Risk & Safety in Engineering

Dr. Jochen Köhler

Content of today

Repitition: Conditional probability and Bayes' Theorem

Introduction to Decision Theory

- The problem
- The decision tree
- Prior decision analysis
- Posterior decision analysis
- Pre-posterior decision analysis

Conditional Probability and Bayes‘ Rule

- Conditional probabilities are of special interest as they provide the basis for utilizing new information in decision making.

- The conditional probability of an event E_1 given that event E_2 has occured is written as:

$$P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Not defined if $P(E_2) = 0$

- The events E_1 and E_2 are said to be statistically independent if:

$$P(E_1 | E_2) = P(E_1)$$

Conditional Probability and Bayes‘ Rule

- From $$P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Commutative

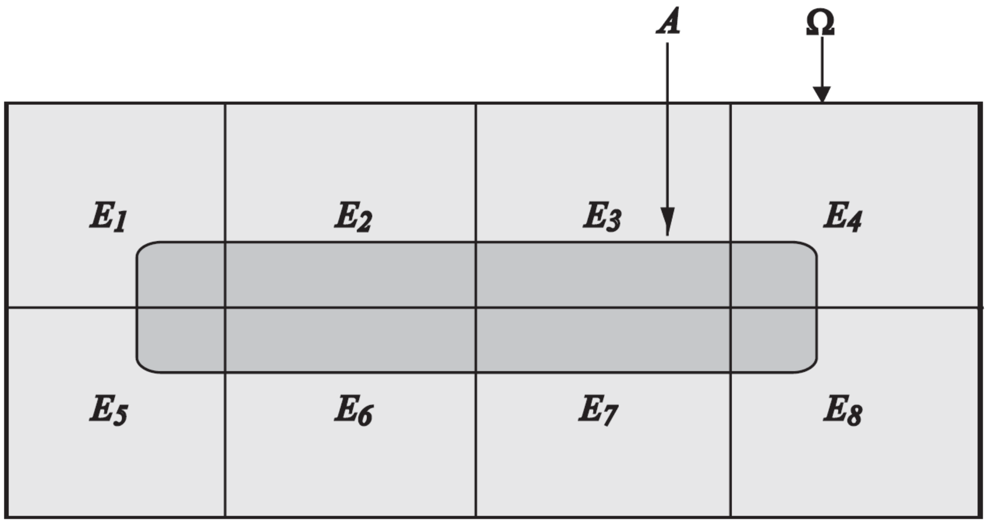
- it follows that $$P(E_1 \cap E_2) = P(E_2)P(E_1 | E_2) = P(E_1)P(E_2 | E_1)$$

- and when E_1 and E_2 are statistically independent there is

$$P(E_1 \cap E_2) = P(E_2)P(E_1)$$

Conditional Probability and Bayes‘ Rule

- Consider the sample space Ω divided up into n mutually exclusive events $E_1, E_2, ..., E_n$



Conditional Probability and Bayes‘ Rule

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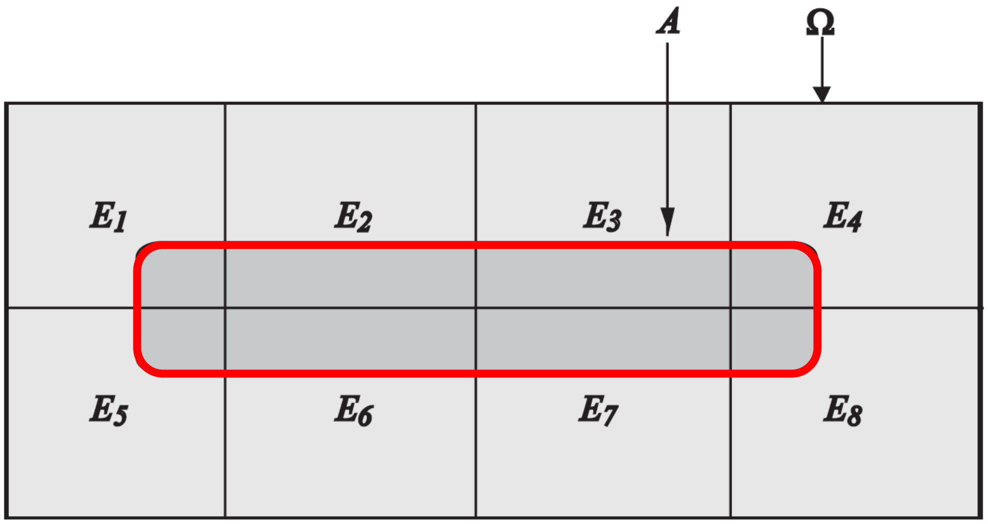

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n)$$

Conditional Probability and Bayes' Rule

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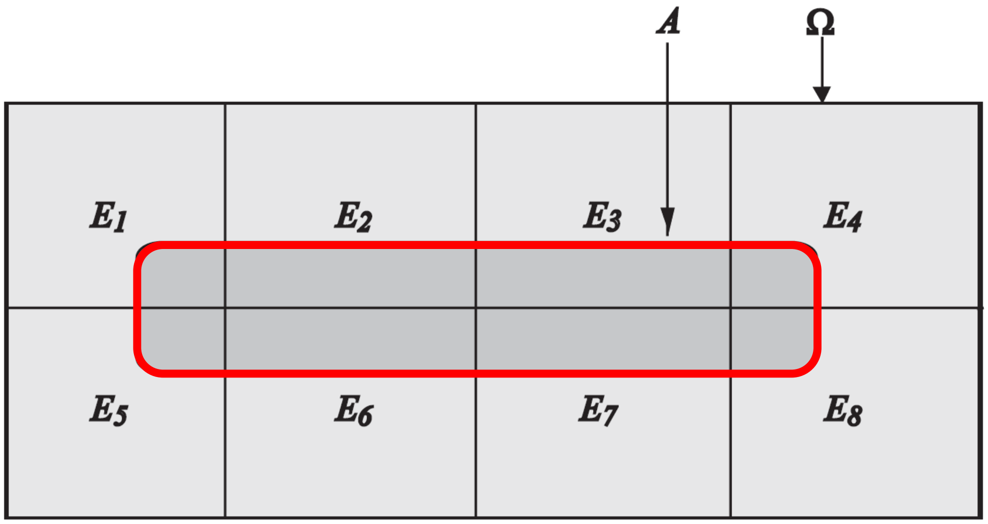

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n)$$

$$= P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \ldots + P(A|E_n)P(E_n)$$

$$= \sum_{i=1}^{n} P(A|E_i)P(E_i)$$

Conditional Probability and Bayes‘ Rule

- As there is $P(A \cap E_i) = P(A|E_i)P(E_i) = P(E_i|A)P(A)$

  we have

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{i=1}^{n} P(A|E_i)P(E_i)}$$

Bayes Rule

Reverend Thomas Bayes (1702-1764)

Conditional Probability and Bayes‘ Rule

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Prior

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Likelihood

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Posterior

Bayes Rule

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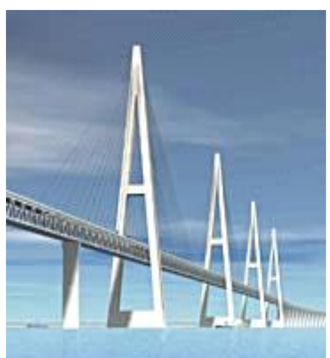
Decision Analysis in Engineering

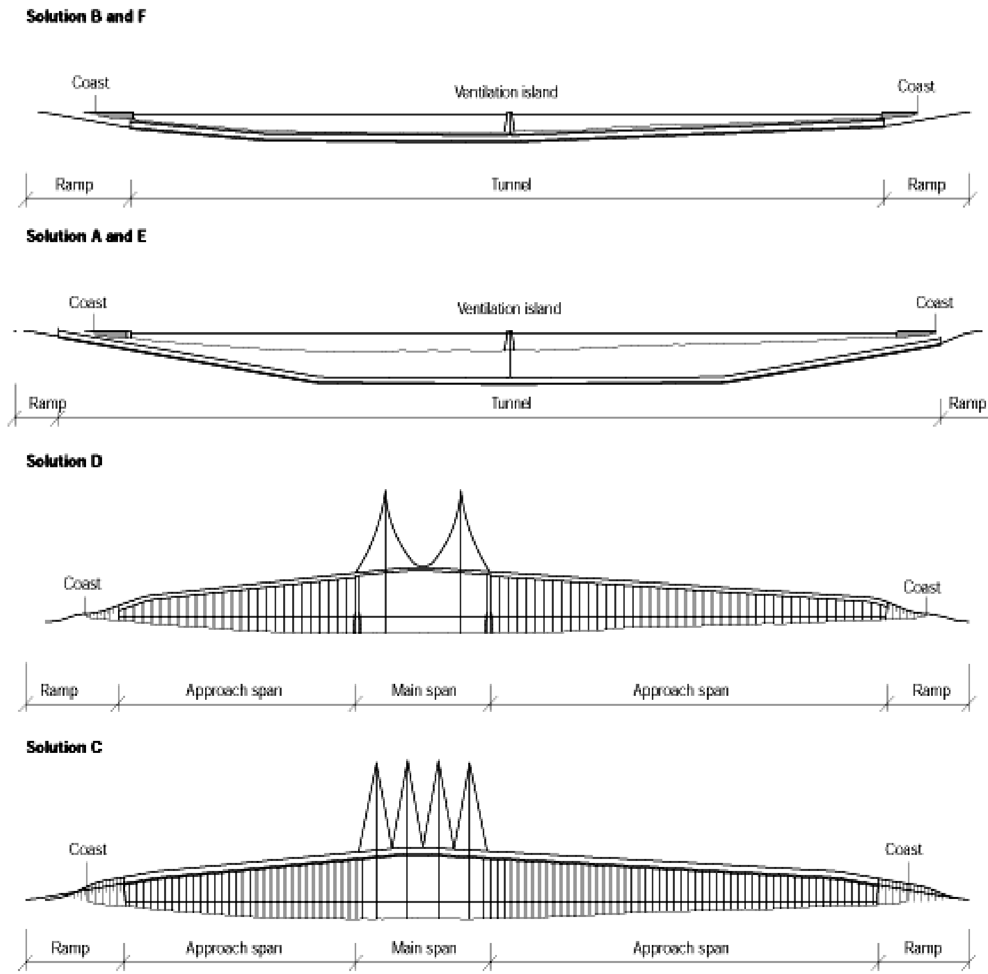
- The basic engineering problem

Several solutions may be identified

The available information is uncertain

A decision must be made !





