Risk and Safety

in

Engineering

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Contents of Today's Lecture

- Probability theory
- Descriptive statistics
- Uncertainties in engineering decision making
- Probabilistic modelling
- Engineering model building

Overview of Probability Theory

- What are we aiming for ?

Data

Model estimation

Probabilistic model

Probabilities of events

Consequences of events

Risks

Decision Making !

Probability theory provides the basis for the consistent treatment of uncertainties in decision making !

We need to be able to quantify the probability of events and to update these based on new information

Interpretation of Probability

States of nature in which we have interest such as:

- a bridge failing due to excessive traffic loads
- a water reservoir being over-filled
- an electricity distribution system "breaking down"
- a project being delayed

are in the following denoted "events".

We are generally interested in quantifying the probability that such events take place within a given "time frame"

Interpretation of Probability

- **There are in principle three different interpretations of probability**

- Frequentistic $$P(A) = \lim \frac{N_A}{n_{\exp}} \quad \textbf{for} \quad n_{\exp} \to \infty$$

- Classical $$P(A) = \frac{n_A}{n_{tot}}$$

- Bayesian $$P(A) = \text{ degree of belief that } A \text{ will occur}$$

Interpretation of Probability

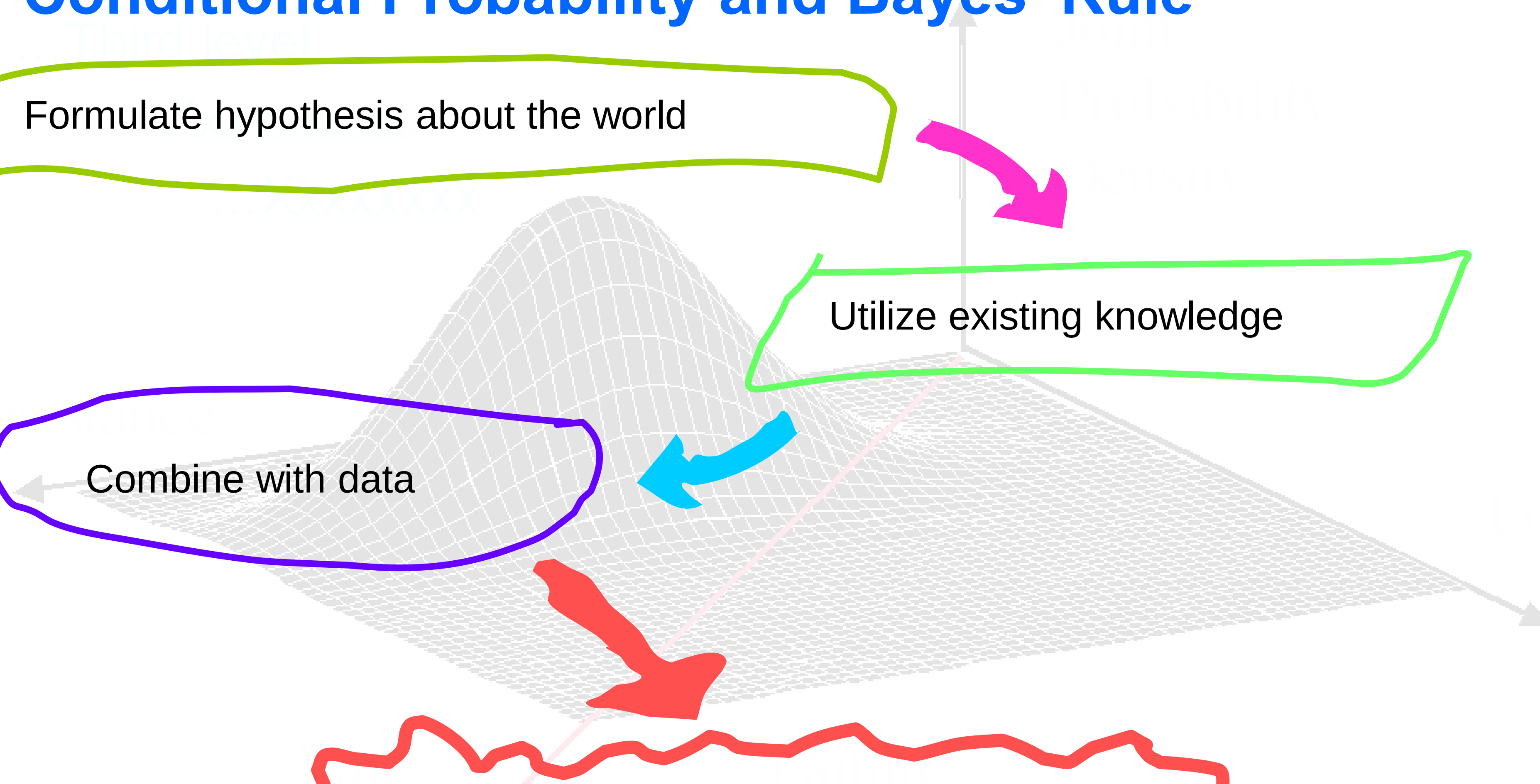
Consider the probability of getting a “head” when flipping a coin

- **Frequentistic** $$P(A) = \frac{510}{1000} = 0.51$$

- **Classical** $$P(A) = \frac{1}{2}$$

- **Bayesian** $$P(A) = 0.5$$

Conditional Probability and Bayes' Rule



Formulate hypothesis about the world

Utilize existing knowledge

Combine with data

Learn how to develop knowledge !

Conditional Probability and Bayes' Rule

Conditional probabilities are of special interest as they provide the basis for utilizing new information in decision making.

The conditional probability of an event E_1 given that event E_2 has occured is written as:

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Not defined if $P(E_2) = 0$

The event E_1 is said to be probabilistically independent of the event E_2 if:

$$P(E_1|E_2) = P(E_1)$$

Conditional Probability and Bayes' Rule

From $$P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

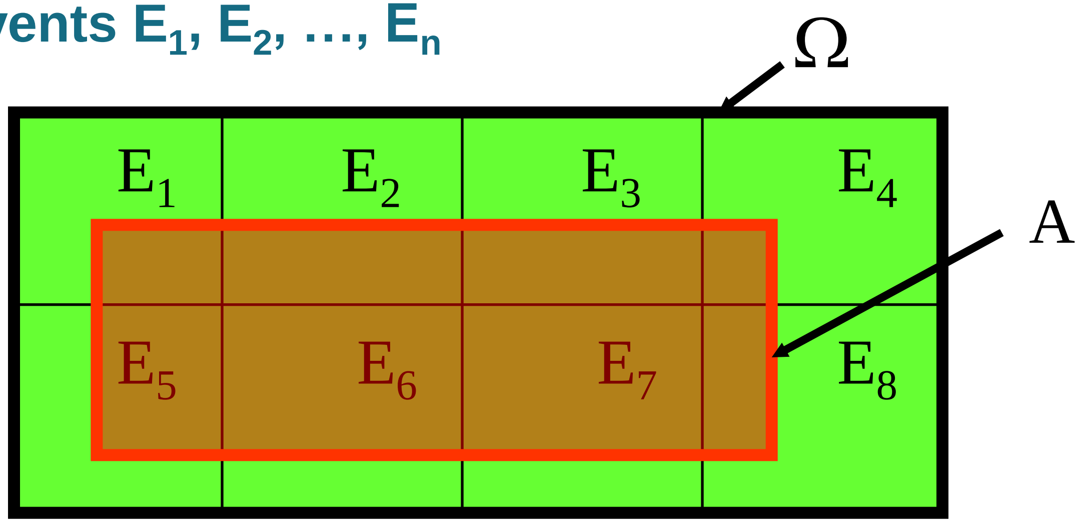
it follows that $$P(E_1 \cap E_2) = P(E_2)P(E_1 | E_2)$$

and when E_1 and E_2 are statistically independent it is

$$P(E_1 \cap E_2) = P(E_2)P(E_1)$$

Conditional Probability and Bayes’ Rule

Consider the sample space Ω divided up into n mutually exclusive events $E_1, E_2, \ldots, E_n$



