}}{l}=-\underline{\underline{\frac{3ql}{40}}}$$

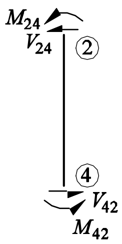
$$V_{32}=V_{23}=-\underline{\underline{\frac{3ql}{40}}}$$



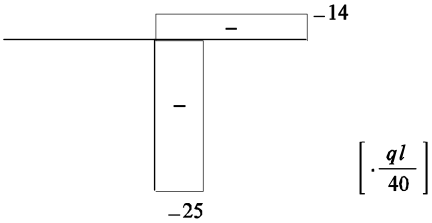
$$V_{24}\cdot l-ql\cdot\frac{l}{2}+M_{24}+M_{42}=0$$

$$V_{24}=\frac{ql}{2}-\frac{\left(M_{24}+M_{42}\right)}{l}=\underline{\underline{\frac{14ql}{40}}}$$

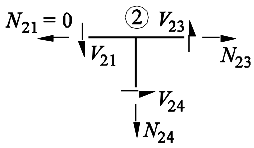
$$V_{42}=V_{24}-ql=-\underline{\underline{\frac{26ql}{40}}}$$



N:



Berechnung von N mit der Konvention Verformungsmethode für $V^+ \downarrow \;—\!\uparrow$



$N_{21} = 0$

$N_{24} + V_{21} - V_{23} = 0 \;\rightarrow\; N_{24} = -\dfrac{25ql}{40}$

$N_{23} + V_{24} = 0 \;\rightarrow\; N_{23} = -\dfrac{14ql}{40}$