Decision Analysis in Engineering

Approach

- Formulation of the decision problem
    - Identification of the decision maker and the preferences of the decision maker
    - Mapping of the decision process
    - Identification of the possible decision alternatives
    - Identification of the contributing uncertainties

- Identification of potential consequences and their utility (cost/benefit)

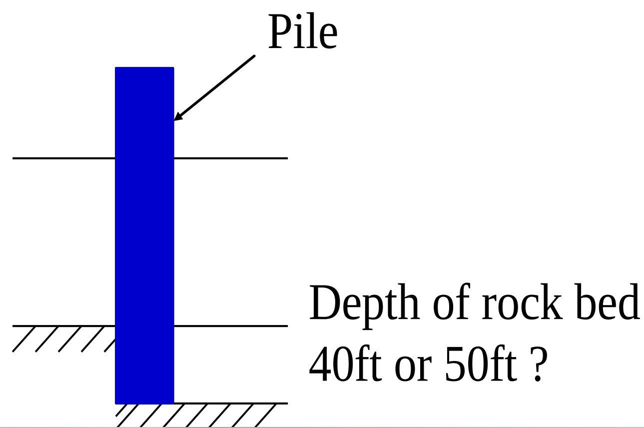
- Assessment of the probabilities of the consequences

- Comparison of the different decision alternatives based on their expected utilities

- Final decision making and reporting of the assumptions underlying the selected alternative

Decision Analysis in Engineering

- The decision tree



Decision alternatives	State	Consequence	Utility
Pile 40m	depth=40m	none	0
	depth=50m	splice two piles	400
Pile 50m	depth=40m	cut pile	100
	depth=50m	none	0

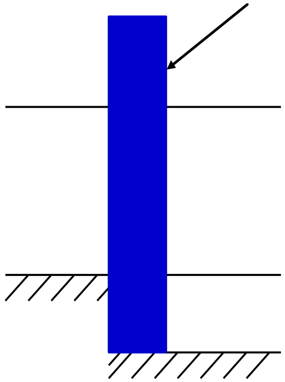
Decision Analysis in Engineering

The different types of decision analysis

- Prior
- Posterior
- Pre-posterior

Illustrated on an example :

What pile length should be applied ?



Decision Analysis in Engineering

The different types of decision analysis

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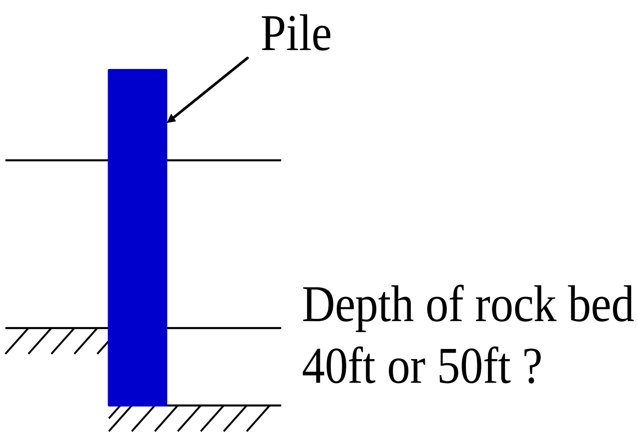


Illustrated on an example :

What pile length should be applied ?

Alternatives :

a_0 : Choose a 40 ft pile
a_1 : Choose a 50 ft pile

States of nature (depth to rock bed)
θ_0 : Rock bed at 40 ft
θ_1 : Rock bed at 50 ft

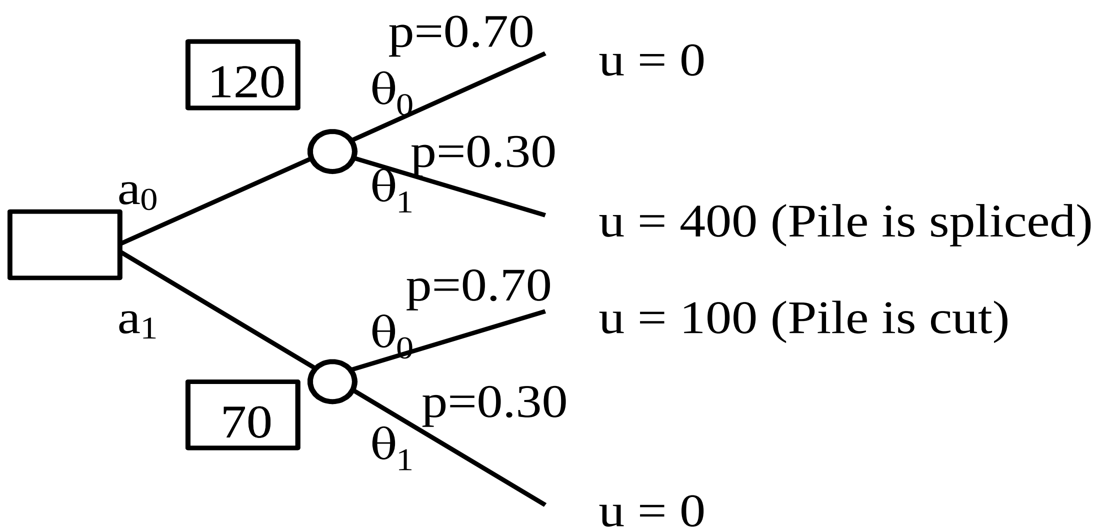
$$P'(\theta_0) = 0.7$$

$$P'(\theta_1) = 0.3$$

Decision Analysis in Engineering

Prior Analysis

$$P'(\theta_0) = 0.7$$
$$P'(\theta_1) = 0.3$$



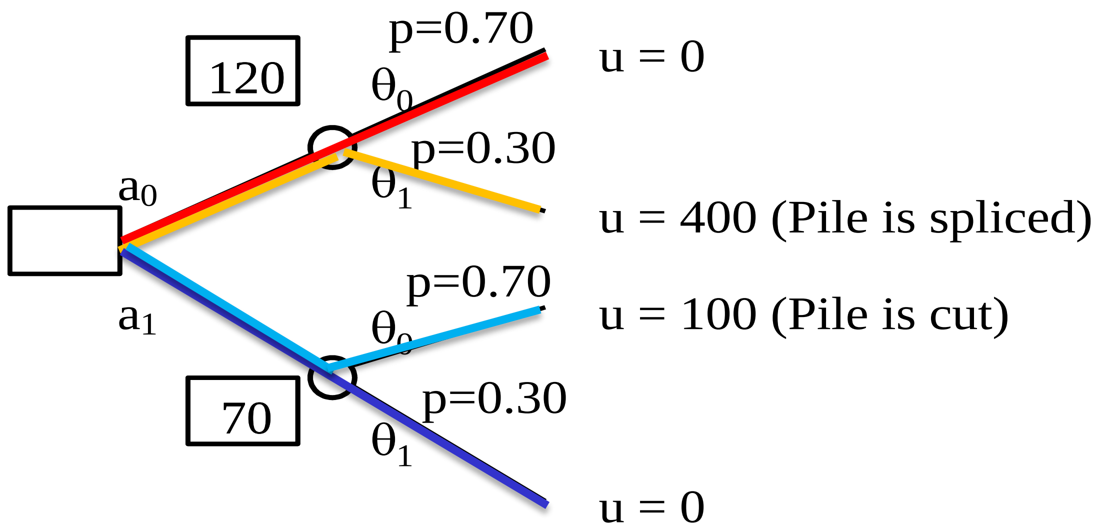
$$E'[u] = \min\{u[a_0]\ , u[a_1]\}$$
$$= \min\{P'[\theta_0]\times u[\theta_0|a_0] + P'[\theta_1]\times u[\theta_1|a_0]\ ,$$
$$P'[\theta_0]\times u[\theta_0|a_1] + P'[\theta_1]\times u[\theta_1|a_1]\}$$
$$= \min\{0.7\times 0 + 0.3\times 400,\ 0.7\times 100 + 0.3\times 0\}$$
$$= \min\{120, 70\} = 70 \quad \Rightarrow \text{Decision for } a_1 \text{ (50ft Pile)}$$

Decision Analysis in Engineering

Prior Analysis

$$P'(\theta_0) = 0.7$$

$$P'(\theta_1) = 0.3$$



$$E'[u] = \min\{u[a_0]\ ,u[a_1]\}$$

$$= \min\{P'[\theta_0]\times u\left[\theta_0|a_0\right] + P'[\theta_1]\times u\left[\theta_1|a_0\right]\ ,$$

$$P'[\theta_0]\times u\left[\theta_0|a_1\right] + P'[\theta_1]\times u\left[\theta_1|a_1\right]\}$$

$$= \quad \min\{0.7\times 0 + 0.3\times 400,\ 0.7\times 100 + 0.3\times 0\}$$

$$= \quad \min\{120, 70\} = 70 \qquad \Rightarrow \ \text{Decision for } \mathrm{a}_1 \ \text{(50ft Pile)}$$