$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n)$$

$$P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \ldots + P(A|E_n)P(E_n) =$$

$$\sum_{i=1}^{n} P(A|E_i)P(E_i)$$

Conditional Probability and Bayes' Rule

as there is $$P(A \cap E_i) = P(A|E_i)P(E_i) = P(E_i|A)P(A)$$

we have

Likelihood **Prior**

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{i=1}^{n} P(A|E_i)P(E_i)}$$

Posterior

Bayes' Rule

Reverend Thomas Bayes
(1702-1764)

Conditional Probability and Bayes' Rule

Example – inspection of degrading concrete structure

A reinforced concrete structure is considered

It is assumed (known) that the probability that corrosion of the reinforcement has initiated is: $P(CI) = 0.01$

The state of the reinforcement of the considered beam is unknown and NDE tests are invoked

The quality of the test is specified by the probabilities

- that the test will indicate corrosion given that corrosion has initiated $P(I|CI)$

- that the test will indicate corrosion given that corrosion has not initiated $P(I|\overline{CI})$

Conditional Probability and Bayes' Rule

Example – inspection of degrading concrete structure

By comparison of a large number of NDE measurements with the real condition of concrete structures it has been found that

$$P(I|CI) = 0.8$$

$$P(I|\overline{CI}) = 0.1$$

We now seek the probability of corrosion given that we get an indication of corrosion by the NDE inspection i.e.

$$P(CI|I) = ?$$

Likelihood

Prior

Posterior

$$P(CI|I) = \frac{P(I|CI)P(CI)}{P(I|CI)P(CI) + P(I|\overline{CI})P(\overline{CI})}$$

$$P(CI|I) = \frac{0.008}{0.107} = 0.075$$

Overview of Descriptive Statistics

- What are we aiming for ?

Descriptive statistics make
no assumptions – only describe !

Data ⇄ Model estimation

↓

Probabilistic model

↓

Probabilities of events

Consequences of events

↘ Risks ↙

↓

Decision Making !

In the first step we
simply want to describe
the data
- graphically
- numerically

Numerical Summaries

- **Central measures:**

Sample mean : $$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

If one number should be given to represent a data set typically the sample mean would be chosen

Median : The 0.5 quantile (obtained from ordered data sets, see quantile plots)

Mode : Most frequent value – obtained from histograms

Numerical Summaries

- **Dispersion measures:**

Sample variance: $$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

S **: standard deviation**

Indicator of variability around the sample mean

Sample coefficient of variation (CoV): $$\nu = \frac{s}{\bar{x}}$$

Indicator of variability relative to the sample mean

Numerical Summaries

- **Other measures:**

Sample skewness:

$$\eta = \frac{1}{n} \cdot \frac{\sum_{i=1}^{n} (x_i - \bar{x})^3}{s^3}$$

Measure of symmetry

Sample kurtosis:

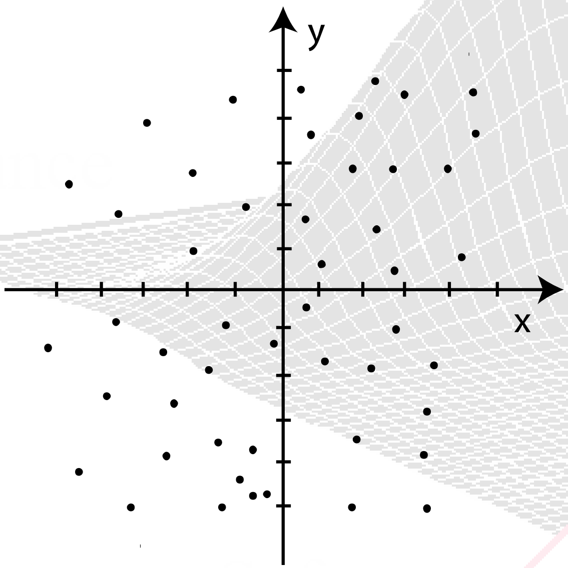
$$\kappa = \frac{1}{n} \cdot \frac{\sum_{i=1}^{n} (x_i - \bar{x})^4}{s^4}$$

Measure of peakedness

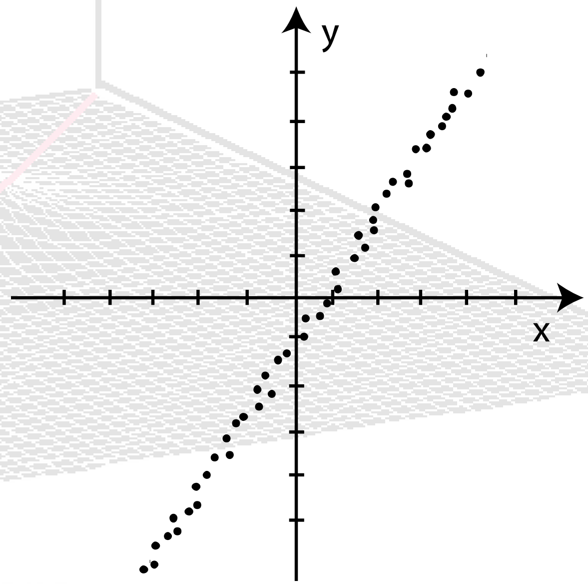
Numerical Summaries

- **Measures of correlation (linear dependency between data pairs):**

2-dimensional scatter plots



Almost no dependency



Almost full dependency

Numerical Summaries

- **Measures of correlation (linear dependency between data pairs):**

Sample covariance:

$$s^2_{XY} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})\cdot(y_i - \bar{y})$$

The sum will get positive contributions in case of low-low or high-high data pairs

Sample coefficient of correlation:

$$r_{XY} = \frac{1}{n}\frac{\sum_{i=1}^{n}(x_i - \bar{x})\cdot(y_i - \bar{y})}{s_X \cdot s_Y}$$

r_{XY} is limited in the interval -1 to +1

Numerical Summaries

- **Summary:**

Central measures:

- sample mean value:	The center of gravity of a data set
- sample median:	The mid value of a data set
- sample mode:	The most frequent value/range of a data set

Dispersion measures:

- sample variance:	The distribution around the sample mean
- sample CoV:	The variability relative to the sample mean

Other measures:

- sample skewness:	The skewness relative to the sample mean
- sample kurtosis:	The peakedness around the sample mean

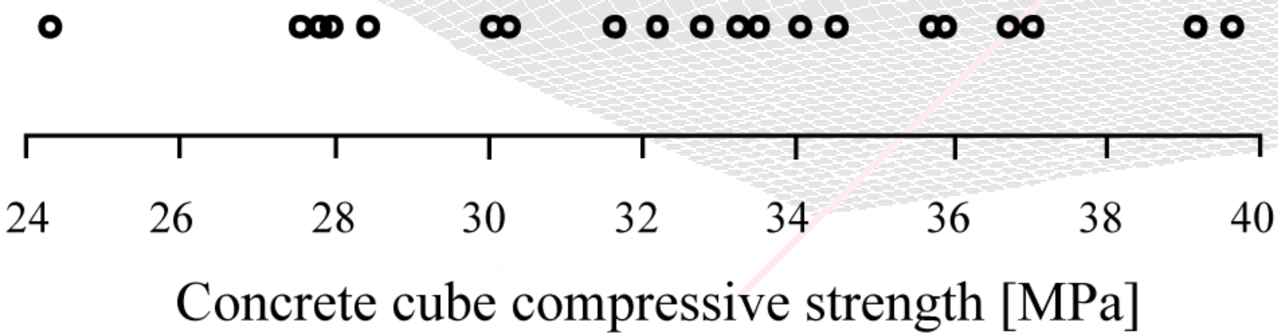
Measures of correlation:

- sample covariance:	Tendency for high-high, low-low and high-low pairs in two data sets
- sample coefficient of correlation :	Normalized coefficient between -1 and +1

