Risk and Safety

in

Engineering

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Contents of Today's Lecture

Methods of structural reliability theory

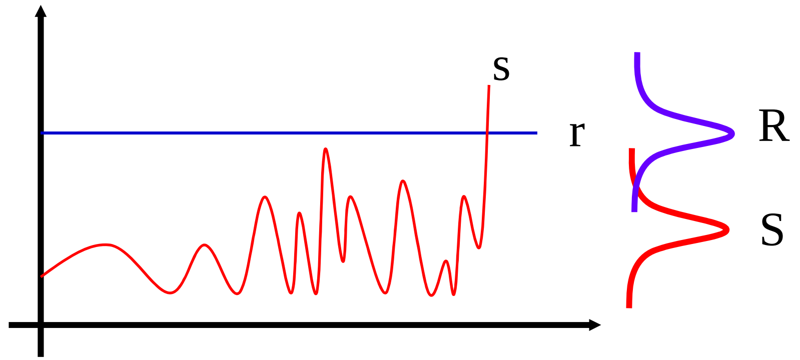
- Linear Normal distributed safety margins
- Non-linear Normal distributed safety margins
- General case - FORM
- SORM improvements
- Monte-Carlo simulation
- Partial safety factors

Structural Reliability Analysis

Reliability of structures cannot be assessed through failure rates because

- Structures are unique in nature

- Structural failures normally take place due to extreme loads exceeding the residual strength



Therefore in structural reliability, models are established for resistances R and loads S individually and the structural reliability is assessed through the probability of failure:

$$P_f = P(R - S \leq 0)$$

Structural Reliability Analysis

If only the resistance is uncertain the failure probability may be assessed by

$$P_f = P(R \leq s) = F_R(s) = P(R/s \leq 1)$$

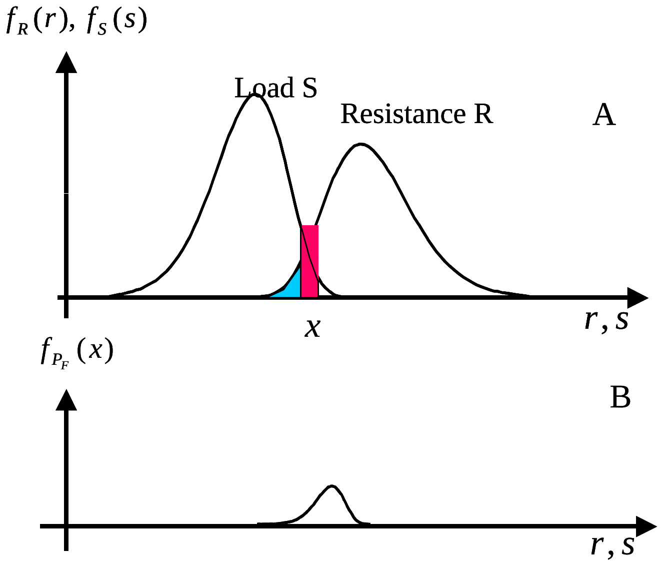
If also the load is uncertain we have

$$P_f = P(R \leq S) = P(R - S \leq 0) = \int_{-\infty}^{\infty} F_R(x) f_S(x) dx$$

where it is assumed that the load and the resistance are independent

This is called the

"Fundamental Case"



Structural Reliability Analysis

In the case where R and S are Normal distributed the **safety margin** M is also Normal distributed