Decision Analysis in Engineering

Posterior Analysis

$$P''(\theta_i) = \frac{P[z_k | \theta_i] P'[\theta_i]}{\sum_j P[z_k | \theta_j] P'[\theta_j]}$$

$$\left(\text{Posteriori-probability of } \theta_i \text{ for given info} \right) = \left(\text{Normalizing constant} \right) \left(\text{Likelihood given } \theta_i \right) \left(\text{A-Priori probability of } \theta_i \right)$$

Decision Analysis in Engineering

Posterior Analysis - Example

$$P''(\theta_i) = \frac{P[z_k|\theta_i]P'[\theta_i]}{\sum_j P[z_k|\theta_j]P'[\theta_j]}$$

Ultrasonic tests to determine the depth to bed rock

True state / Test result	θ_0 40 ft – depth	θ_1 50 ft – depth
z_0 - 40 ft indicated	0.6	0.1
z_1 - 50 ft indicated	0.1	0.7
z_2 - 45 ft indicated	0.3	0.2

Likelihoods of the different indications/test results given the various possible states of nature – ultrasonic test methods $P\left[z_k|\theta_j\right]$

Decision Analysis in Engineering

Posterior Analysis

$$P''(\theta_i) = \frac{P[z_k|\theta_i]P'[\theta_i]}{\sum_j P[z_k|\theta_j]P'[\theta_j]}$$

It is assumed that a test gives a 45 ft indication :

$$P''[\theta_0] = P[\theta_0|z_2] \propto P[z_2|\theta_0]P[\theta_0] = 0.3\ x\ 0.7 = 0.21$$

$$P''[\theta_1] = P[\theta_1|z_2] \propto P[z_2|\theta_1]P[\theta_1] = 0.2\ x\ 0.3 = 0.06$$

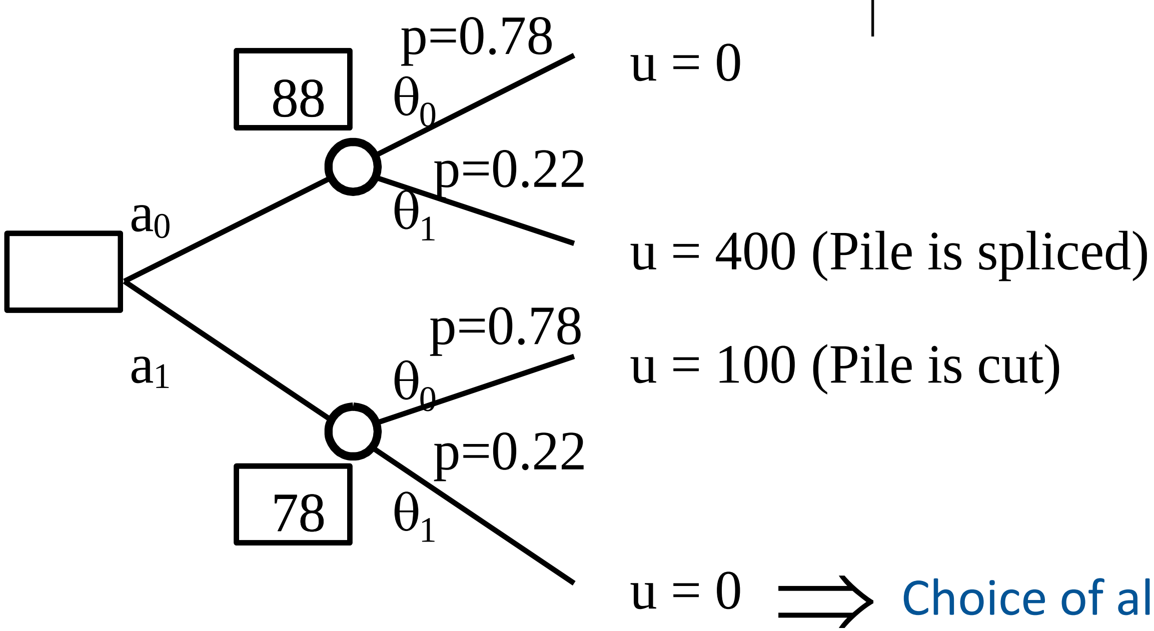
$$P''[\theta_0|z_2] = \frac{0.21}{0.21+0.06} = 0.78$$

$$P''[\theta_1|z_2] = \frac{0.06}{0.21+0.06} = 0.22$$

Decision Analysis in Engineering

Posterior Analysis

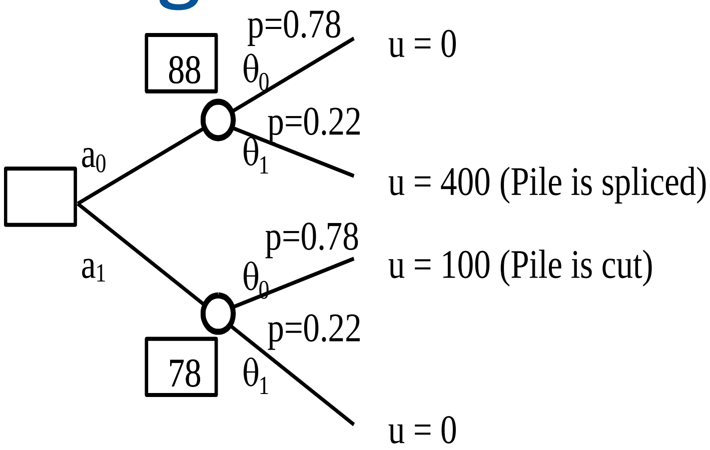
a test gives a 45 ft indication



$\Longrightarrow$ Choice of alternative a_1 (50ft Pile)

Decision Analysis in Engineering

Posterior Analysis



$$E''\left[u\middle|z_2\right] = \min_j \{E''\left[u(a_j)\middle|z_2\right]\}$$

$$= \min\{ P''[\theta_0]\times 0 + P''[\theta_1]\times 400, P''[\theta_0]\times 100 + P''[\theta_1]\times 0\}$$

$$= \min\{0.78\times 0 + 0.22\times 400, 0.78\times 100 + 0.22\times 0\}$$

$$= \min\{88\ \ , 78\} = 78$$

$\Longrightarrow$ Choice of alternative a_1 (50ft Pile)