Graphical Representations

- Assume that we have a set of data (observations of concrete compressive strength)

The simplest representation of the data is the one-dimensional scatter plot



i	Unordered x_i	Ordered x_i^o
1	35.8	24.4
2	39.2	27.6
3	34.6	27.8
4	27.6	27.9
5	37.1	28.5
6	33.3	30.1
7	32.8	30.3
8	34.1	31.7
9	27.9	32.2
10	24.4	32.8
11	27.8	33.3
12	33.5	33.5
13	35.9	34.1
14	39.7	34.6
15	28.5	35.8
16	30.3	35.9
17	31.7	36.8
18	32.2	37.1
19	36.8	39.2
20	30.1	39.7

Graphical Representations

- **Histograms**

The data are grouped into intervals

i	Unordered x_i	Ordered x_i^o
1	35.8	24.4
2	39.2	27.6
3	34.6	27.8
4	27.6	27.9
5	37.1	28.5
6	33.3	30.1
7	32.8	30.3
8	34.1	31.7
9	27.9	32.2
10	24.4	32.8
11	27.8	33.3
12	33.5	33.5
13	35.9	34.1
14	39.7	34.6
15	28.5	35.8
16	30.3	35.9
17	31.7	36.8
18	32.2	37.1
19	36.8	39.2
20	30.1	39.7

Interval	Midpoint	Number of observations	Frequency [%]	Cumulative frequency
23-26	24.5	1	5	0.05
26-29	27.5	4	20	0.25
29-32	30.5	3	15	0.40
32-35	33.5	6	30	0.70
35-38	36.5	4	20	0.90
38-41	39.5	2	10	1.00

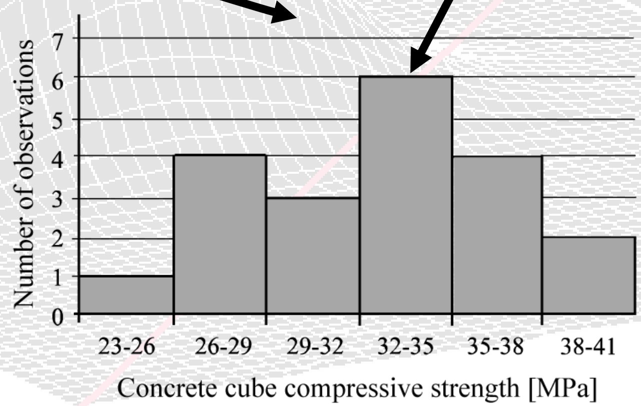
Graphical Representations

- **Histograms**

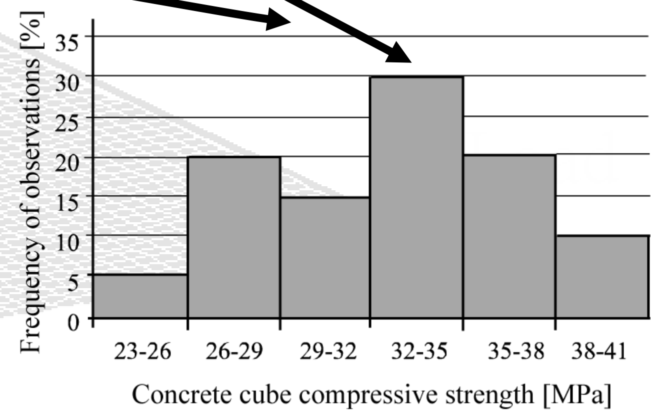
The grouped data are plotted

Interval	Midpoint	Number of observations	Frequency [%]	Cumulative frequency
23-26	24.5	1	5	0.05
26-29	27.5	4	20	0.25
29-32	30.5	3	15	0.40
32-35	33.5	6	30	0.70
35-38	36.5	4	20	0.90
38-41	39.5	2	10	1.00

mode



Simple histogram



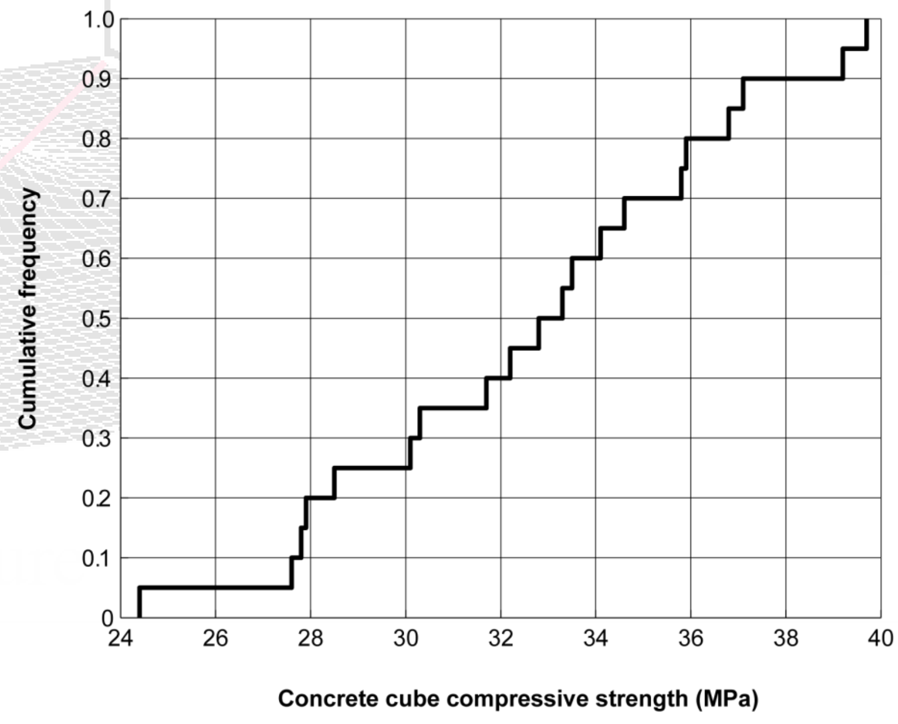
Frequency distribution

Graphical Representations

- **Histograms**

The grouped data are plotted

Interval	Midpoint	Number of observations	Frequency [%]	Cumulative frequency
23-26	24.5	1	5	0.05
26-29	27.5	4	20	0.25
29-32	30.5	3	15	0.40
32-35	33.5	6	30	0.70
35-38	36.5	4	20	0.90
38-41	39.5	2	10	1.00



Graphical Representations

- **Quantile plots**

A quantile is related to a percentage.

E.g. : the 0.65 quantile of a given data set of observations is the observation for which 65% of all observations in the data set have smaller values.

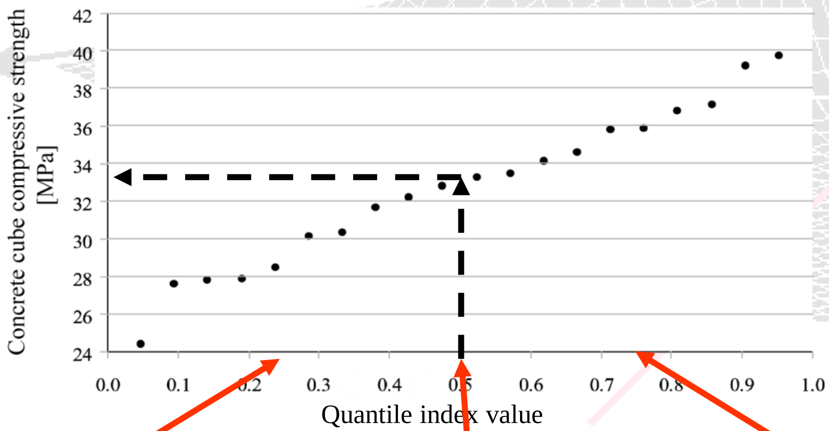
Quantile plots are generated by plotting the ordered data against the respective quantile index values.

