Risk & Safety in Engineering

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Contents of Today's Lecture

Methods of structural reliability theory

- General case - FORM
- Monte-Carlo simulation
- Partial safety factors

Structural Reliability Analysis

In the general case the resistance and the load may be defined in terms of functions where X are **basic random variables**

$$R = f_1(\mathbf{X})$$
$$S = f_2(\mathbf{X})$$

The safety margin can be written as where $g(x)$ is called the **limit state function**

$$M = R - S = f_1(\mathbf{X}) - f_2(\mathbf{X}) = g(\mathbf{X})$$

Failure occurs when

$$g(\mathbf{x}) \leq 0$$

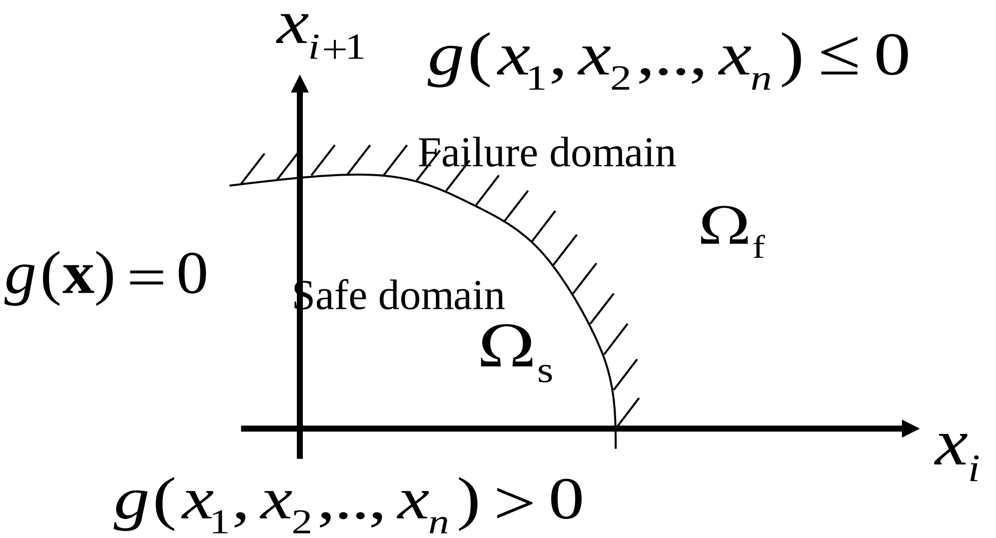
Structural Reliability Analysis

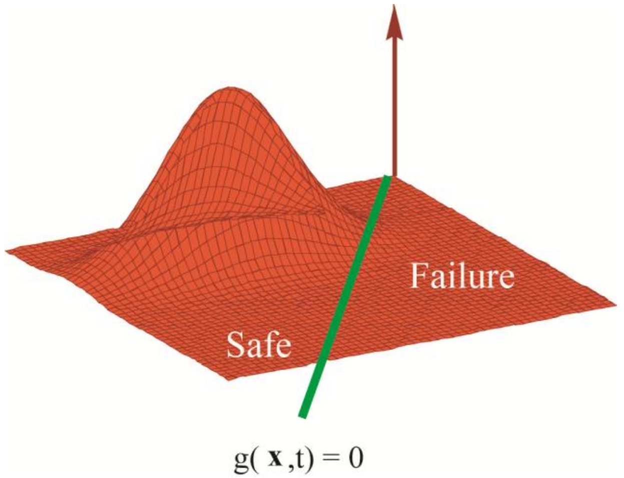
Setting $g(\mathbf{x}) = 0$ defines a (n-1) dimensional surface in the space spanned by the n basic variables X

This is the failure surface separating the sample space of X into a safe domain and a failure domain

The failure probability may in general terms be written as

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \leq 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$





Failure event

$$\mathbf{F} = \{g(\mathbf{x}) \leq 0\}$$

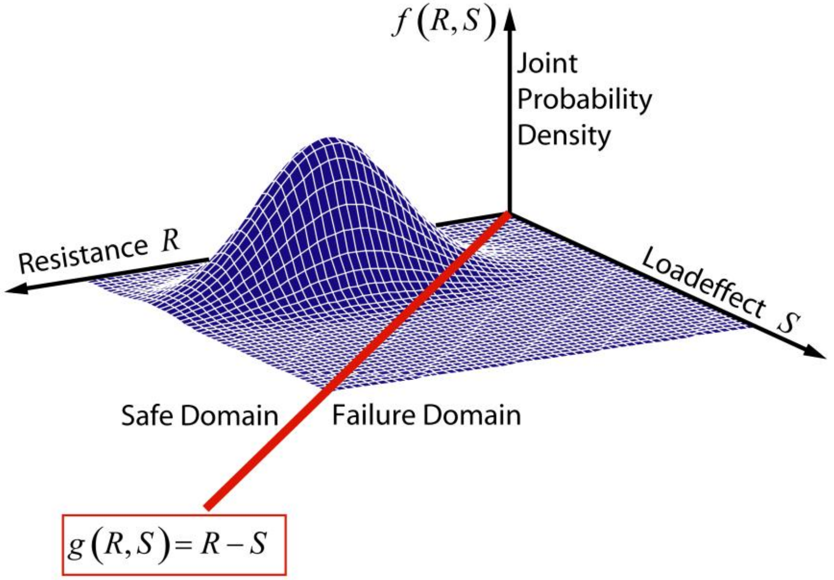
Basics of Structural Reliability Methods

The probability of failure can be assessed by

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \leq 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function for the basic random variables $\boldsymbol{X}$

For the 2-dimensional case the failure probability simply corresponds to the integral under the joint probability density function in the area of failure



Basics of Structural Reliability Methods

The probability of failure can be calculated using

- numerical integration
  (Simpson, Gauss, Tchebyschev)

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \leq 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

but for problems involving dimensions higher than say 6 the numerical integration becomes cumbersome

Other methods are necessary !