Decision Analysis in Engineering

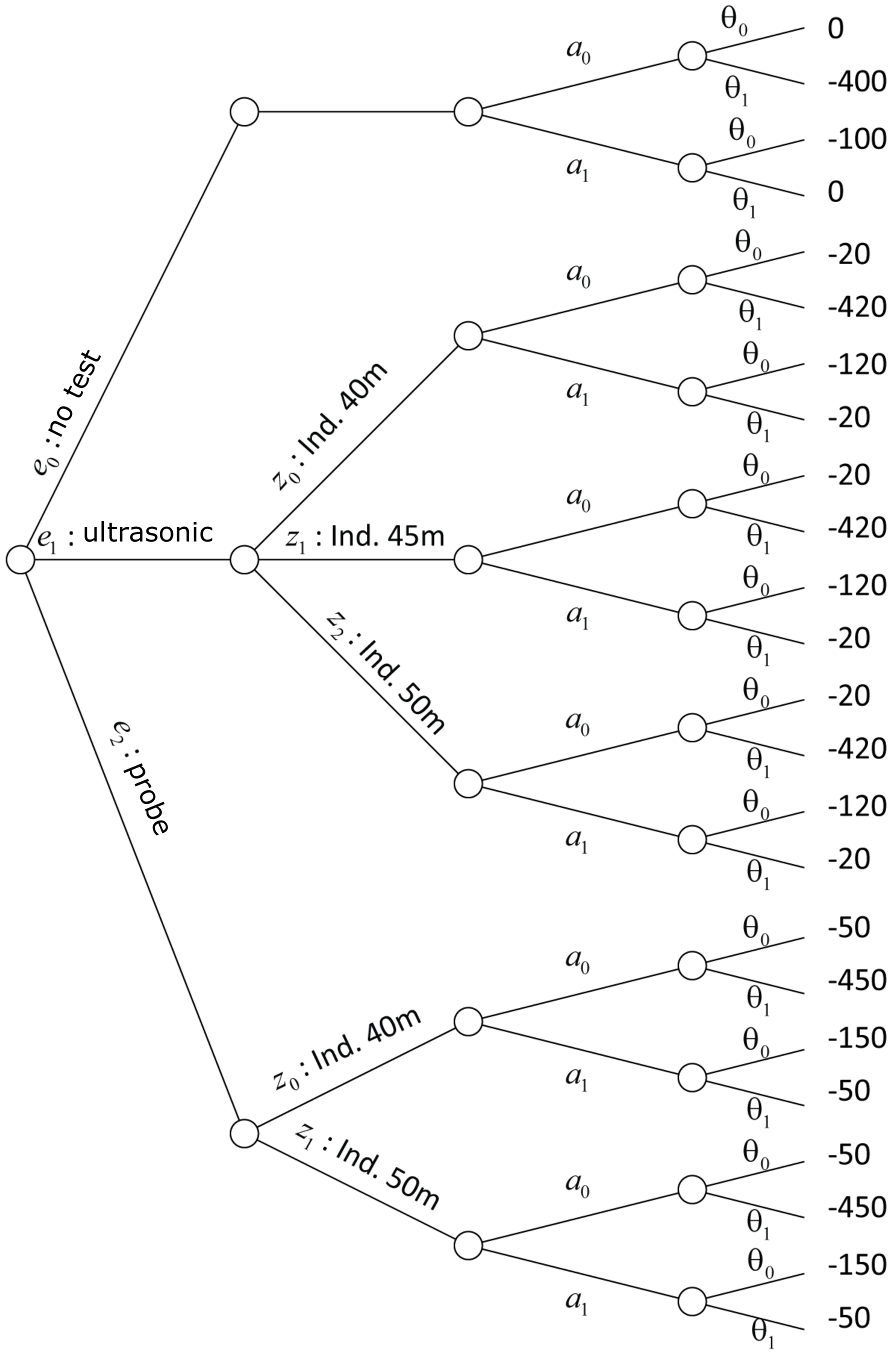
Pre-Posteriori Analyse

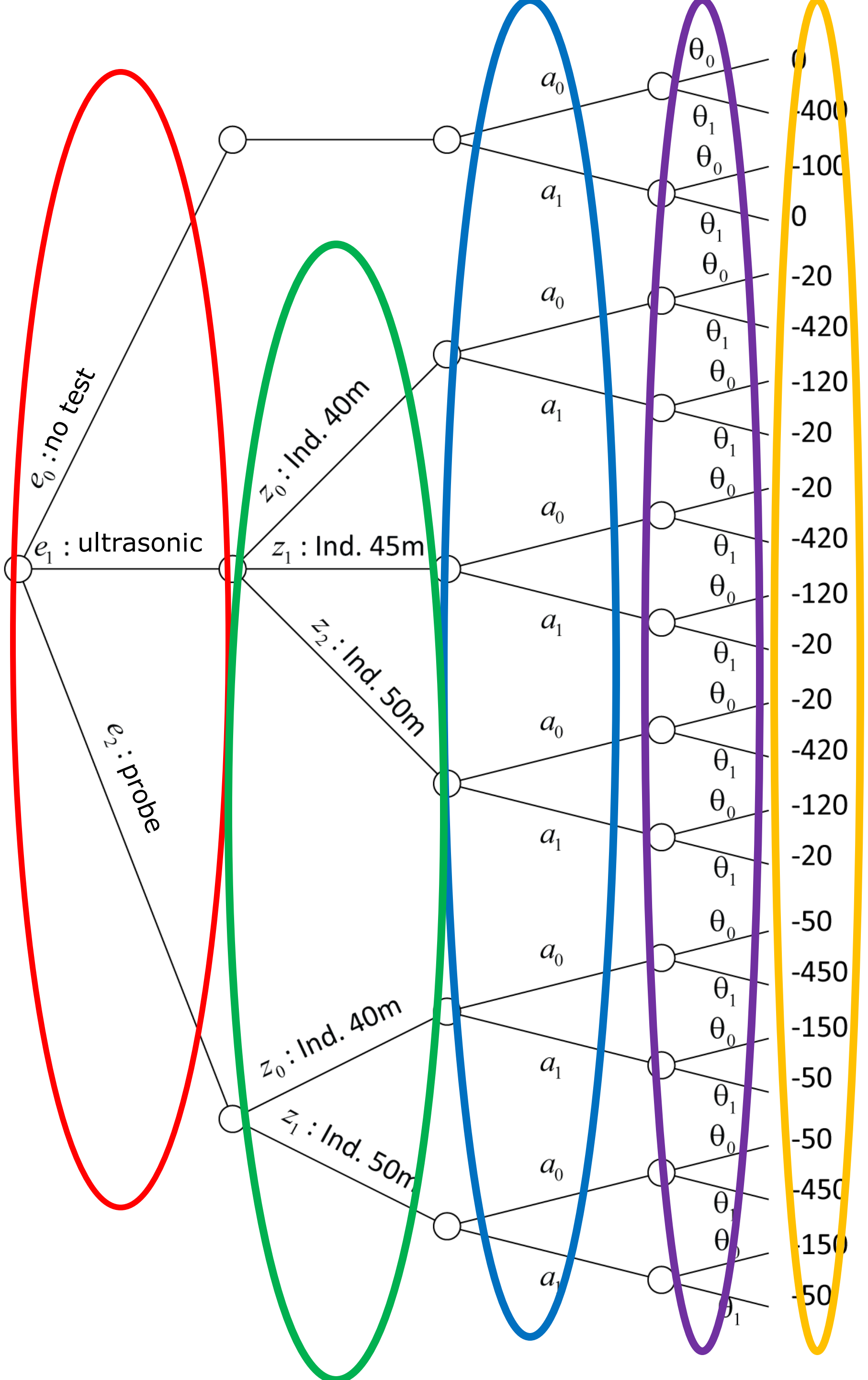
- The same example
- Three different options for tests
  - e_0 no test-> no cost
  - e_1 ultrasonic -> 20 GE
  - e_2 probe -> 50 GE

- How can be decided which test option carries the best efficiency ??

Pre-Posteriori Analysis

Decision model

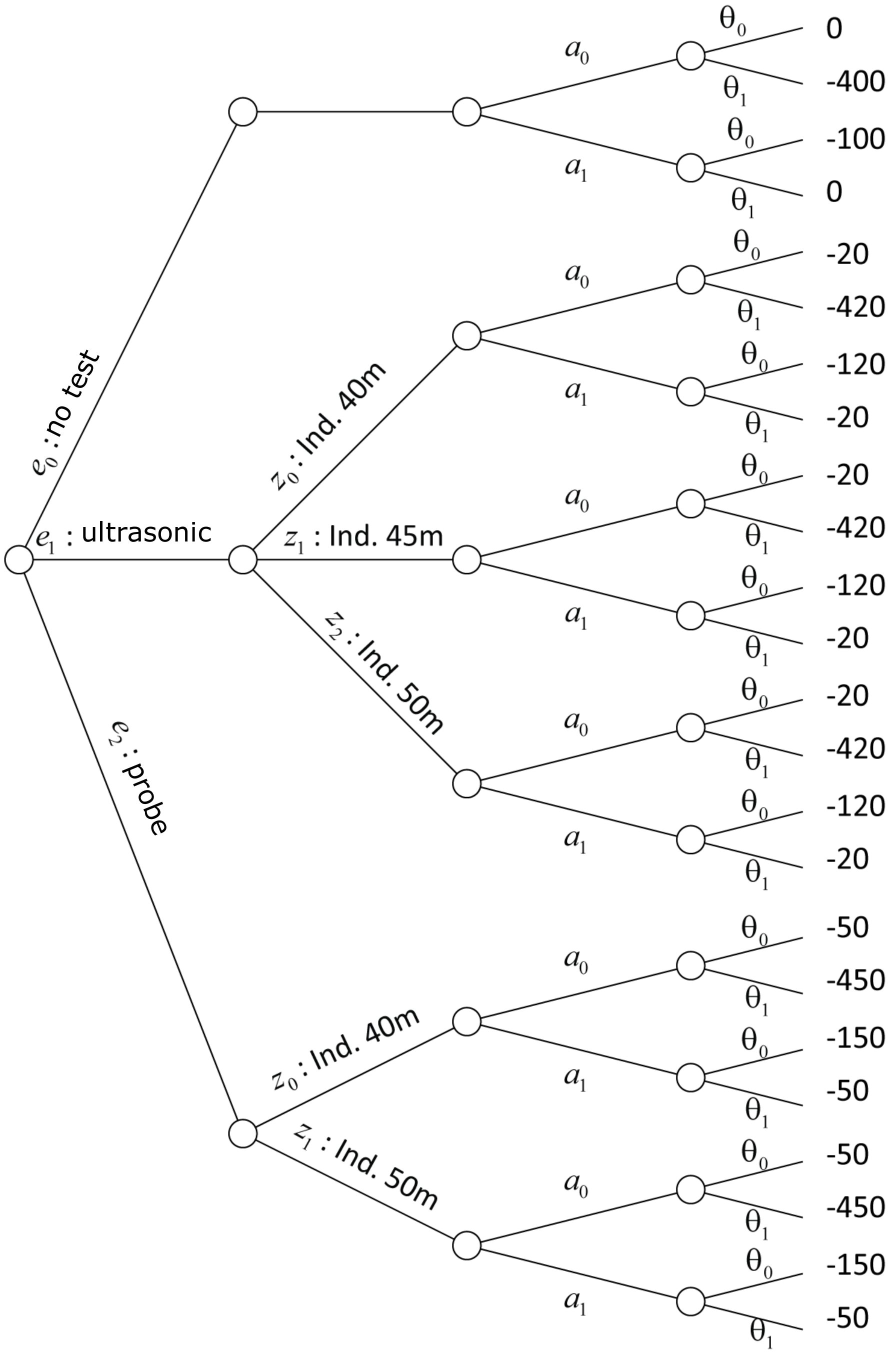




Pre-Posteriori Analysis

Decision model:

- Test options
- Test results
- Decision alternatives
- states
- consequences

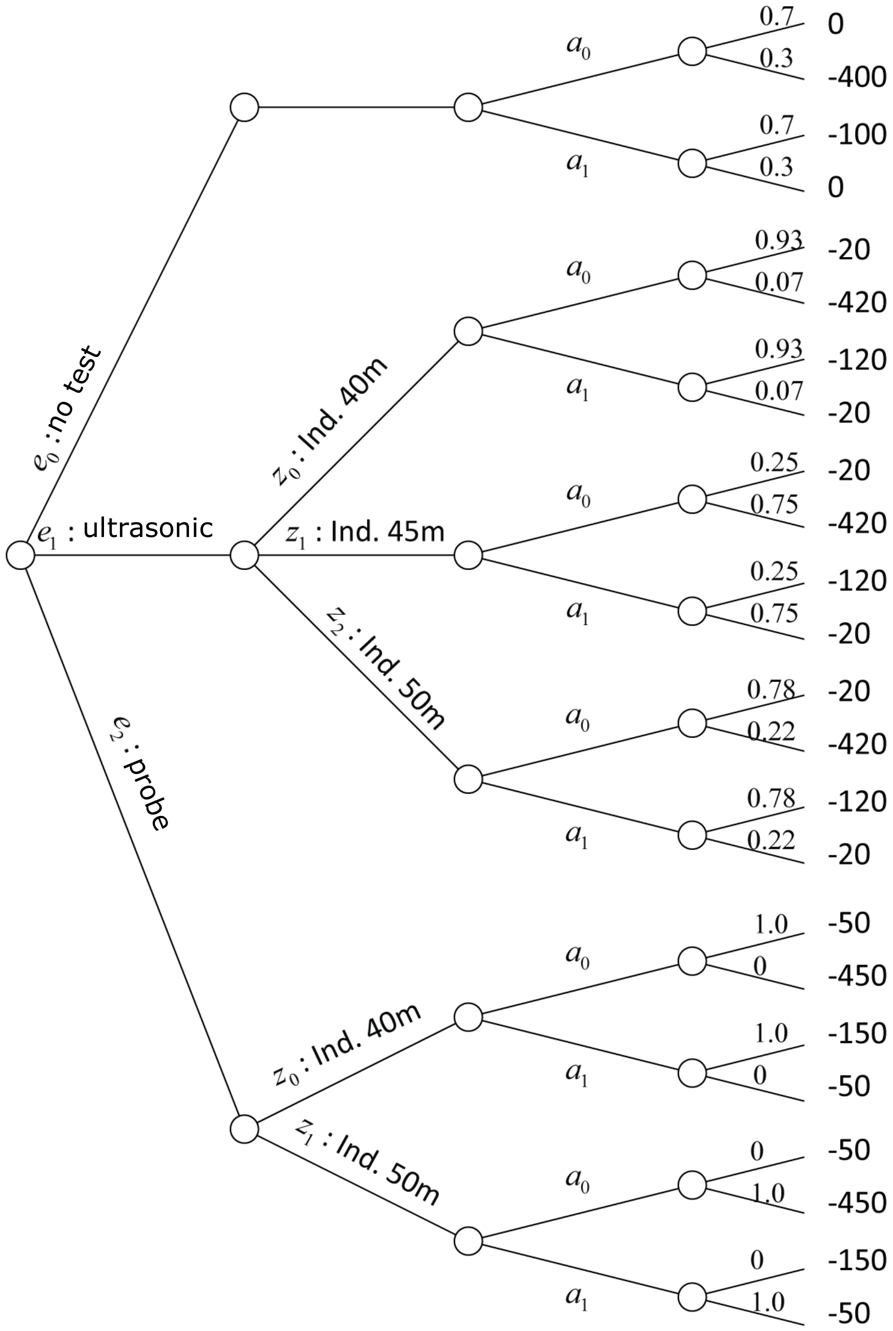


Pre-Posteriori Analysis

Solution -> from right to left:

1. Assessment of probabilities

Pre-Posteriori Analysis



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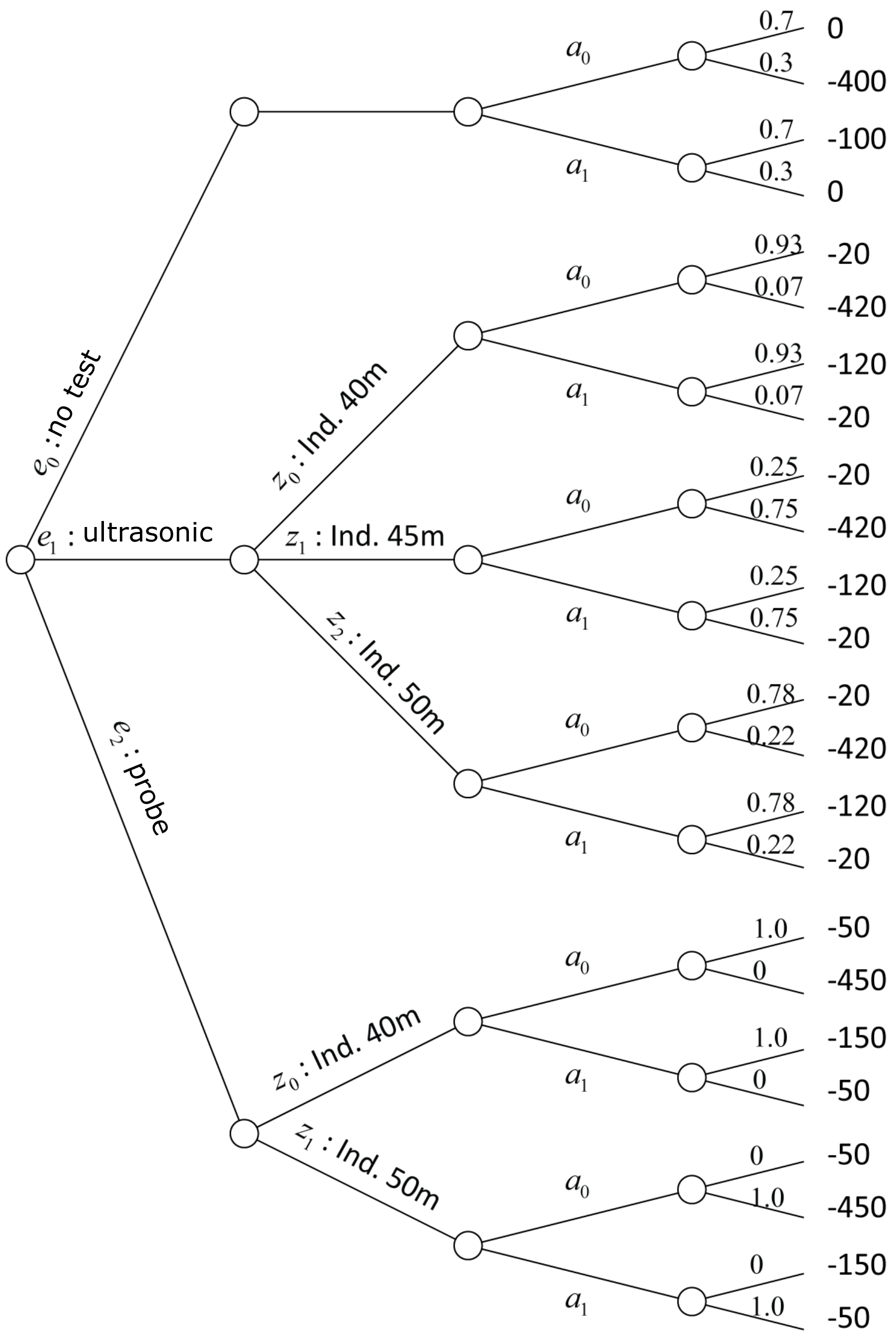
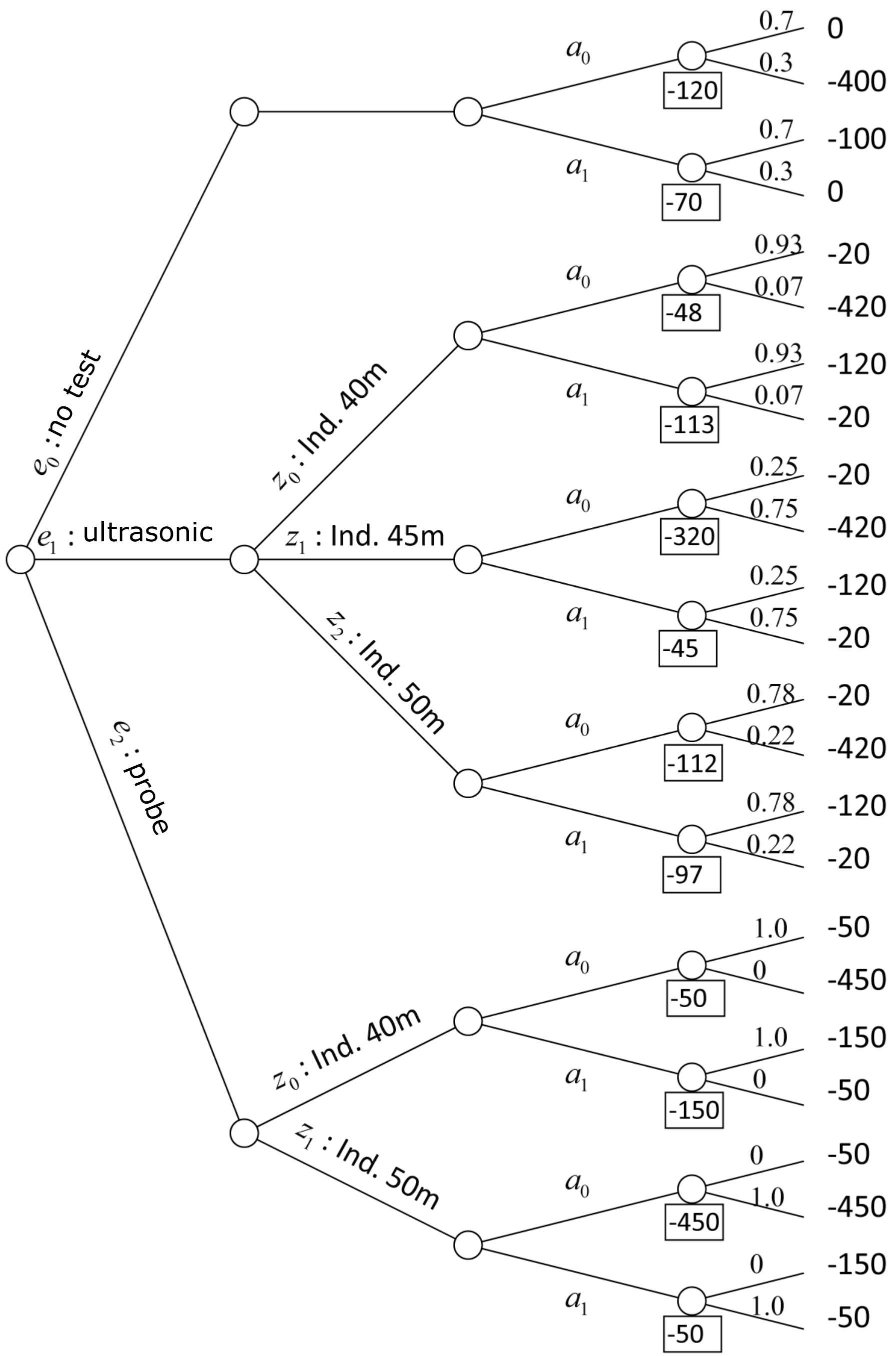