Graphical Representations

- **Quantile plots**

The quantile index is calculated from the ordered data set as: $v = \dfrac{i}{1+n}$

i	Ordered x_i^o	**v**
1	24.4	0.048
2	27.6	0.095
3	27.8	0.143
4	27.9	0.190
5	28.5	0.238
6	30.1	0.286
7	30.3	0.333
8	31.7	0.381
9	32.2	0.429
10	32.8	0.476
11	33.3	0.524
12	33.5	0.571
13	34.1	0.619
14	34.6	0.667
15	35.8	0.714
16	35.9	0.762
17	36.8	0.810
18	37.1	0.857
19	39.2	0.905
20	39.7	0.952



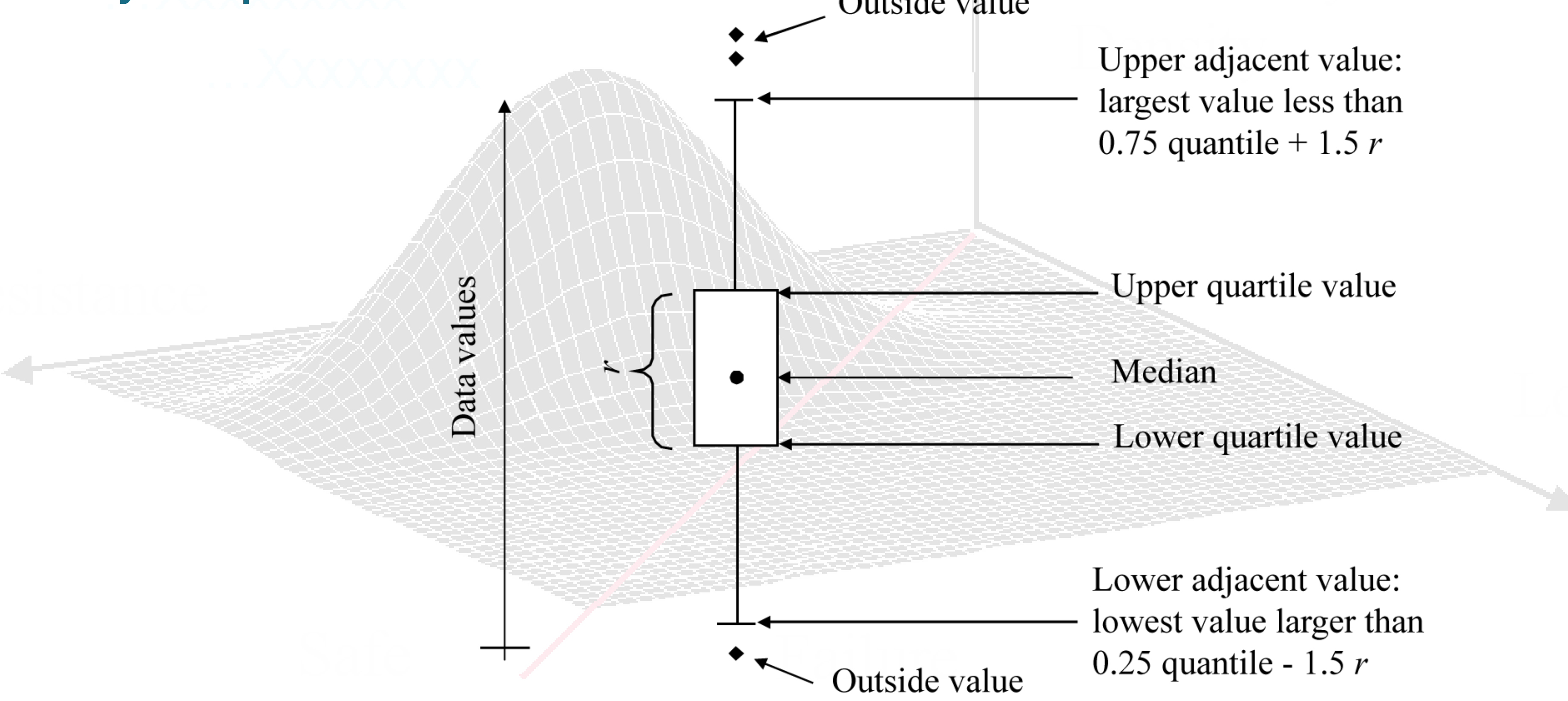
Lower quartile = 0.25 quantile value

Upper quartile = 0.75 quantile value

Median = 0.5 quantile value

Graphical Representations

- **Tukey Box plots**

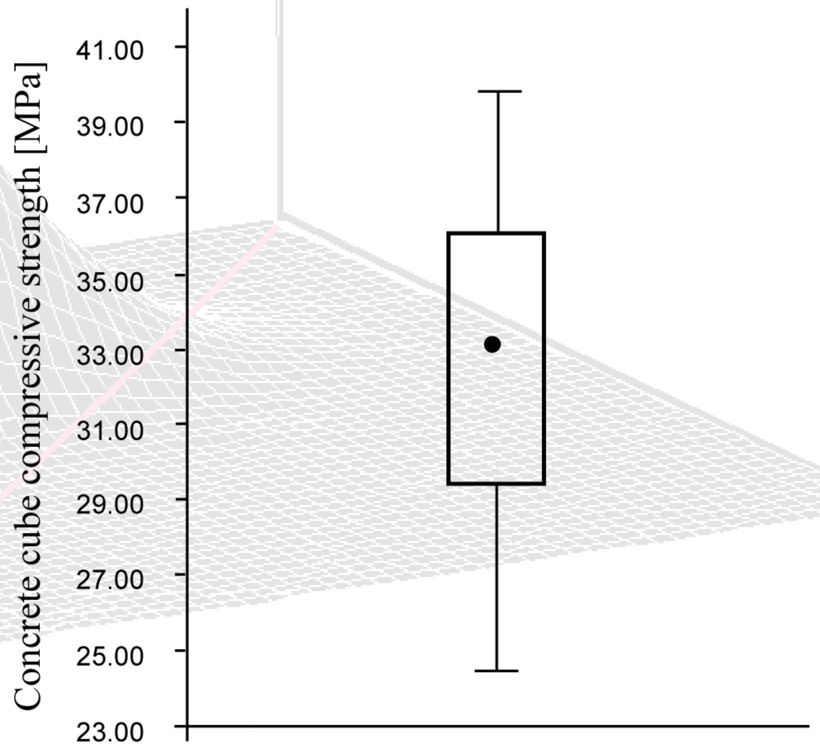


r : Inter-quartile range (50% of data)

Graphical Representations

- **Tukey Box plots**

Statistic	Value
Lower quartile	29.30
Lower adjacent value	24.40
Median	33.05
Upper adjacent value	39.70
Upper quartile	35.85



Graphical Representations

- **Summary**

One-dimensional scatter plots

Illustrate the range and distribution of a dataset along one axis; indicate symmetry.

Histograms

Illustrate how the data are distributed over a certain range indicate mode and symmetry.

Quantile plots

Illustrate median, distribution and symmetry.

Tukey – Box plots

Illustrate median, upper/lower quartiles, symmetry and distribution.

Overview of Uncertainty Modelling

- Why uncertainty modelling

Uncertain phenomenon

Data

Model estimation

Probabilistic model

Probabilities of events

Consequences of events

Risks

Decision Making !