Basics of Structural Reliability Methods

When the limit state function is linear

$$g(\mathbf{x}) = a_0 + \sum_{i=1}^{n} a_i \cdot x_i$$

the saftey margin M is defined through

$$M = a_0 + \sum_{i=1}^{n} a_i \cdot X_i$$

with

mean value

$$\mu_M = a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}$$

and

variance

$$\sigma_M^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \rho_{ij} a_i a_j \sigma_i \sigma_j$$

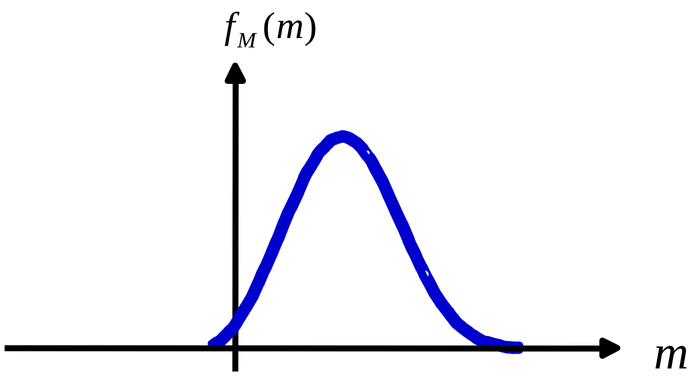
Basics of Structural Reliability Methods

The failure probability can then be written as

$$P_F = P(g(\mathbf{X}) \leq 0) = P(M \leq 0)$$

The reliability index is defined as

$$\beta = \frac{\mu_M}{\sigma_M}$$ (Basler and Cornell)



Provided that the safety margin is Normal distributed

the failure probability is determined as

$$P_F = \Phi(-\beta)$$

Basics of Structural Reliability Methods

Example:

Consider a steel rod with resistance *r* subjected to a tension force *s*

$$g(\mathbf{X}) = R - S$$

r* and *s* are modeled by the random variables *R* and *S

$$\mu_R = 350, \sigma_R = 35$$
$$\mu_S = 200, \sigma_S = 40$$

The probability of failure is wanted

$$P(R - S \leq 0)$$

The safety margin is

$$M = R - S \begin{cases} \mu_M = 350 - 200 = 150 \\ \sigma_M = \sqrt{35^2 + 40^2} = 53.15 \end{cases}$$

The reliability index is then

$$\beta = \frac{150}{53.15} = 2.84$$

and the probability of failure

$$P_F = \Phi(-2.84) = 2.4 \cdot 10^{-3}$$

Basics of Structural Reliability Methods

The problem of lack of invariance

The estimate of the failure probability depends on the formulation of the safety margin / limit state function:

for the previous example:

$$\mu_R = 350, \sigma_R = 35$$
$$\mu_S = 200, \sigma_S = 40$$

$$M = R - S \qquad M = \ln\frac{R}{S} = \ln R - \ln S$$

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \qquad \beta' = \frac{\mu_{\ln(R/S)}}{\sigma_{\ln(R/S)}} \cong \frac{\ln\mu_R - \ln\mu_s}{\sqrt{V_R^2 + V_S^2}}$$

$$\beta = \frac{150}{53.15} = 2.84 \qquad \beta' \simeq \frac{\ln(350) - \ln(200)}{\sqrt{(35/350)^2 + (40/200)^2}} = 2.5$$

Basics of Structural Reliability Methods

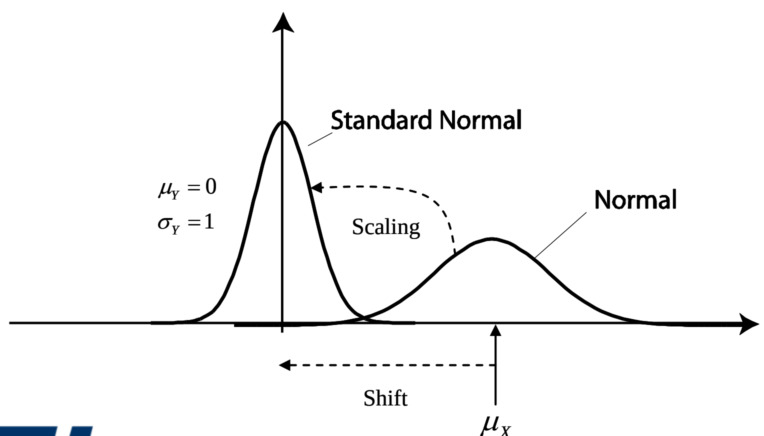
- **The reliability problem is generalized in order:**
  - To resolve the lack of invariance problem
  - To facilitate the estimation of failure probabilities based on non-linear limit state functions
  - To facilitate the consideration of correlated random variables
  - To facilitate the consideration of non-normal random variables