Solution -> from right to left:

1. Assessment of probabilities

2. Computation of expected utility

$$E\left[u\middle|a,z,e\right]=\sum_i P\left[\theta_i\middle|z,e\right]\times u\left[\theta_i\middle|a,z,e\right]$$



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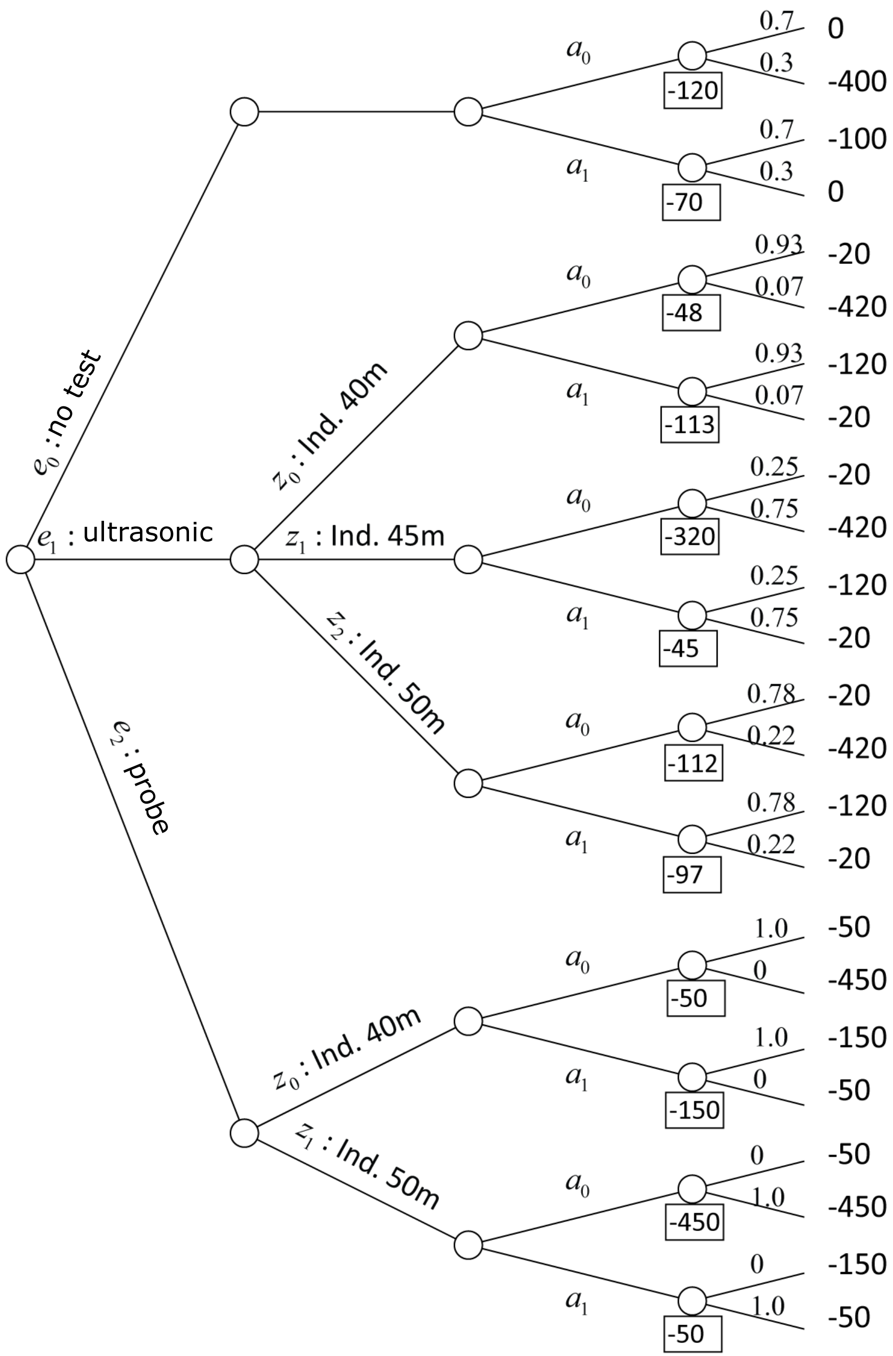

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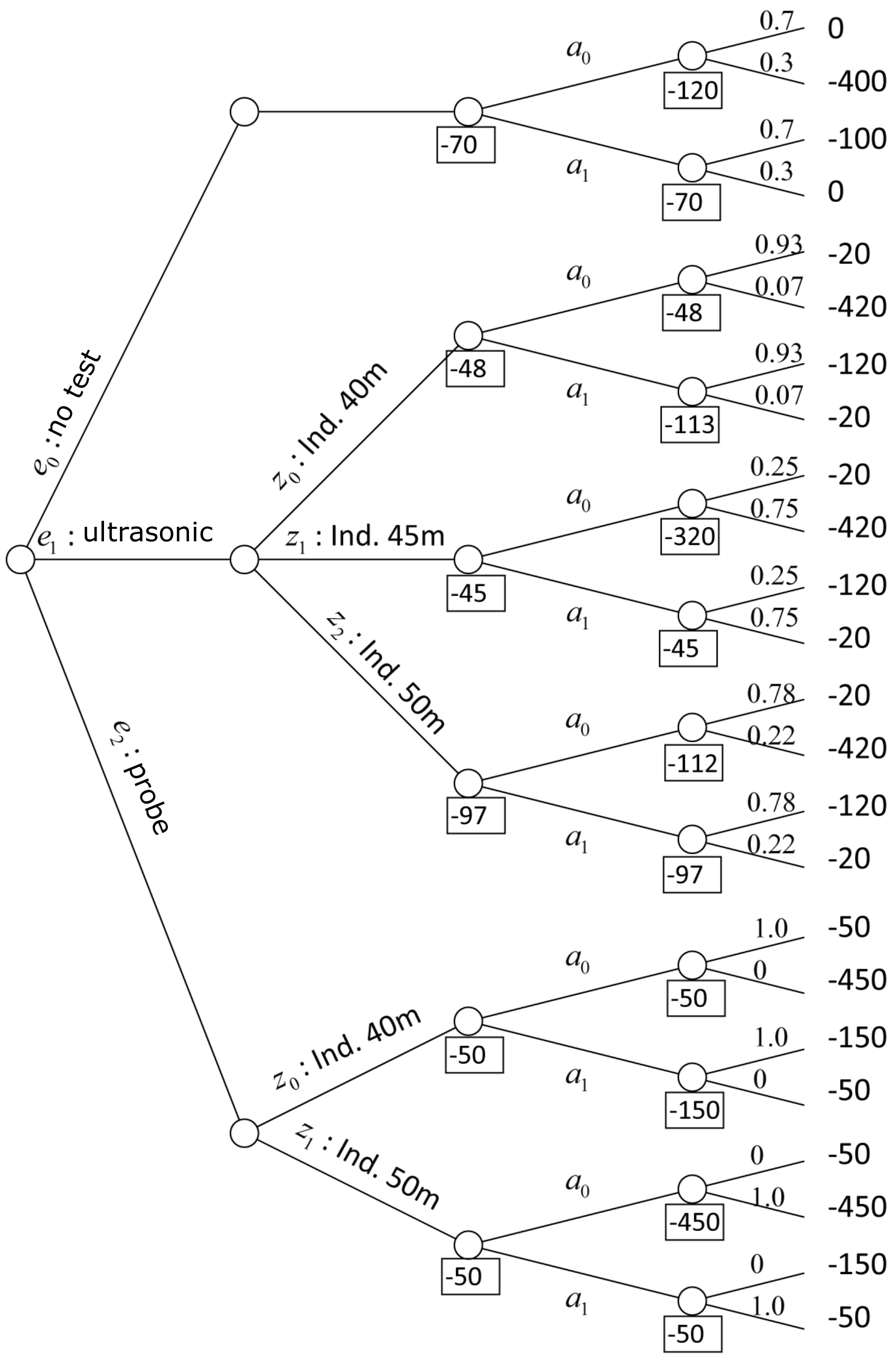
3. What decision would be taken?