Uncertainties in Engineering Problems

Different types of uncertainties influence decision making

- **Inherent natural variability – aleatory uncertainty**
  - result of throwing dices
  - variations in material properties
  - variations of wind loads
  - variations in rain fall

- **Model uncertainty – epistemic uncertainty**
  - lack of knowledge (future developments)
  - inadequate/imprecise models (simplistic physical modelling)

- **Statistical uncertainties – epistemic uncertainty**
  - sparse information/small number of data

Random Variables

- **Probability distribution and density functions**

A random variable is denoted with capital letters : X

A realization of a random variable is denoted with small letters : x

We distinguish between

- *continuous random variables*   :   can take any value in a given range

- *discrete random variables*   :   can take only discrete values

Random Variables

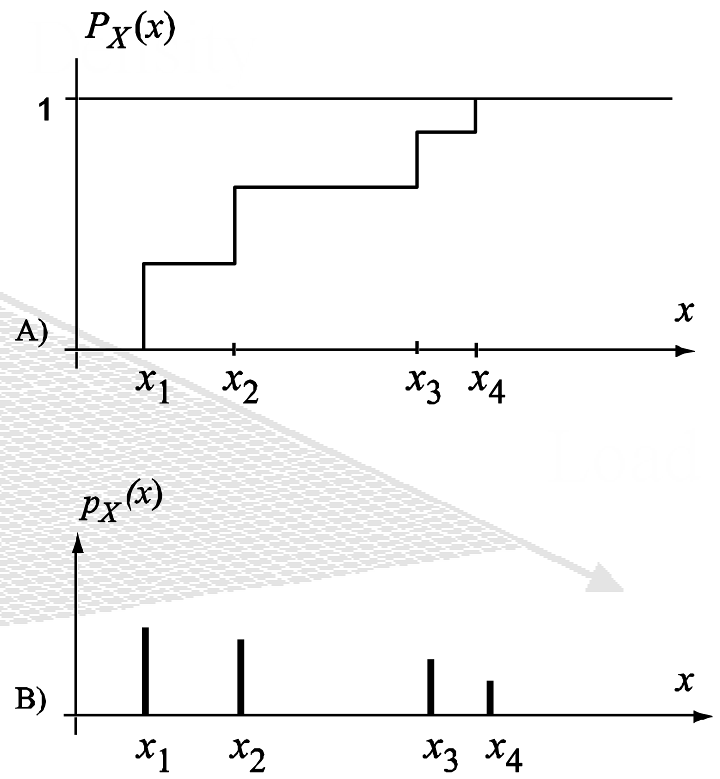
- **Probability distribution and density functions**

The probability that the outcome of a discrete random variable X is smaller than x is denoted the *probability distribution function*

$$P_X(x) = \sum_{x_i < x} p_X(x_i)$$

The *probability density function* for a discrete random variable is defined by

$$p_X(x_i) = P(X = x)$$



Random Variables

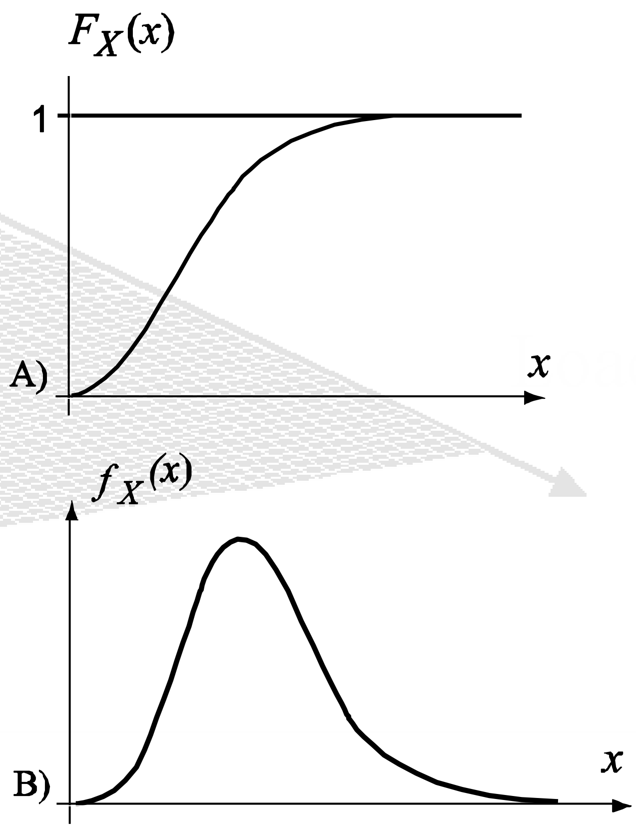
- **Probability distribution and density functions**

The probability that the outcome of a continuous random variable X is smaller than x is denoted the *probability distribution function*

$$F_X\left(x\right) = P\left(X < x\right)$$

The probability density function for a continuous random variable is defined by

$$f_X\left(x\right) = \frac{\partial F_X\left(x\right)}{\partial x}$$



Random Variables

- **Moments of random variables and the expectation operator**

Probability distribution and density function can be described in terms of their parameters $\mathbf{p}$ or their moments

Often we write

$$F_X(x, \mathbf{p}) \qquad f_X(x, \mathbf{p})$$

Parameters

The parameters can be related to the moments and vice versa

Random Variables

- **Moments of random variables and the expectation operator**

  The i^{th} moment m_i for a continuous random variable X is defined through

$$m_i = \int_{-\infty}^{\infty} x^i \cdot f_X(x)dx$$

  The *expected value E[X]* of a continuous random variable *X* is defined accordingly as the first moment

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$$

Random Variables

- **Moments of random variables and the expectation operator**

The i^th^ moment m_i for a discrete random variable X is defined through

$$m_i = \sum_{j=1}^{n} x_j^i \cdot p_X(x_j)$$

The *expected value E[X]* of a discrete random variable *X* is defined accordingly as the first moment

$$\mu_X = E[X] = \sum_{j=1}^{n} x_j \cdot p_X(x_j)$$

Random Variables

- **Moments of random variables and the expectation operator**

  The *standard deviation* σ_X of a continuous random variable is defined as the second central moment i.e. for a continuous random variable *X* we have

$$\sigma_X{}^2 = \mathrm{Var}\left[\mathrm{X}\right] = E\left[(X-\mu_X)^2\right] = \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f_X(x)dx$$

Variance

Mean value

for a discrete random variable we have correspondingly

$$\sigma_X^2 = Var\left[X\right] = \sum_{j=1}^{n} (x_j-\mu_X)^2 \cdot p_X(x_j)$$

Random Variables

- **Moments of random variables and the expectation operator**

The ratio between the standard deviation and the expected value of a random variable is called the *Coefficient of Variation CoV* and is defined as

$$CoV\left[X\right]=\frac{\sigma_X}{\mu_X}$$

Dimensionless

a useful characteristic to indicate the variability of the random variable around its expected value

Random Variables

- **Typical probability distribution functions in engineering**

  Normal : Sum of random effects

  Lognormal: Product of random effects

  Exponential: Waiting times

  Gamma: Sum of waiting times

  Beta: Flexible modeling function

Distribution type	Parameters	Moments
Rectangular a ≤ x ≤ b $f_X(x)=\frac{1}{b-a}$ $F_X(x)=\frac{x-a}{b-a}$	a b	$\mu=\frac{a+b}{2}$ $\sigma=\frac{b-a}{\sqrt{12}}$
Normal $f_X(x)=\frac{1}{\sigma\sqrt{2\pi}}exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$ $F_X(x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx$	μ $\sigma>0$	μ σ
Shifted Lognormal x > ε $f_X(x)=\frac{1}{(x-\varepsilon)\zeta\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln(x-\varepsilon)-\lambda}{\zeta}\right)^2\right)$ $F_X(x)=\Phi\left(\frac{ln(x-\varepsilon)-\lambda}{\zeta}\right)^2$	λ $\zeta>0$ ε	$\mu=\varepsilon+\exp\left(\lambda+\frac{\zeta^2}{2}\right)$ $\sigma=\exp\left(\lambda+\frac{\zeta^2}{2}\right)\sqrt{\exp(\zeta^2)-1}$
Shifted Exponential x ≥ ε $f_X(x)=\lambda\exp(-\lambda(x-\varepsilon))$ $F_X(x)=1-e^{-\lambda(x-e)}$	ε $\lambda>0$	$\mu=\varepsilon+\frac{1}{\lambda}$ $\sigma=\frac{1}{\lambda}$
Gamma x ≥ 0 $f_X(x)=\frac{b^p}{\Gamma(p)}\exp(-bx)x^{p-1}$ $F_X(x)=\frac{\Gamma(bx,p)}{\Gamma(p)}$	$p>0$ $b>0$	$\mu=\frac{p}{b}$ $\sigma=\frac{\sqrt{p}}{b}$
Beta a ≤ x ≤ b, r, t ≥ 1 $f_X(x)=\frac{\Gamma(r+t)}{\Gamma(r)\cdot\Gamma(t)}\frac{(x-a)^{r-1}(b-x)^{t-1}}{(b-a)^{r+t-1}}$ $F_X(x)=\frac{\Gamma(r+t)}{\Gamma(r)\cdot\Gamma(t)}\cdot\int_a^u\frac{(u-a)^{r-1}(b-u)^{t-1}}{(b-a)^{r+t-1}}du$	a b $r>1$ $t>1$	$\mu=a+(b-a)\frac{r}{r+1}$ $\sigma=\frac{b-a}{r+t}\sqrt{\frac{rt}{r+t+1}}$