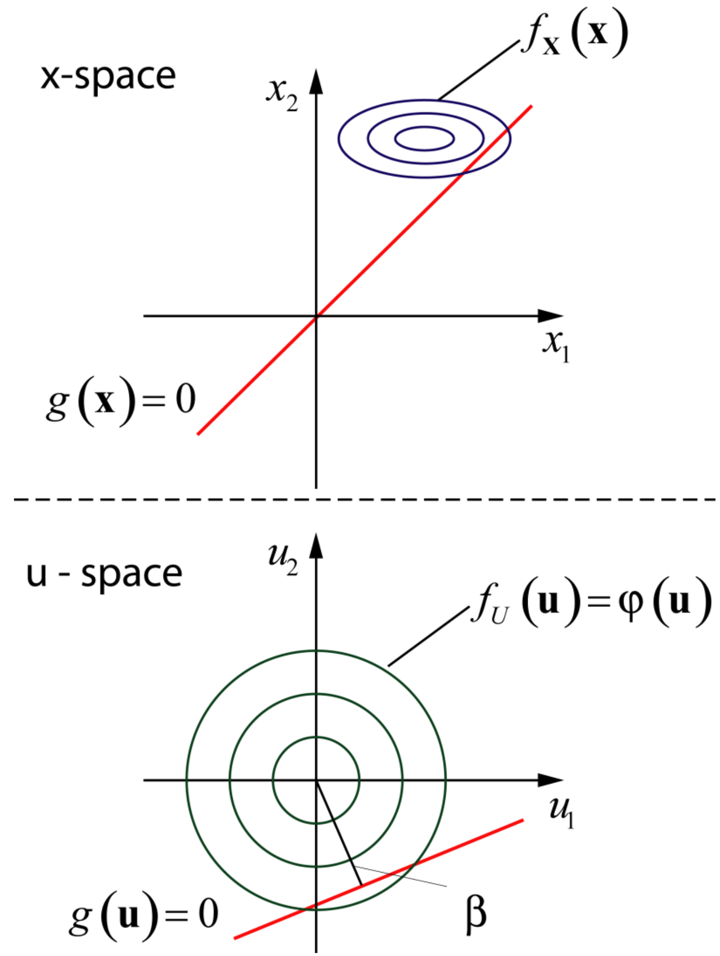
Basics of Structural Reliability Methods

The reliability index β has the geometrical interpretation of being the shortest distance between the failure surface and the origin in standard Normal distributed space U

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}$$

in which case the components of U have zero means and variances equal to 1





Basics of Structural Reliability Methods

Usually the limit state function is non-linear

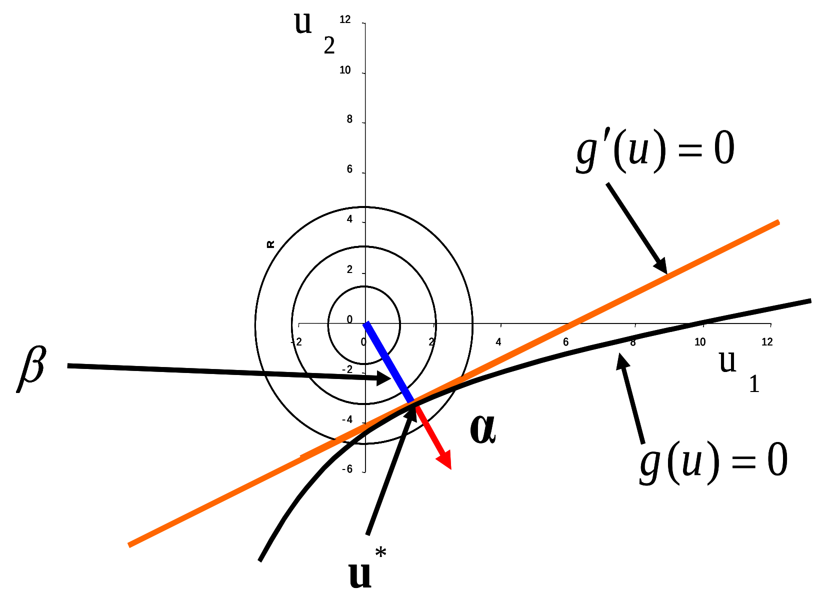
- this small phenomenon caused the so-called invariance problem

Hasofer & Lind suggested to linearize the limit state function in the design point

- this solved the invariance problem

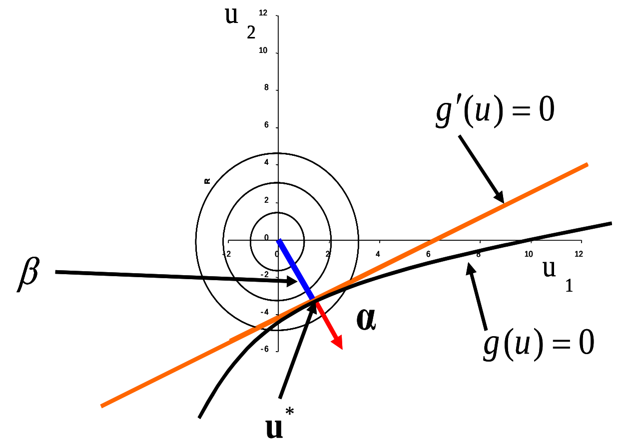
The reliability index may then be determined by the following optimization problem

Can however easily be linearized !



$$\beta = \min_{\mathbf{u} \in \{g(\mathbf{u})=0\}} \sqrt{\sum_{i=1}^{n} u_i^2}$$

Basics of Structural Reliability Methods



The optimization problem can be formulated as an iteration problem

1) the design point is determined as $\mathbf{u}^* = \beta \cdot \boldsymbol{\alpha}$

