Risk and Safety
in
Engineering

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Contents of Today's Lecture

- Introduction to time variant reliability analysis
- The Poisson process
- The Normal process
- Assessment of the mean out-crossing rate
- Hierarchical modelling in time variant reliability analysis
- Simplifications

Time variant reliability

It is important to emphasize that probabilities are always somehow related to a “time measure”

Typically the “time measure” can be

- a particular number of experiments
- a time interval (e.g. a year)
- a spatial characteristic (e.g. length, area or volume)

In most cases we can formulate the reliability problems in a way that the “time measure” does not explicitly enter the probabilistic modelling of basic random variables

In order to understand “when to do what” we will consider some basic aspects of time variant reliability

Time variant reliability

As mentioned earlier we can often model reliability problems such that time does not enter the probabilistic modelling of the basic random variables.

This is e.g. the case when loads are ergodic – in which case reliability problems relating to extreme load events may be formulated using extreme value distributions for the extreme load realisations (corresponding to a certain time interval)

In such cases we may directly use FORM analysis for the assessment of the relevant probabilities.

Time variant reliability

In order to be able to introduce the basics of time variant reliability problems it is useful to introduce two special types of stochastic processes, namely the:

- **Poisson process** → Used extensively to describe statistical characteristics of events (usually rare)
- **Normal process** → Used extensively to describe the time variant behaviour of uncertain properties

Time variant reliability

The process *N(t)* denoting the number of points in the time interval (0;*T*[is called a simple Poisson process if it satisfies the following conditions:

- **The probability of one event in the interval (*t;t+Δt*[is asymptotically proportional to *Δt***
- **The probability of more than one event in the interval (*t;t+Δt*[is a function of a higher order term of *Δt* for *Δt→0***
- **Events in disjoint intervals are stochastically independent**

The simple Poisson process may be described completely by its density

$$\nu(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(\text{one event in } (t;t+\Delta t[)$$

For homogeneous Poisson processes, *ν(t)* = constant

Time variant reliability

The probability of n events of a simple Poisson process in the interval $(0;t[$ can be shown to be given by:

$$P_n(t) = \frac{(\int_0^t \nu(\tau)d\tau)^n}{n!} \exp(-\int_0^t \nu(\tau)d\tau)$$

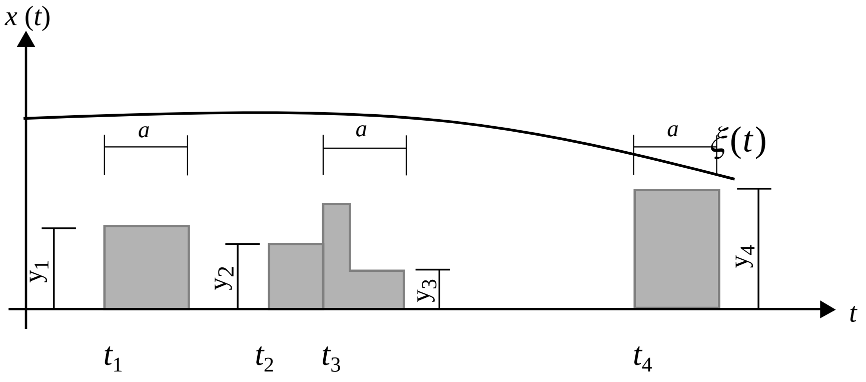
From this, we can derive the probability of no events as:

$$P_0(t) = \exp(-\int_0^t \nu(\tau)d\tau)$$

and then the probability distribution for the time till the first event as:

$$F_{T_1}(t) = 1 - P_0(t) = 1 - \exp(-\int_0^t \nu(\tau)d\tau)$$

Time variant reliability



Filtered Poisson processes may be derived from the simple Poisson process

We assume that events are generated along the time axis – in accordance with a simple Poisson process