Random Variables

- **The Normal distribution**

The analytical form of the Normal distribution may be derived by repeated use of the result regarding the probability density function for the sum of two random variables

The normal distribution is very frequently applied in engineering modelling when a random quantity can be assumed to be composed as a sum of a number of individual contributions.

A linear combination *S* of *n* Normal distributed random variables $X_i, i = 1, 2, .., n$ **is thus also a Normal distributed random variable**

$$S = a_0 + \sum_{i=1}^{n} a_i X_i$$

Random Variables

- **The Normal distribution:**

  In the case where the mean value is equal to zero and the standard deviation is equal to 1 the random variable is said to be *standardized*.

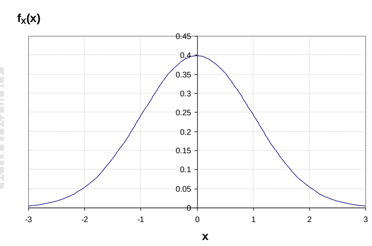
$$Z = \frac{X - \mu_X}{\sigma_X}$$

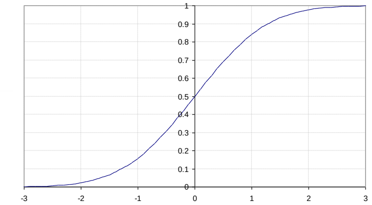
Standardized random variable

$$f_Z(z) = \varphi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right)$$

$$F_Z(z) = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{1}{2} x^2\right) dx$$

Standard normal





Stochastic Processes and Extremes

- Random quantities may be "time variant" in the sense that they take new values at different times or at new trials.

  - If the new realizations occur at discrete times and have discrete values the random quantity is called a **random sequence**

    failure events, traffic congestions,…

  - If the new realizations occur continuously in time and take continuous values the random quantity is called a **random process** or **stochastic process**

    wind velocity, wave heights,…

Stochastic Processes and Extremes

- Random sequences

  The Poisson counting process is one of the most commonly applied families of probability distributions applied in reliability theory

  The process *N(t)* denoting the number of events in a time interval $[0;t[$ is called a Poisson process if the following conditions are fulfilled:

  1) the probability of one event in the interval $[t,t+\Delta t[$ is asymptotically proportional to Δt.

  2) the probability of more than one event in the interval $[t,t+\Delta t[$ is a function of higher order of Δt for $\Delta t \rightarrow 0$.

  3) events in disjoint intervals are mutually independent.

Stochastic Processes and Extremes

- **Random sequences**

The probability distribution function of the (waiting) time till the first event T_1 is now easily derived recognizing that the probability of $T_1 > t$ is equal to $P_0(t)$ we get:

$$F_{T_1}(t_1) = 1 - P_0(t_1)$$

$$= 1 - exp(-\int_0^t \nu(\tau) d\tau)$$

$$F_{T_1}(t_1) = 1 - exp(-\nu t)$$

Homogeneous case !

Exponential probability distribution

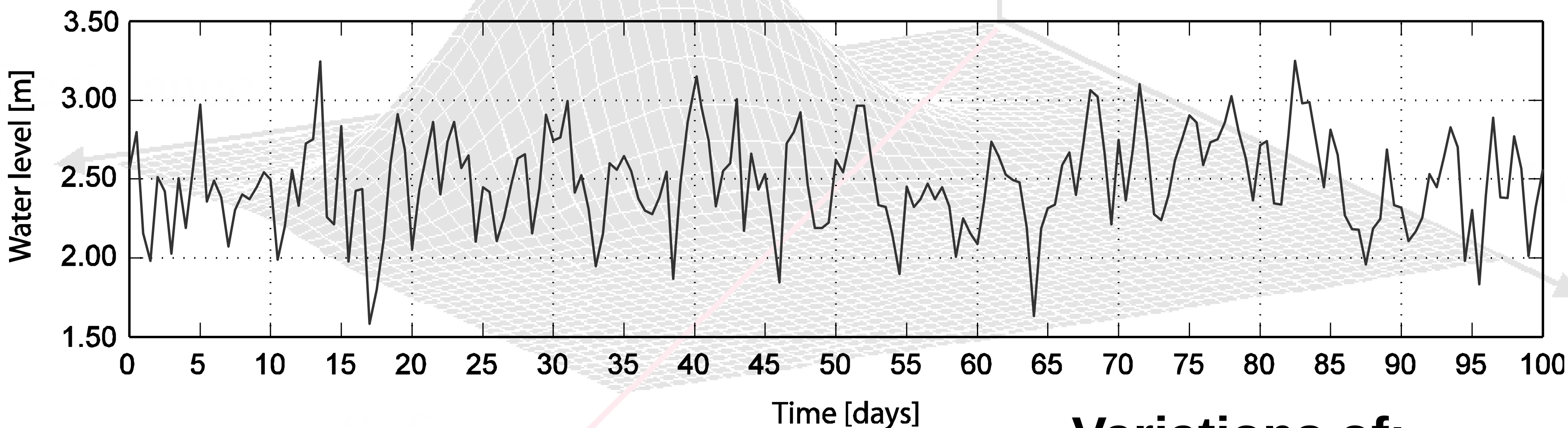
Exponential probability density

$$f_{T_1}(t_1) = \nu \cdot exp(-\nu t)$$

Stochastic Processes and Extremes

- **Continuous random processes**

  A continuous random process is a random process which has realizations continuously over time and for which the realizations belong to a continuous sample space.



Variations of:

water levels

wind speed

rain fall

Stochastic Processes and Extremes

Extreme Value Distributions

In engineering we are often interested in extreme values i.e. the smallest or the largest value of a certain quantity within a certain time interval e.g.:

- **The largest earthquake in 1 year**
- **The highest wave in a winter season**
- **The largest rainfall in 100 years**