2) the normal vector to the limit state function is determined as

$$\alpha_i = \frac{-\dfrac{\partial g}{du_i}(\beta \cdot \boldsymbol{\alpha})}{\left[\sum_{j=1}^{n} \dfrac{\partial g}{du_i}(\beta \cdot \boldsymbol{\alpha})^2\right]^{1/2}}, \quad i = 1,2,..n$$

3) the safety index is determined as $g(\beta \cdot \alpha_1, \beta \cdot \alpha_2, ...\beta \cdot \alpha_n) = 0$

4) a new design point is determined as $u^* = (\beta \cdot \alpha_1, \beta \cdot \alpha_2, ...\beta \cdot \alpha_n)^T$

5) the above steps are continued until convergence in β is attained

Basics of Structural Reliability Methods

Example :

Consider the steel rod with cross-sectional area *a* and yield stress *r*

$$h = r \cdot a$$

The rod is loaded with the tension force s

The limit state function can then be written as

$$g(\mathbf{x}) = r \cdot a - s$$

***r*, *a* and *s* are uncertain and modeled by normal distributed random variables**

$$\mu_R = 350, \sigma_R = 35 \quad \mu_S = 1500, \sigma_S = 300$$

$$\mu_A = 10, \sigma_A = 1$$

we would like to calculate the probability of failure

Example (cont.)

The initial step is to transform the basic random variables into standardized Normal distributed space.

$$U_R = \frac{R-\mu_R}{\sigma_R} \qquad U_A = \frac{A-\mu_A}{\sigma_A} \qquad U_S = \frac{S-\mu_S}{\sigma_S}$$

Then we write the limit state function in terms of the realizations of the standardized Normal distributed random variables.

$$g(u) = (u_R\sigma_R + \mu_R)(u_A\sigma_A + \mu_A) - (u_S\sigma_S + \mu_S)$$

$$= (35u_R + 350)(u_A + 10) - (300u_S + 1500)$$

$$= 350\mathrm{u}_\mathrm{R} + 350u_A - 300u_S + 35u_R u_A + 2000$$

Iteration: 1) Arbitrary first design point: $\mathbf{u}^* = \beta \cdot \boldsymbol{\alpha}$ e.g. $\beta = 3, \quad \boldsymbol{\alpha} = [-0.58, \ -0.58, \ 0.58]^T$

2) Normal vector at this point:

$$\alpha_i = \frac{-\dfrac{\partial g}{du_i}(\beta \cdot \boldsymbol{\alpha})}{\left[\sum_{j=1}^{n} \dfrac{\partial g}{du_i}(\beta \cdot \boldsymbol{\alpha})^2\right]^{1/2}}, \quad i = 1, 2, ..n \quad \Rightarrow \boldsymbol{\alpha}^{(k+1)}$$

3) the safety index is determined as $g(\beta \cdot \alpha_1, \beta \cdot \alpha_2, ... \beta \cdot \alpha_n) = 0$

4) a new design point is determined as $u^* = (\beta \cdot \alpha_1, \beta \cdot \alpha_2, ... \beta \cdot \alpha_n)^T$

Example (cont.)

The reliability index is calculated as

$$\beta = \frac{-2000}{350\alpha_R + 350\alpha_A - 300\alpha_S + 35\beta\alpha_R\alpha_A}$$

the components of the α-vector are then calculated as

$$\begin{cases} \alpha_R = -\frac{1}{k}(350 + 35\beta\alpha_A) \\ \alpha_A = -\frac{1}{k}(350 + 35\beta\alpha_R) \\ \alpha_S = \frac{300}{k} \end{cases}$$

where

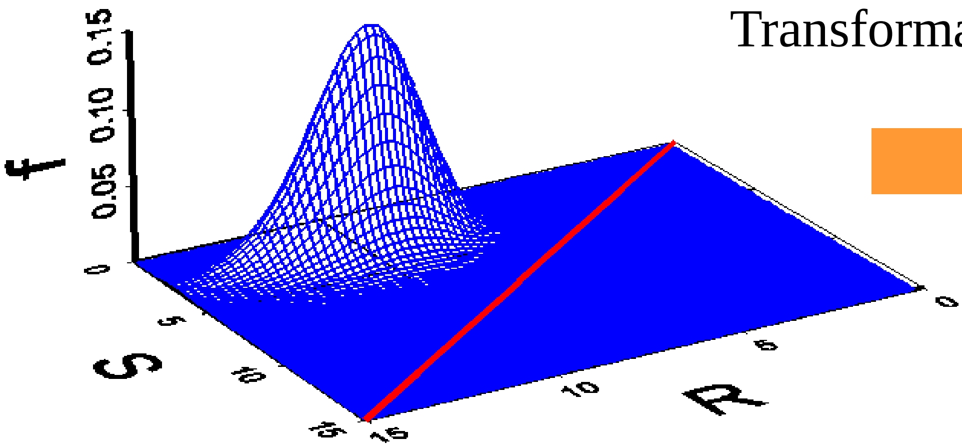
$$k \rightarrow \sqrt{\alpha_R^2 + \alpha_A^2 + \alpha_S^2} \overset{!}{=} 1$$