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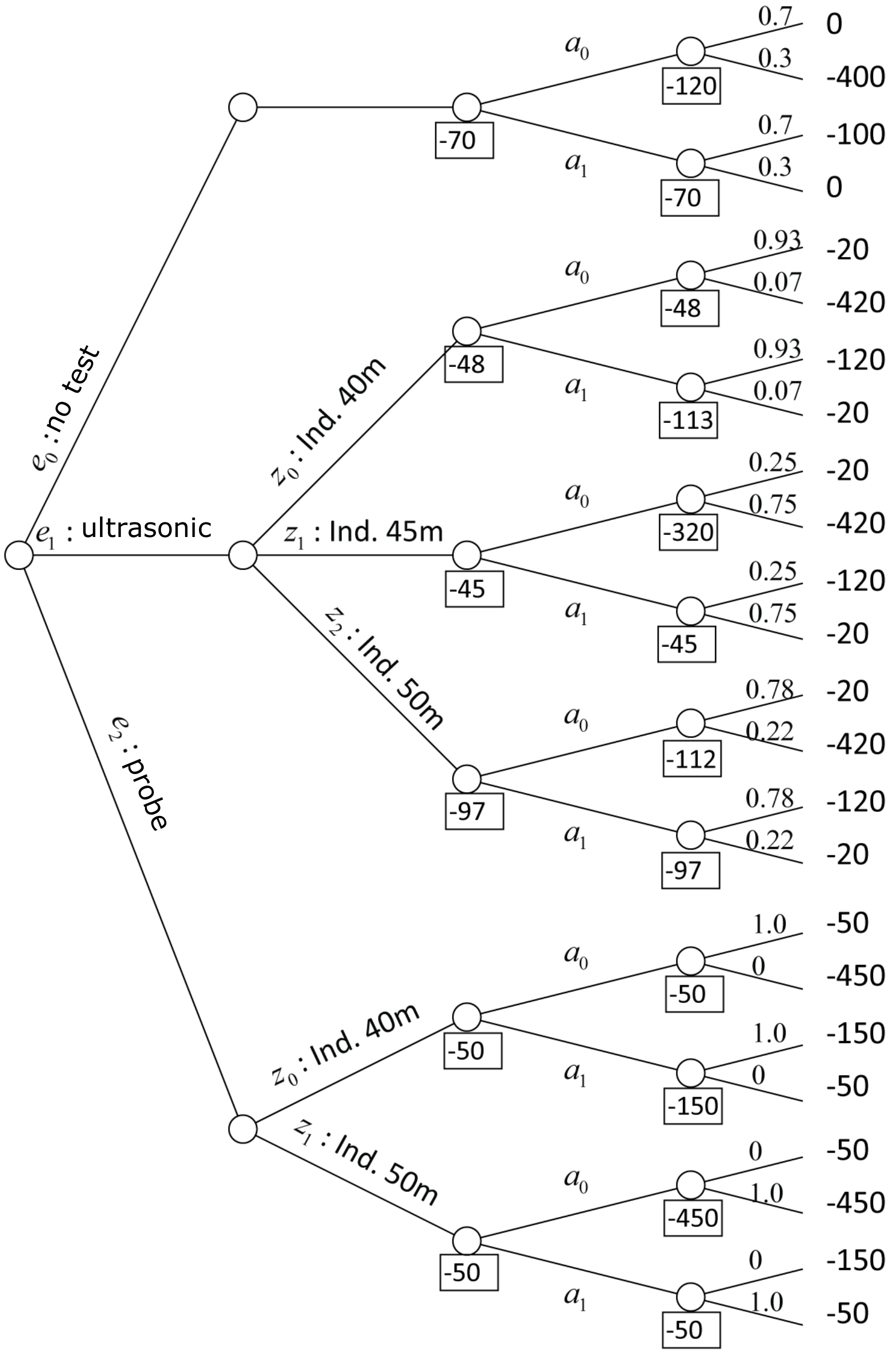

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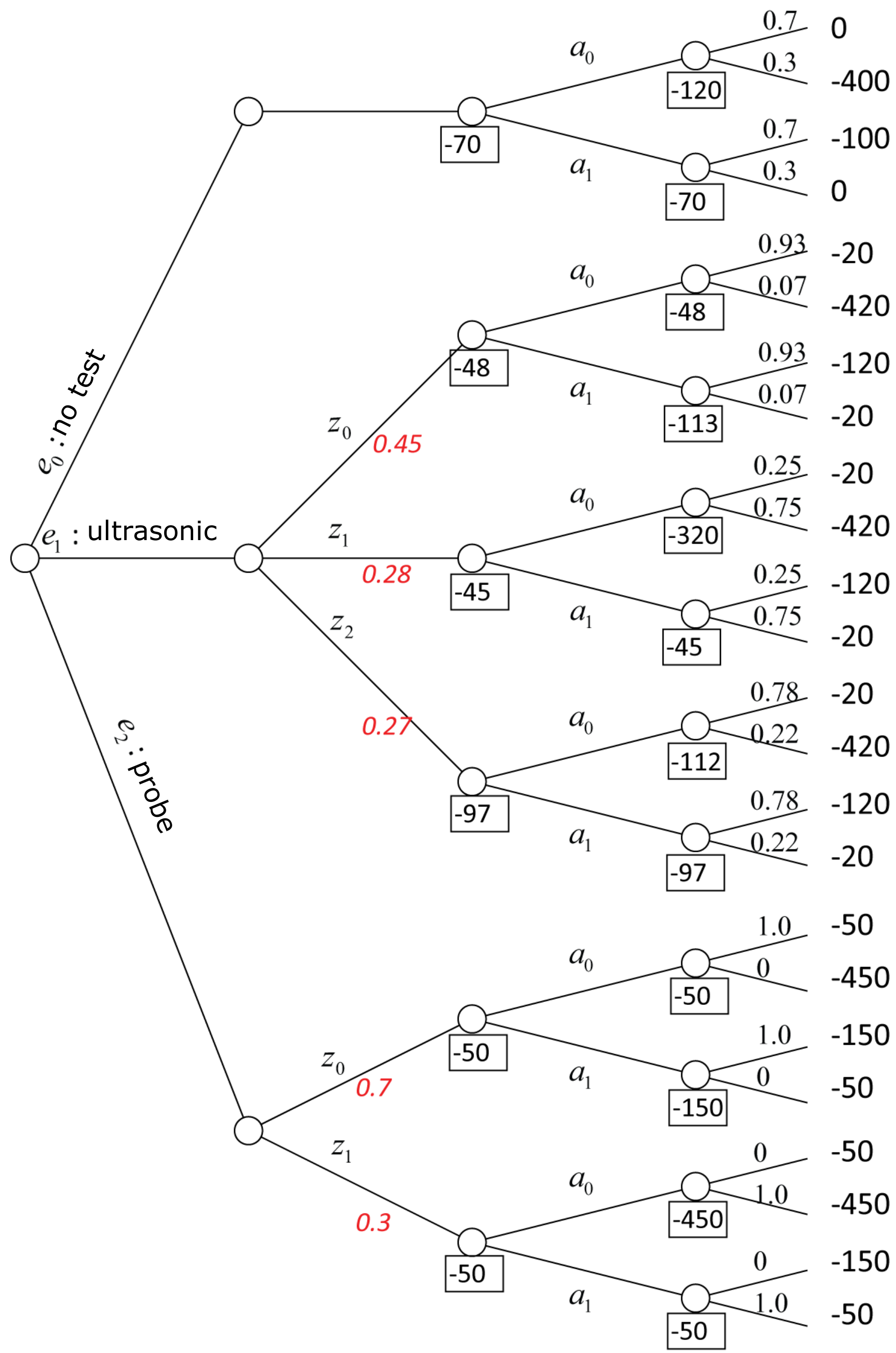
3. What decision would be taken?

4. How probable are the test results?

e.g.

$$P\left[z_0\middle|e_1\right]=P\left[z_0\middle|e_1,\theta_0\right]P'\left[\theta_0\right]+P\left[z_0\middle|e_1,\theta_1\right]P'\left[\theta_1\right]$$
$$=(0.6)(0.7)+(0.1)(0.3)=0.45$$



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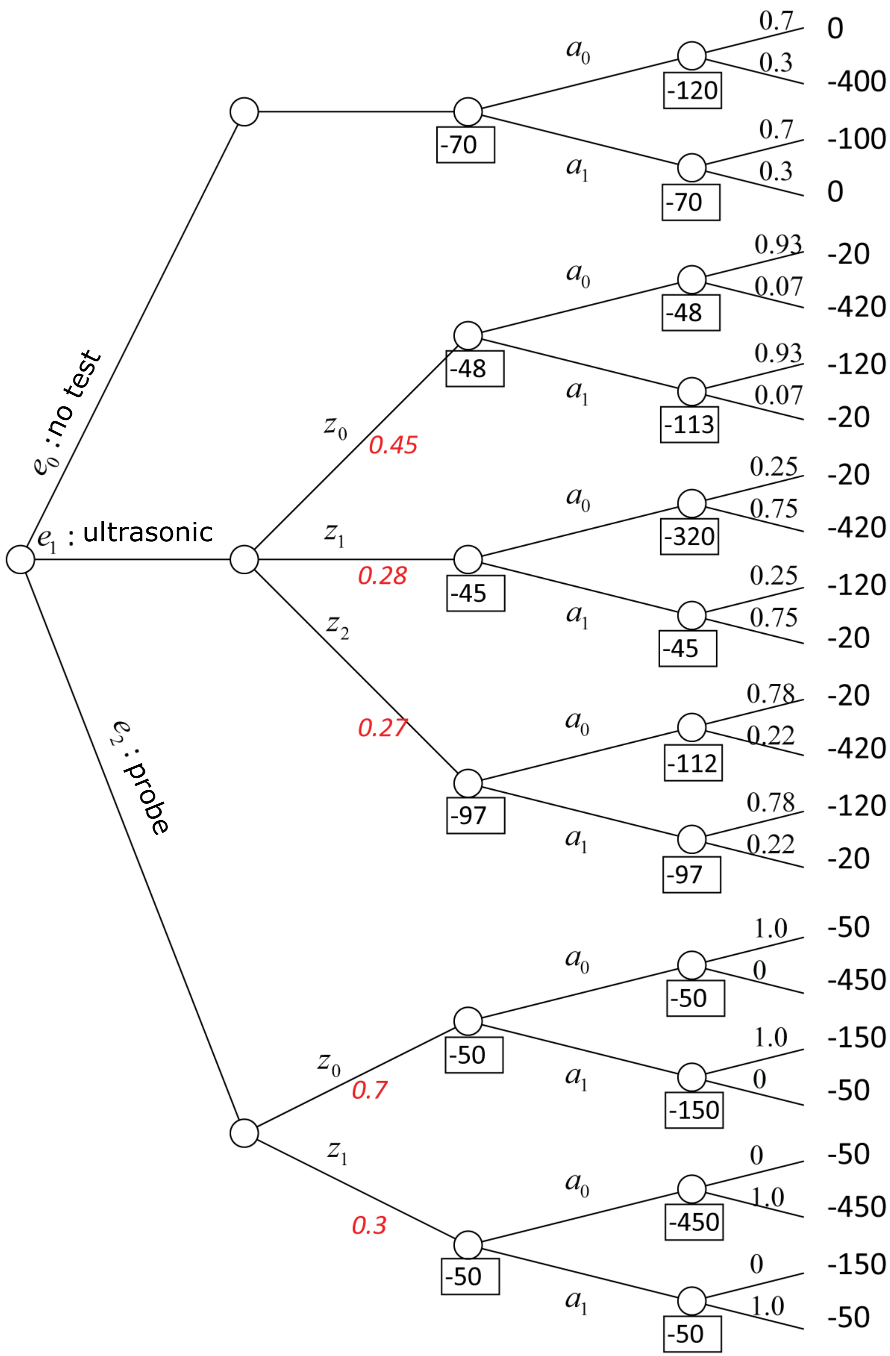