Stochastic Processes and Extremes

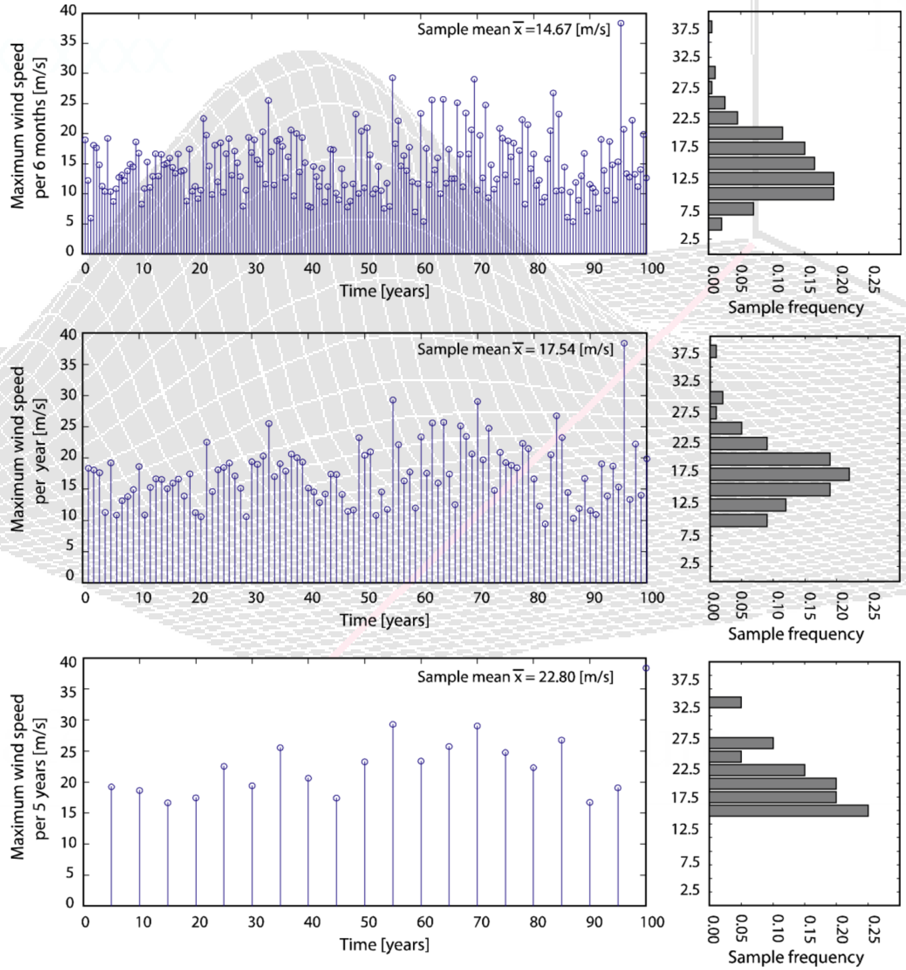
Extreme Value Distributions

We could also be interested in the smallest or the largest value of a certain quantity within a **certain volume or area** unit e.g.:

- The largest concentration of pesticides in a volume of soil
- The weakest link in a chain
- The smallest thickness of concrete cover

Stochastic Processes and Extremes

Extremes of a random process:



Stochastic Processes and Extremes

Return period for extreme events:

The return period for extreme events T_R may be defined as

$$T_R = n \cdot T = \frac{1}{(1 - F_{X,T}^{\max}(x))} T$$

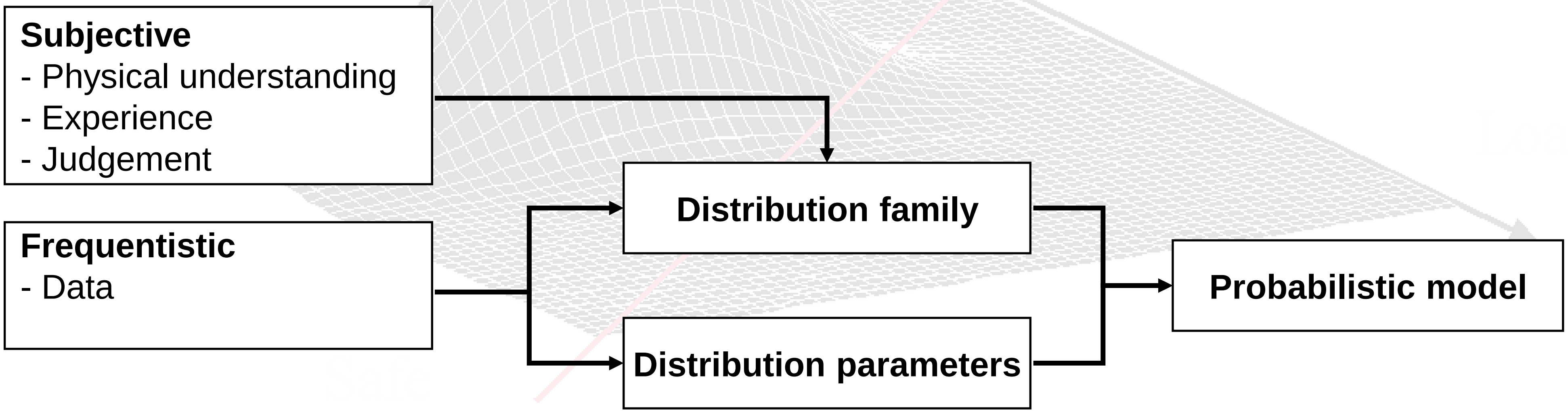
If the probability of exceeding *x* during a reference period of 1 year is 0.01 then the return period for exceedances is

$$T_R = n \cdot T = \frac{1}{0.01} \cdot 1 = 100 \cdot 1 = 100$$

Overview of Estimation and Model Building

Different types of information is used when developing engineering models

- subjective information
- frequentististic information



Overview of Estimation and Model Building

Model building may be seen to consist of five steps

1) **Assessment and statistical quantification of the available data**

2) **Selection of distribution function**

3) **Estimation of distribution parameters**

4) **Model verification**

5) **Model updating**

Overview of Estimation and Model Building

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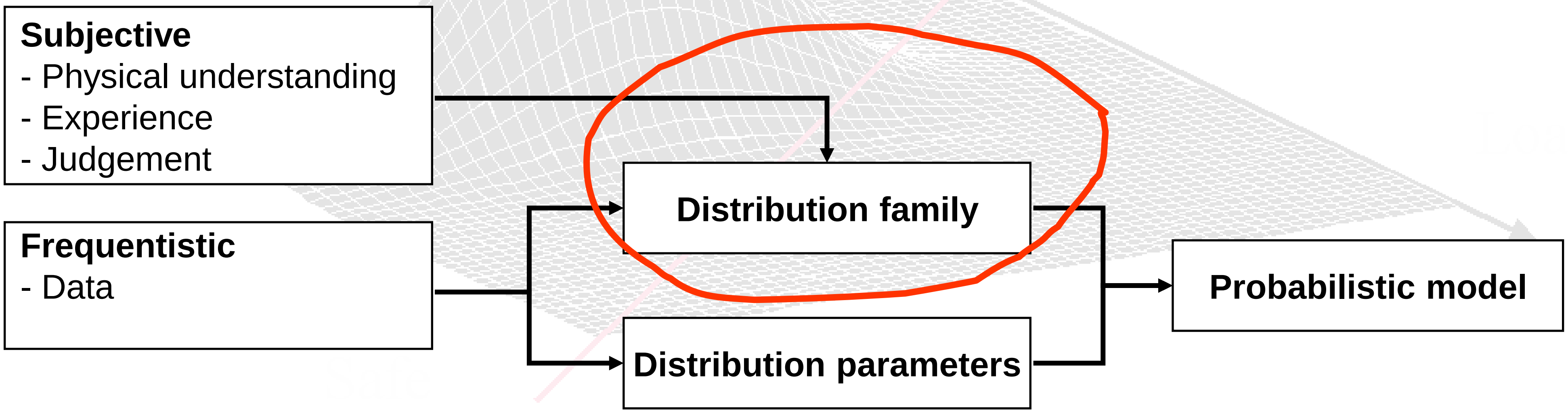


Estimation and Model Building

Selection of probability distribution function

In engineering application it is often the case that

the available data is too sparse

to be able to support/reject the hypothesis of a given probability distribution – with a reasonable significance

Therefore it is necessary to use common sence i.e. :