Basics of Structural Reliability Methods

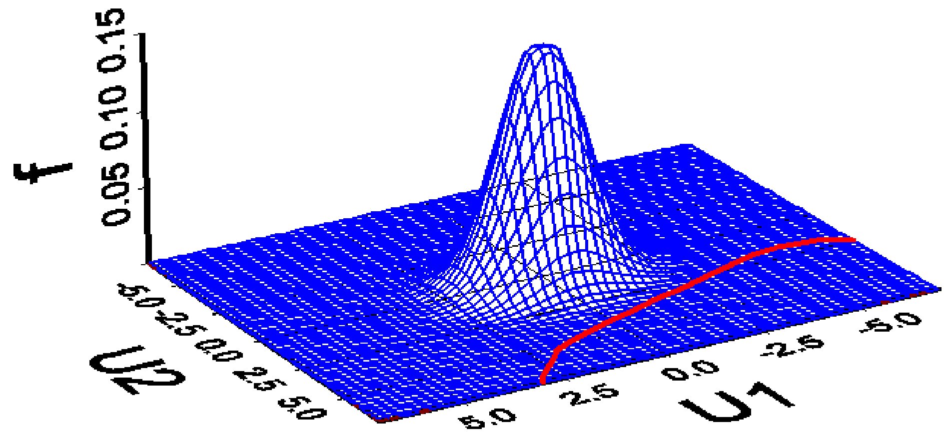
Following the iteration scheme we get the following iteration history

Iteration	Start	1	2	3	4	5
β	3.0000	3.6719	3.7399	3.7444	3.7448	3.7448
α_R	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_A	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_S	0.5800	0.5916	0.6084	0.6086	0.6087	0.6087