Then we associate with all such events a response function

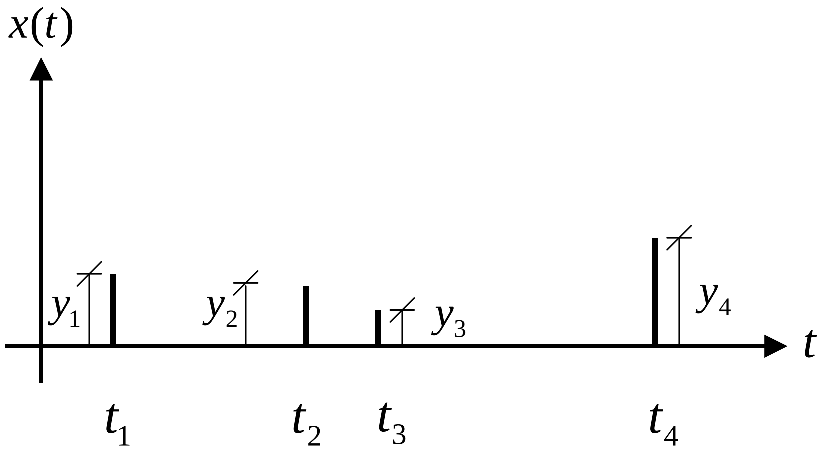
$$\omega(t, t_k, Y_k)$$

which is defined to be equal to 0 for $t < t_k$

The filtered Poisson process is now established as:

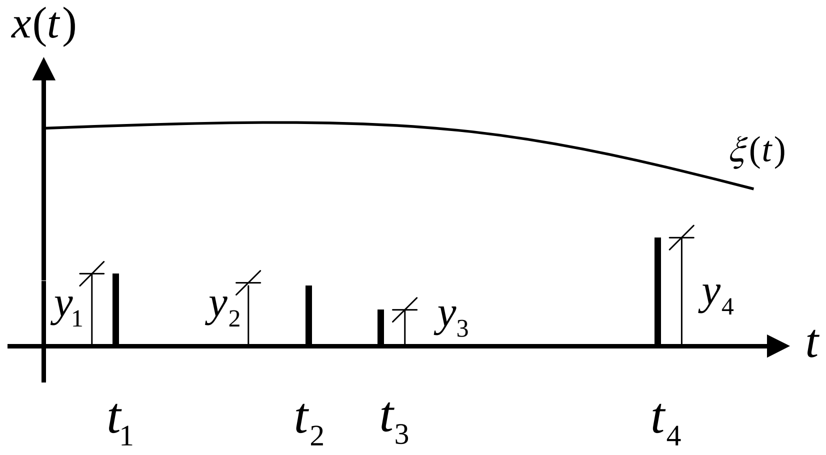
$$X(t) = \sum_{k=1}^{N(t)} \omega(t, t_k, Y_k)$$

If now we let the duration approach zero, we get the Poisson spike process



Time variant reliability

Here we consider the Poisson spike process with mutually independent random spikes Y_i



If e.g. failure can be described as the event of a spike above the threshold x

It is recognised that these events are also events of a Poisson process with intensity

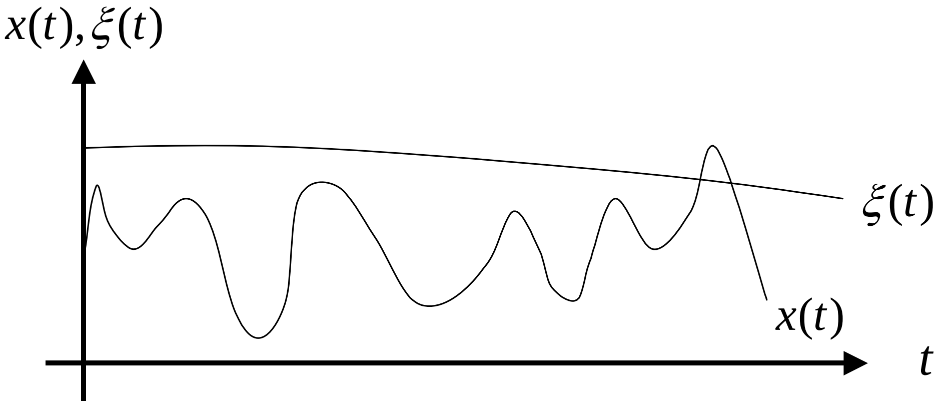
$$\nu^*(t) = \nu(t)(1 - F_Y(\xi(t)))$$

The probability distribution of the time till failure thus becomes exponentially distributed