$$M = R - S$$

Then the **failure probability** is

$$P_F = P(R - S \leq 0) = P(M \leq 0)$$

with the mean value of M

$$\mu_M = \mu_R - \mu_S$$

and standard deviation of M

$$\sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2}$$

The **failure probability** is then

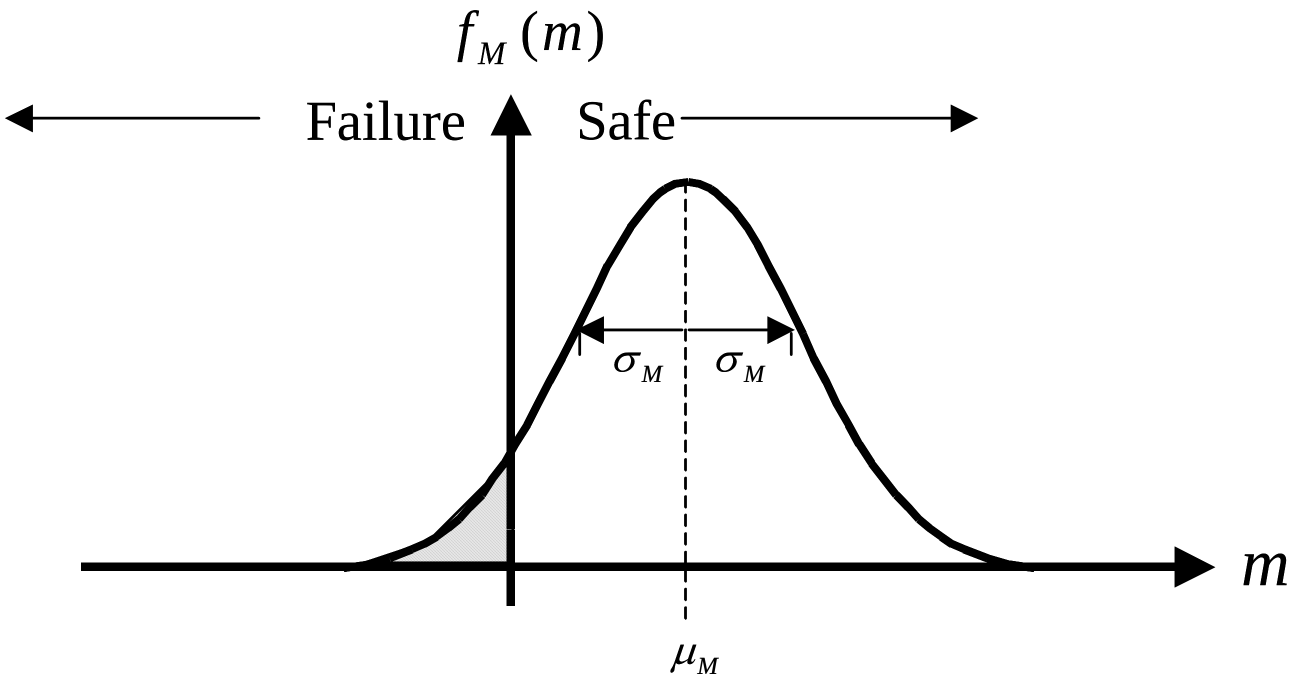
$$P_F = \Phi\left(\frac{0 - \mu_M}{\sigma_M}\right) = \Phi(-\beta)$$

where the **reliability index** is

$$\beta = \mu_M / \sigma_M$$

Structural Reliability Analysis

The Normal distributed safety margin *M*



Structural Reliability Analysis

In the general case the resistance and the load may be defined in terms of functions where X are **basic random variables**

$$R = f_1(\mathbf{X})$$
$$S = f_2(\mathbf{X})$$

The safety margin can be written as where $g(x)$ is called the **limit state function**