Basics of Structural Reliability Methods



Transformation



g(Z): linear

$\mu_{Z1}, \mu_{Z2} \in \mathbb{R}$

$\sigma_{Z1}, \sigma_{Z2} \in \mathbb{R}$

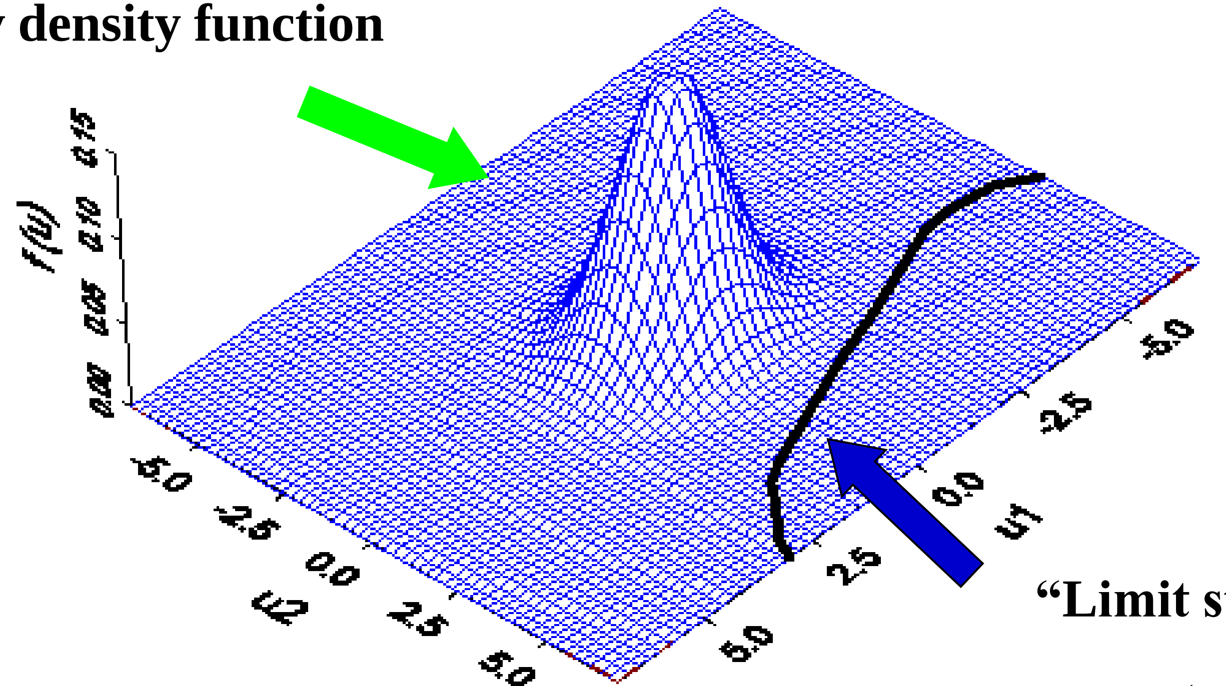
g(U): non linear

$\mu_{U1} = \mu_{U2} = 0$

$\sigma_{U1} = \sigma_{U2} = 1$

Basics of Structural Reliability Methods

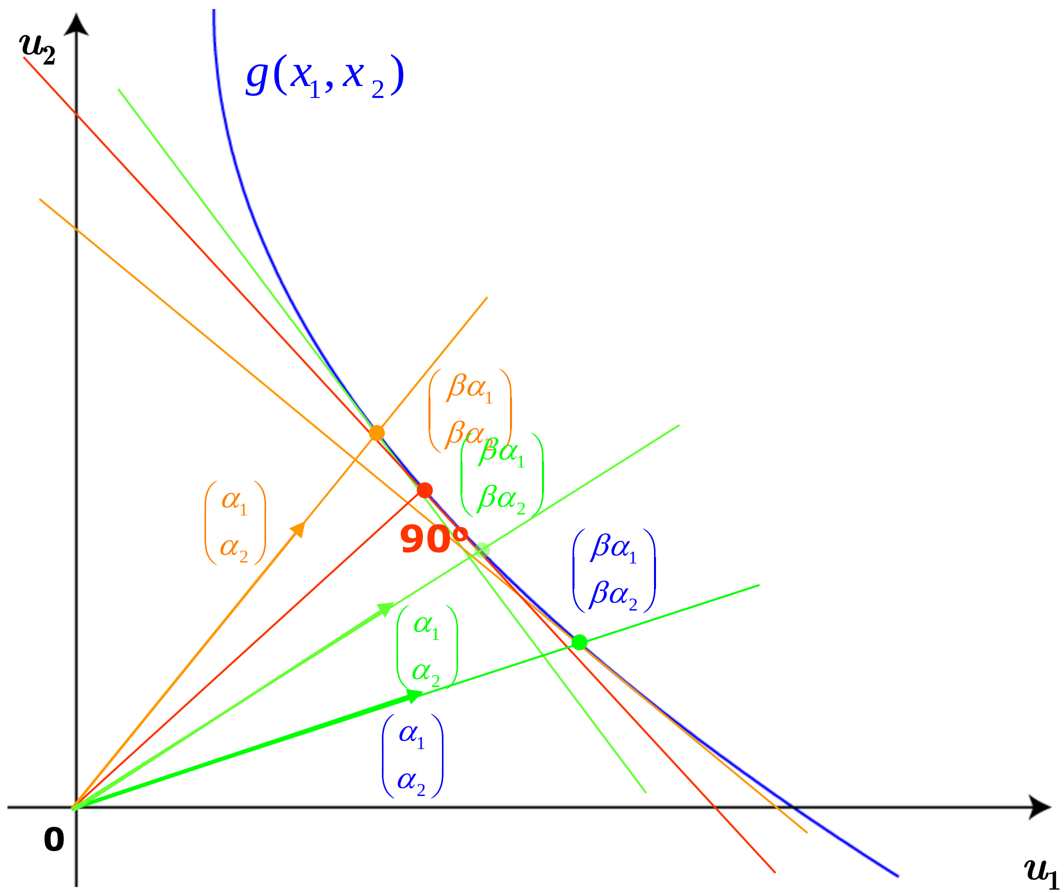
joint probability density function



"Limit state function"

g(U) = R-S

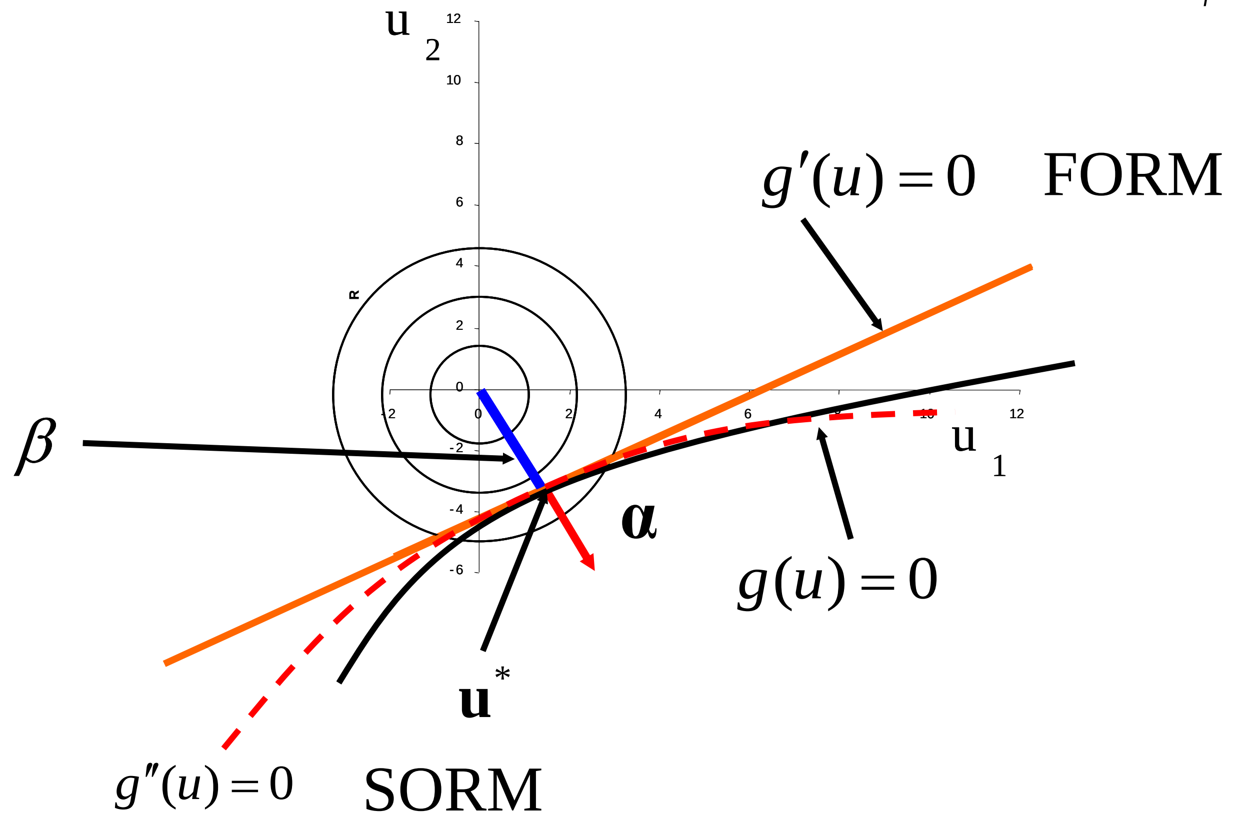
Basics of Structural Reliability Methods