$$P_f(t) = 1 - \exp(-\int_0^t \nu(\tau)(1 - F_Y(\xi(\tau)))d\tau)$$

Time variant reliability

A random process $X(t)$ is said to be Normal (Gaussian) if any set of random variables $X(t_i)$, $i = 1,2,..n$ is jointly Normal distributed



Realisation of Normal process

The mean number of out-crossings of a random process above the threshold $\xi(t)$ can be derived as (Rice's formula)

$$\nu^{+}(\xi(t)) = \int_{\dot{\xi}(t)}^{\infty} \varphi_{X,\dot{X}}(\xi,\dot{x})(\dot{x}-\dot{\xi})\,d\dot{x}$$

First order partial derivative of x

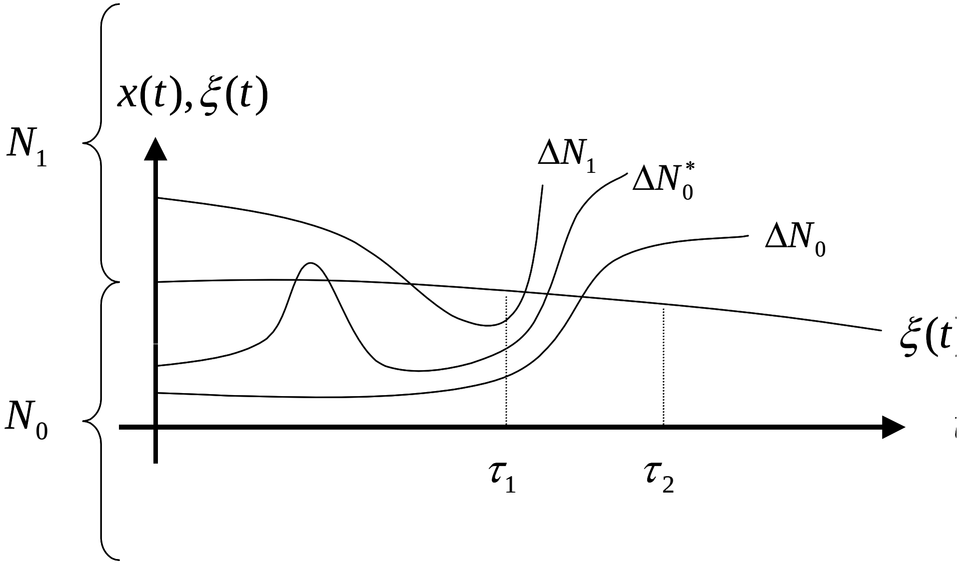
The sample paths must be at least one time differentiable in respect to ***t***

Time variant reliability

For random processes in general we have

$$P_f(T) = P_f(0) + (1 - P_f(0)) \int_0^T f_0(\tau)\, d\tau$$

where $f_0(t)$ is the first passage density function



$$P_f(T) = \frac{N_1 + \sum_{]0,T]} \Delta N_0}{N} = \frac{N_1}{N} + \frac{N_0}{N} \sum_{]0,T]} \frac{\Delta N_0}{N_0}$$

$$f_0(\tau)\Delta\tau = \frac{\Delta N_0}{N_0} + O(\Delta\tau)$$

$$P_f(T) = P_f(0) + (1 - P_f(0)) \sum_{]0,T]} f_0(\tau)\Delta\tau + O(\Delta\tau)$$

$$P_f(T) \le \frac{N_1}{N} + \frac{N_0}{N} \sum_{]0,T]} \frac{\Delta N_0 + \Delta N_0^*}{N_0} = P_f(0) + (1 - P_f(0)) \int_0^T \nu_{X|S_0}^{+}(\xi(t))\, d\tau$$