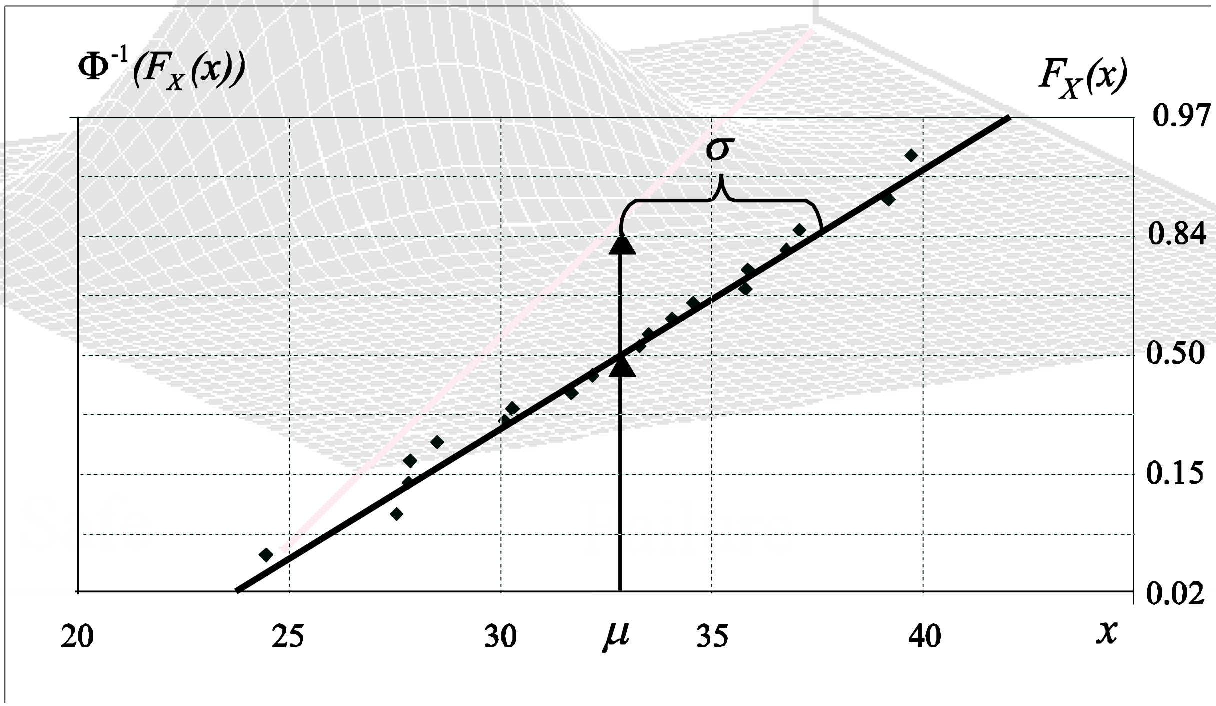
First to consider physical reasons for selecting a given distribution

Thereafter to check if the available data are in gross contradiction with the selected distribution

Estimation and Model Building

Model selection by use of probability paper

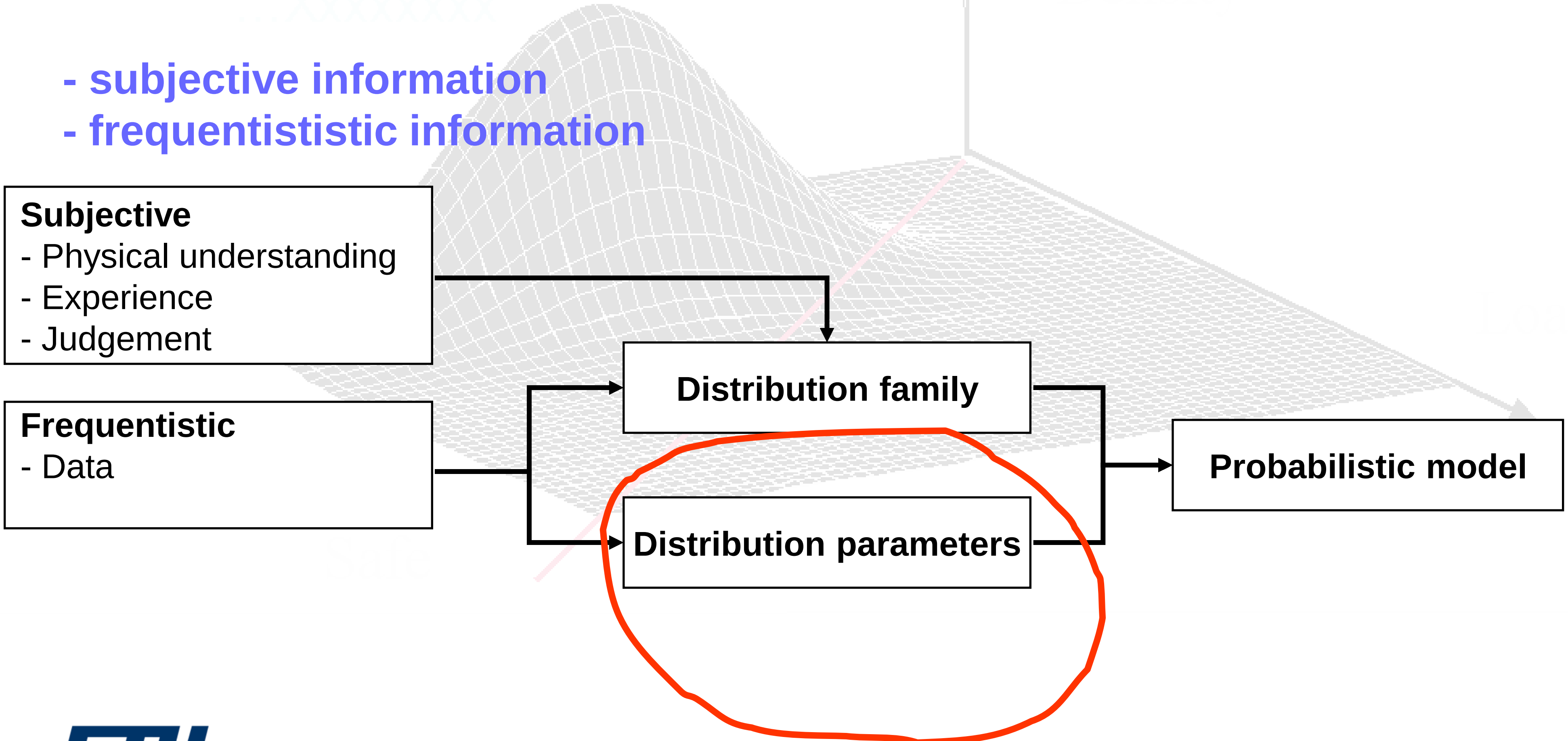
Plotting the sample probability distribution function in the probability paper yields



Overview of Estimation and Model Building

Different types of information is used when developing engineering models

- **subjective information**
- **frequentististic information**



Estimation of Distribution Parameters

We assume that we have identified a plausible family of probability distribution functions – as an example :

Normal Distribution

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

Weibull distribution

$$f_X(x) = \frac{k}{u-\varepsilon}\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^{k-1}\exp\left(-\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^{k}\right)$$

and thus now need to determine/estimate its parameters

$$\boldsymbol{\theta} = (\theta_1, \theta_2, .., \theta_k)^T$$

Estimation of Distribution Parameters

The method of moments (MoM)

To start with we assume that we have data on the basis of which we can estimate the distribution parameters $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_{2,}, .., \hat{x}_n)^T$

The idea behind the method of moments is to determine the distribution parameters such that the sample moments (from the data) and the analytical moments (from the assumed distribution) are identical.

$$m_j = \frac{1}{n}\sum_{i=1}^{n} x_i^j$$

Sample moments

$$\lambda_j = \int_{-\infty}^{\infty} x^j \cdot f_X(x|\boldsymbol{\theta})dx$$

$$= \lambda_j(\theta_1, \theta_2, .., \theta_k)$$

Analytical moments

Estimation of Distribution Parameters

The Maximum Likelihood Method (MLM)

The idea behind the method of maximum likelihood is that

the parameters are determined such that the likelihood of the observations is maximized.

The likelihood can be understood as the probability of occurrence of the observed data conditional on the model.

The Maximum Likelihood Method may seem to be more complicated than the Method of Moments but has a number of attractive properties.

Estimation of Distribution Parameters

Summary

Method of Moments provides point estimates of the parameters.

- No information about the uncertainty with which the parameter estimates are associated.

Maximum Likelihood Method provides point estimates of the parameters.

- Full distribution information – normal distributed parameters, mean values and covariance matrix.