Basics of Structural Reliability Methods

SORM Improvements

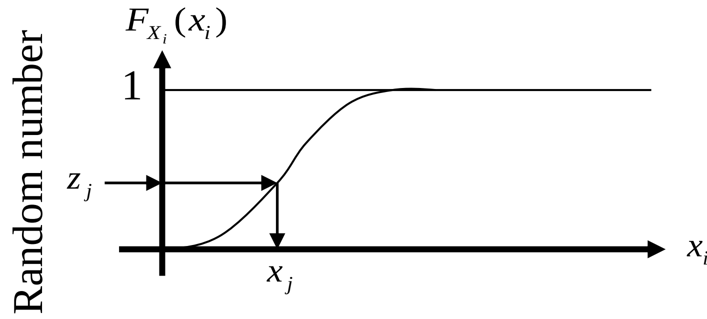
$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \le 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$



Basics of Structural Reliability Methods

Simulation methods may also be used to solve the integration problem

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \leq 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$



1. n_{MCS} realizations $\hat{x}_i$ of the vector X are generated
2. for each realization the value of the limit state function is evaluated
3. the realizations where the limit state function is zero or negative are counted

$$n_{f,MCS} = \sum_{i=1}^{n_{MCS}} I[g(\hat{x}_i) \leq 0]$$

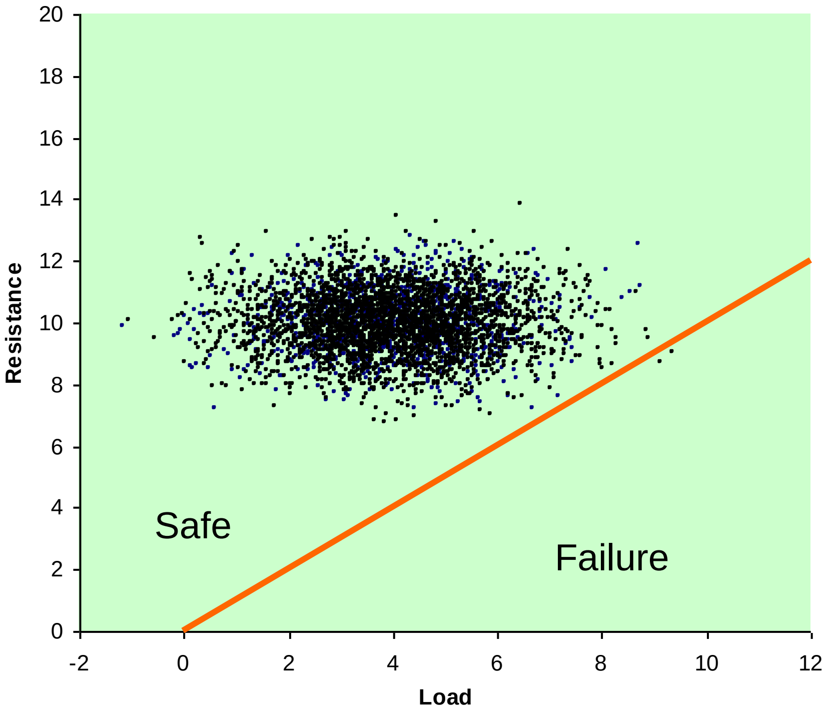
4. the failure probability is estimated as

$$\Pr(F) \approx p_{MCS} = {}^{n_{f,MCS}}/_{n_{MCS}}$$

Basics of Structural Reliability Methods

- **Estimation of failure probabilities using Monte Carlo Simulation**
  - ***m* random outcomes of R und S are generated and the number of outcomes $n_{f,MCS}$ in the failure domain are recorded and summed**
  - **The failure probability p_{MCS} is then**

$$\Pr(F) \approx p_{MCS} = {}^{n_{f,MCS}}/_{n_{MCS}}$$



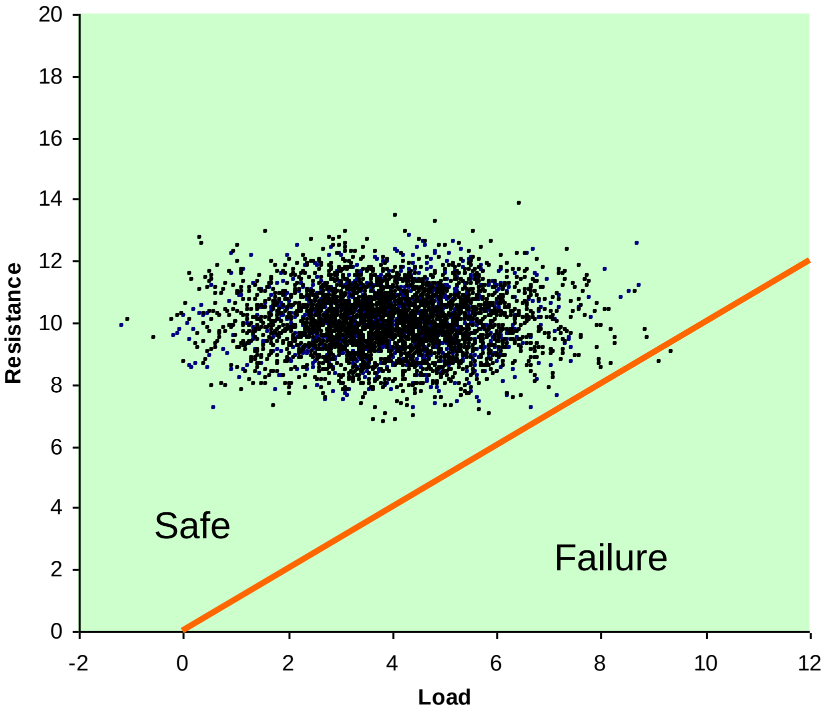
Basics of Structural Reliability Methods