Time variant reliabilty

For Normal processes two cases may be considered

the stationary process and constant threshold case

$$\nu^+(\xi) = \int_0^\infty \dot{x}\frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\exp(-\frac{1}{2}(\frac{\xi^2}{\sigma_X^2}+\frac{\dot{x}^2}{\sigma_{\dot{X}}^2}))d\dot{x}$$

$$= \frac{1}{2\pi}\frac{\sigma_{\dot{X}}}{\sigma_X}\exp(-\frac{1}{2}(\frac{\xi^2}{\sigma_X^2}))$$

the non-stationary process and/or non-constant threshold case

$$\nu^+(\xi(t)) = \omega_0\varphi(\eta)(\varphi(\frac{\dot{\eta}}{\omega_0}) - \frac{\dot{\eta}}{\omega_0}\Phi(-\frac{\dot{\eta}}{\omega_0}))$$

$$\eta(t) = \frac{\xi(t)-\mu_X(t)}{\sigma_X(t)}$$

$$\omega_0(t)^2 = \frac{1}{\sigma_X(t)^2}\left[\frac{\partial^2}{\partial t_1\partial t_2}c(t_1,t_2)+\sigma_{\dot{X}}(t)^2\right]$$

Time variant reliability

Approximations to time variant problems

The first passage time is of interest in reliability applications –
which is extremely hard to assess

$$P_f(T) = P_f(0) + (1 - P_f(0)) \int_0^T f_0(\tau) d\tau$$

Whenever the reliability problem can be formulated in terms of a conditional probability of failure – given a certain event the Poisson spike model can be used to approximate the first passage probability

$$P_f(t) = 1 - \exp(-\int_0^t \nu(\tau)(1 - F_Y(\xi(\tau))) d\tau)$$

In cases where the probability of failure (e.g. per annum) may be calculated, the simple Poisson process may be used to approximate the first passage probability

$$F_{T_1}(t) = 1 - P_0(t) = 1 - \exp(-\int_0^t \nu(\tau) d\tau)$$

Time variant reliability

It is normally the case that the time variant includes not only random processes but also random sequences and random variables

In such cases the hierarchical model should be used

Out-crossing rate of the process

$$F_{T_1}(t_1) = 1 - E_R\left[\exp\left(-E_Q\left[\int_0^t \nu(\tau, R, Q)\,d\tau\right)\right]\right]$$

Non-ergodic variables

Random sequences

Time variant reliability

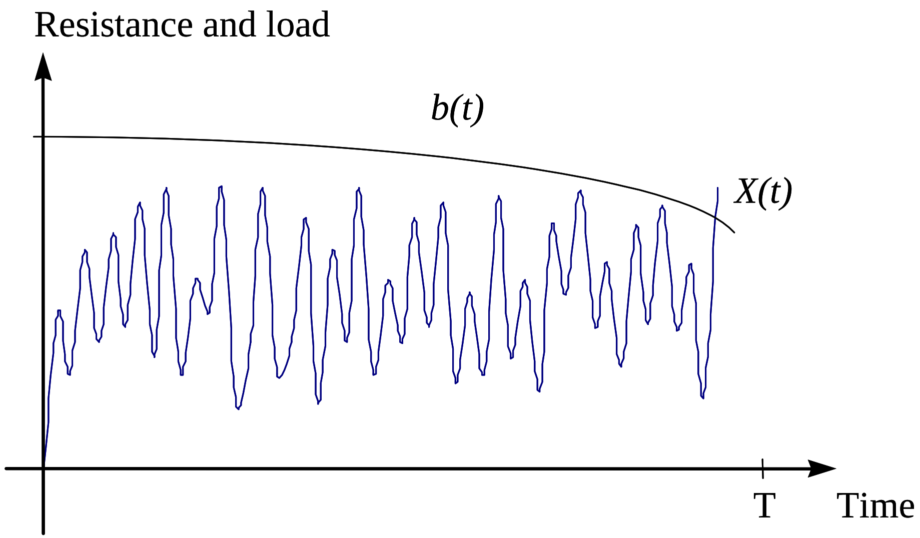
In practice it is useful to differentiate between cases where the threshold (normalised)