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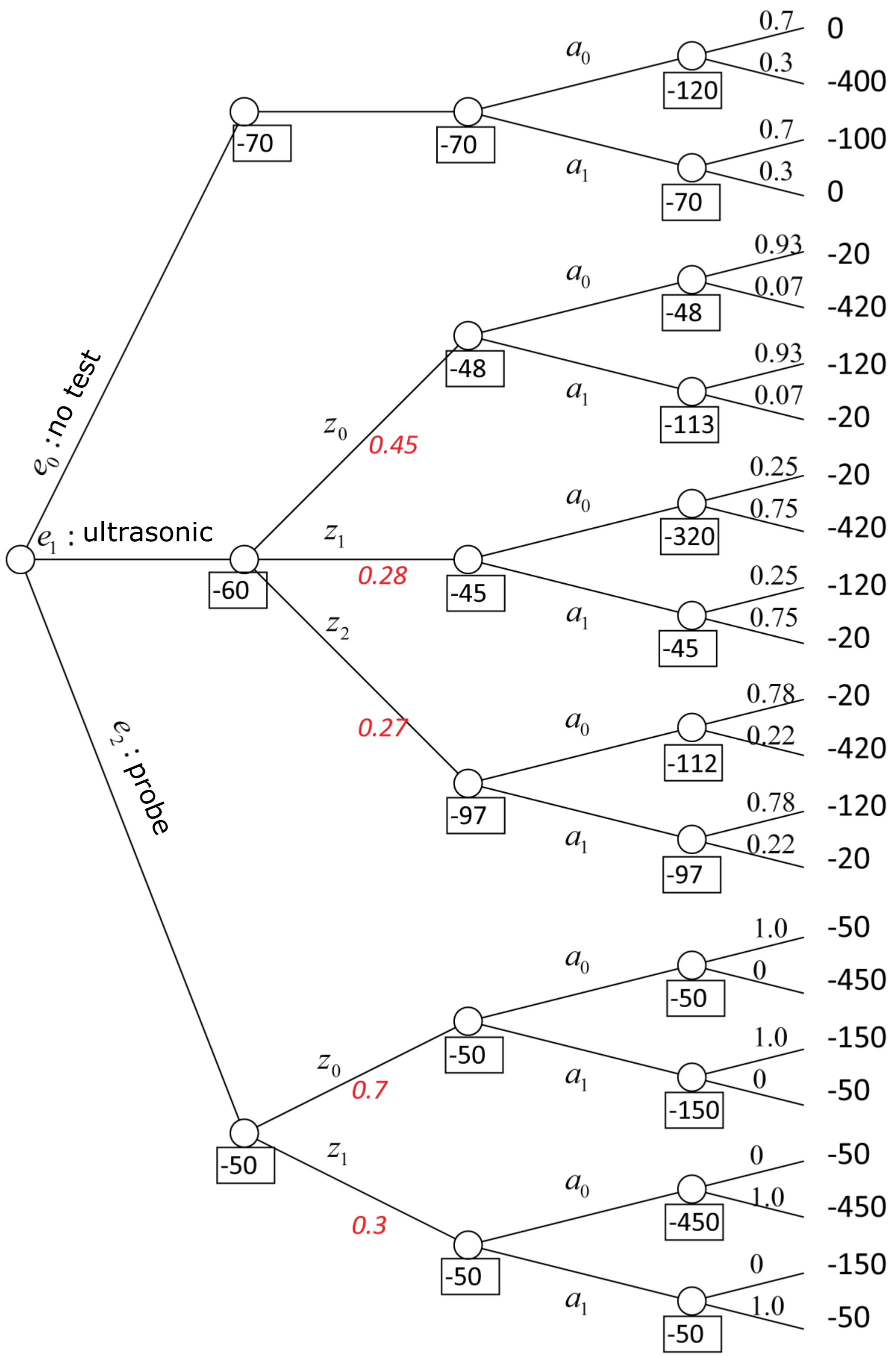
5. Expected utility per test option



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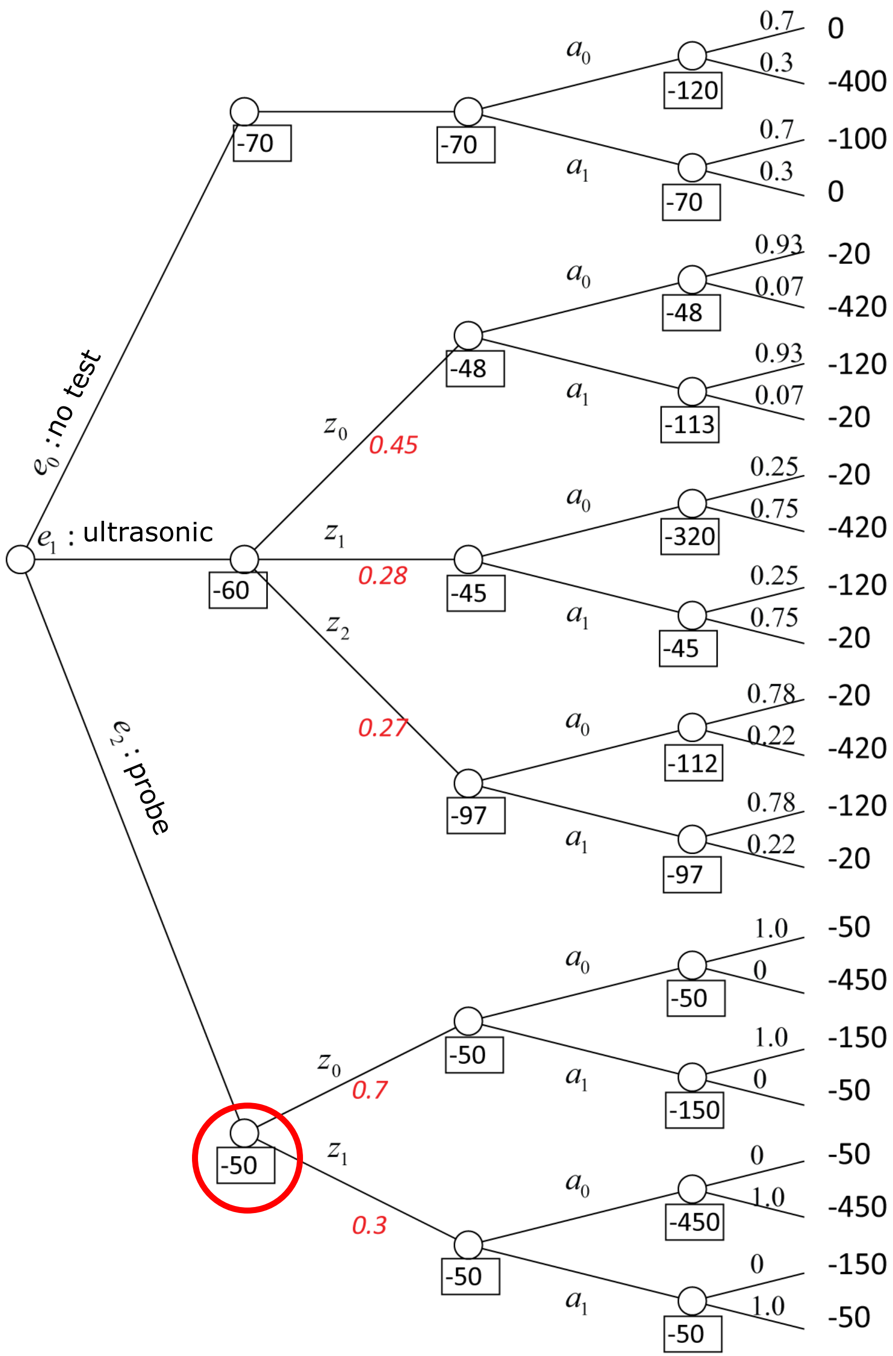

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Decision Analysis in Engineering

Summary

Different types of decision analysis are used.

- A-Priori Analysis: For prior knowledge
- Posteriori Analysis: if new knowledge becomes available
- Pre-Posteriori Analysis: to choose the most efficient test method.

The basic principles of the different types have been illustrated.

It is clear that the application of these principles can get rather complex when applied to real engineering problems.