- **Estimation of failure probabilities using Monte Carlo Simulation**

  - **Uncertainty**

$$\sigma_{MCS} = \sqrt{\Pr(F) - \Pr(F)^2 / n_{MCS}}$$

$$\sigma_{MCS} \approx \sqrt{p_{MCS} - p_{MCS}^2 / n_{MCS}}$$



Basics of Structural Reliability Methods

Partial safety factors

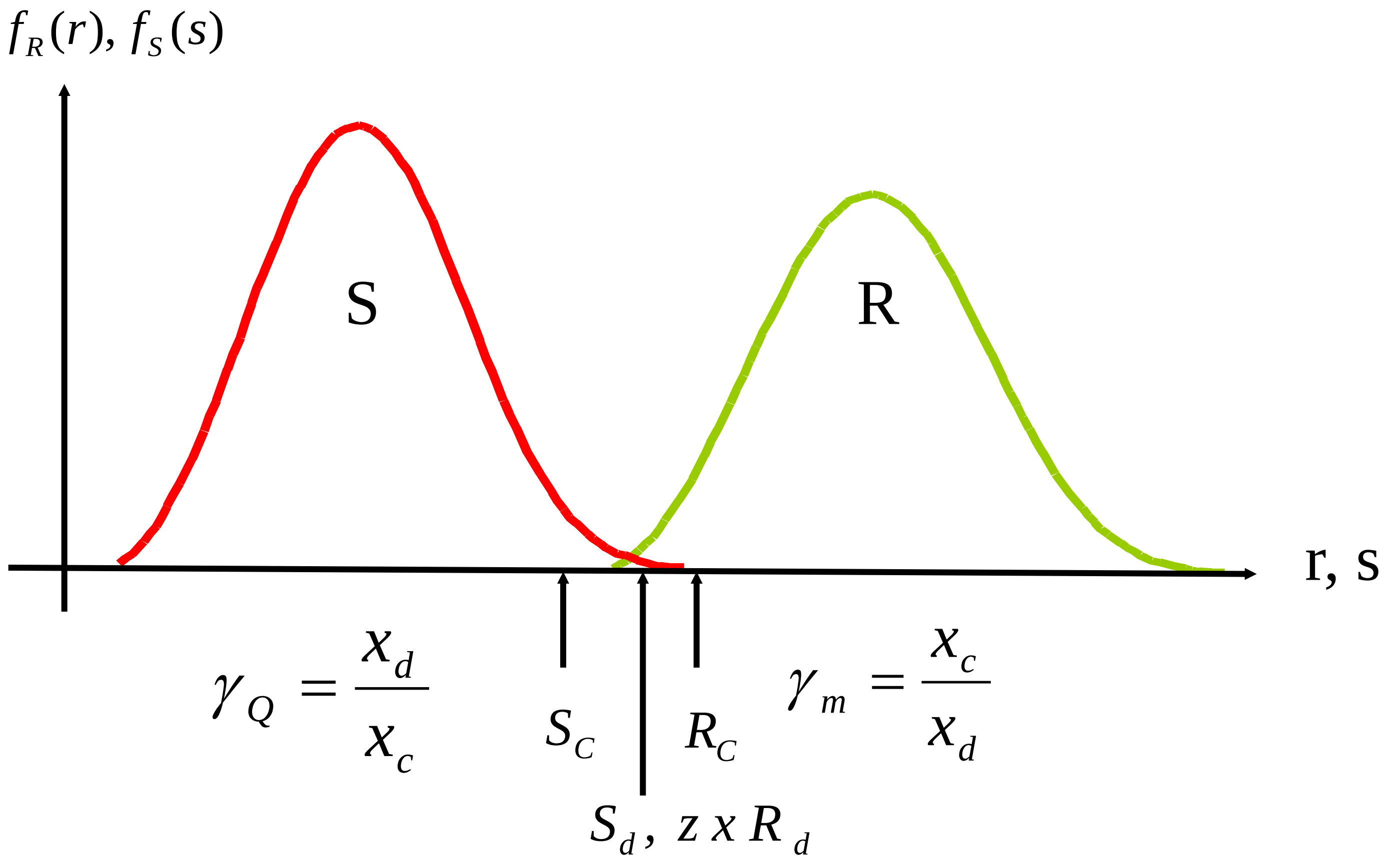
Design codes prescribe design equations where the design variables (e.g. cross-sections) are to be determined as a function of

$$zR_c / \gamma_m - \left(\gamma_{G_a} G_c + \gamma_Q Q_C\right) = 0$$

- Characteristic values $R_C \quad G_C \quad Q_C$
- Partial safety factors $\gamma_m \quad \gamma_G \quad \gamma_Q$

The design variables are selected such that the design equation is close to zero

Basics of Structural Reliability Methods



Basics of Structural Reliability Methods

Example

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α_S	0.5800	0.5916	0.6084	0.6086	0.6087	0.6087

$$\mu_R = 350, \sigma_R = 35$$
$$\mu_A = 10, \sigma_A = 1$$
$$\mu_S = 1500, \sigma_R = 300$$

Design value for r

$$r_d = u_R^* \cdot \sigma_R + \mu_R = -0.561 \cdot 3.7448 \cdot 35 + 350.0 = 276.56$$

Characteristic value for r

$$r_c = -1.64 \cdot \sigma_R + \mu_R = -1.64 \cdot 35 + 350 = 292.60$$

Partial safety factor

$$\gamma_R = \frac{292.60}{276.56} = 1.06$$