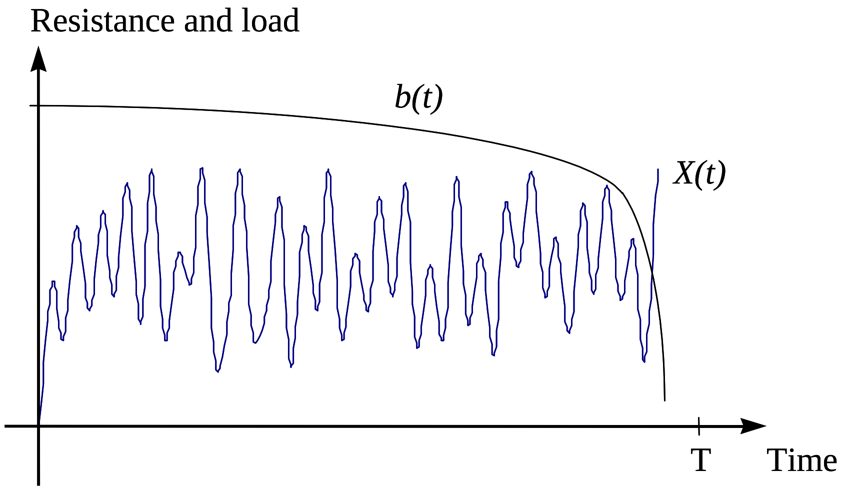
is slowly decreasing

rapidly decreasing

Out-crossing approach

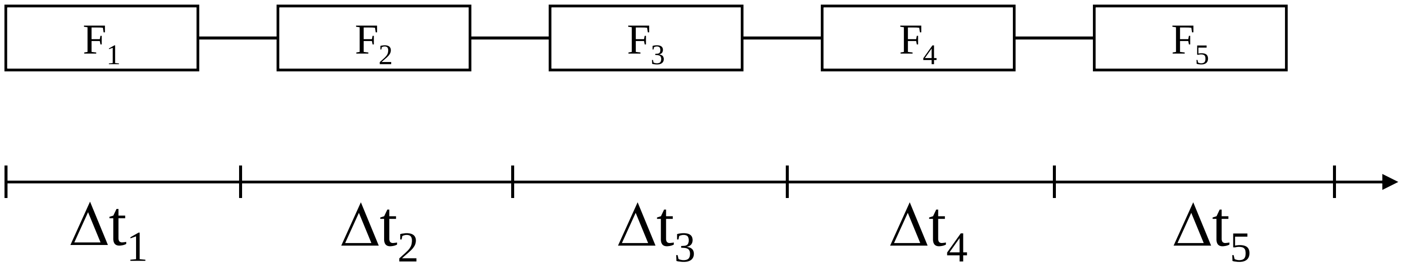
End point modelling





Time variant reliability

Time variant reliability problems may also be addressed approximately by consideration of systems reliability analysis



$$
\begin{aligned}
P_f &= P(F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5) \\
&= 1 - P(\overline{F_1} \cap \overline{F_2} \cap \overline{F_3} \cap \overline{F_4} \cap \overline{F_5})
\end{aligned}
$$

Time variant reliability

A final aspect of time variant reliability is the assessment of the annual probability of failure for components subject to accumulated deterioration

In these cases the reliability analysis provides the failure probability as a function of the experienced service life

If failures at consecutive times may be assumed to be fully dependent the annual failure probabilities may be established by subtraction