$$M = R - S = f_1(\mathbf{X}) - f_2(\mathbf{X}) = g(\mathbf{X})$$

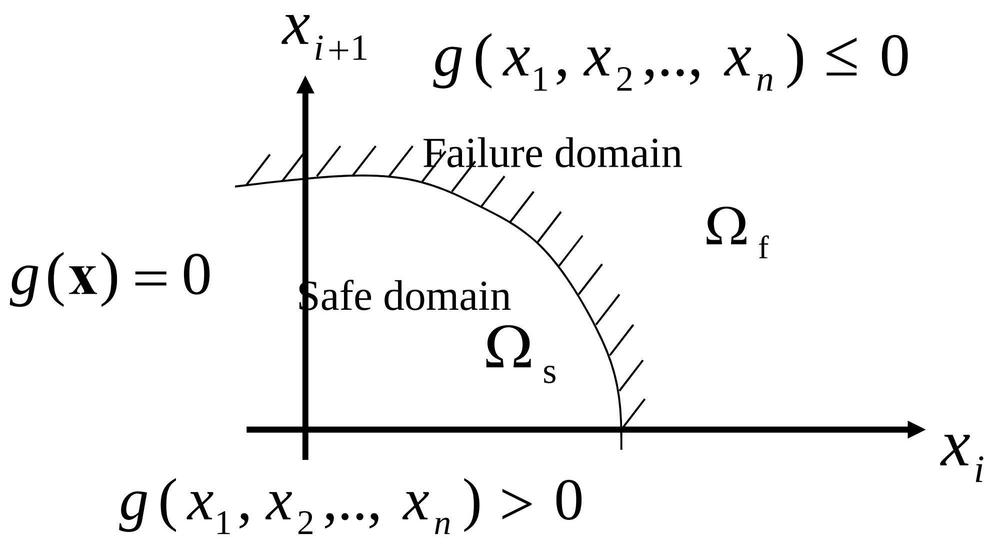
Failure occurs when

$$g(\mathbf{x}) \leq 0$$

Structural Reliability Analysis

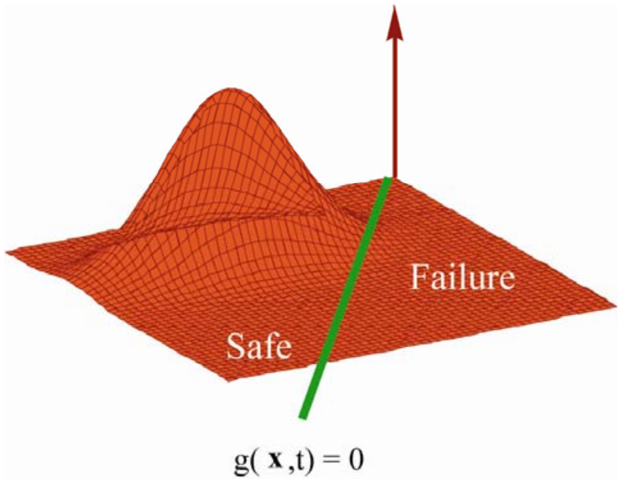
Setting $g(\mathbf{x}) = 0$ defines a (n-1) dimensional surface in the space spanned by the n basic variables X

This is the failure surface separating the sample space of X into a safe domain and a failure domain



The failure probability may in general terms be written as

$$P_f = \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$



Failure event

$$\mathbf{F} = \{g(\mathbf{x}) \leq 0\}$$

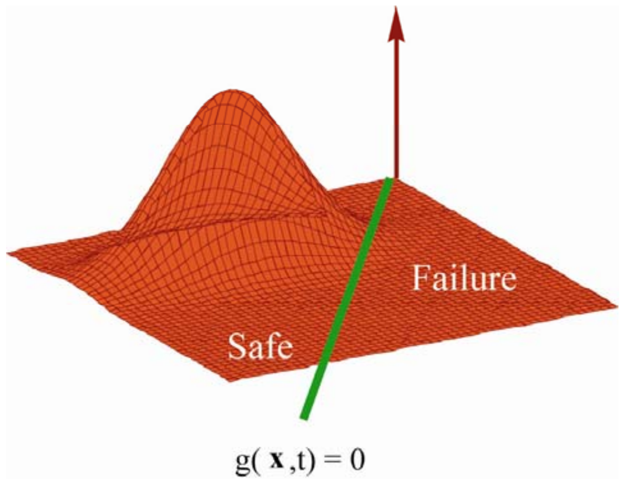
Basics of Structural Reliability Methods

The probability of failure can be assessed by

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \le 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function for the basic random variables $\boldsymbol{X}$

For the 2-dimensional case the failure probability simply corresponds to the integral under the joint probability density function in the area of failure



Basics of Structural Reliability Methods

The probability of failure can be calculated using

- numerical integration
  (Simpson, Gauss, Tchebyschev)

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \le 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

but for problems involving dimensions higher than say 6 the numerical integration becomes cumbersome

Other methods are necessary !

Basics of Structural Reliability Methods

When the limit state function is linear

$$g(\mathbf{x}) = a_0 + \sum_{i=1}^{n} a_i \cdot x_i$$

the saftey margin M is defined through

$$M = a_0 + \sum_{i=1}^{n} a_i \cdot X_i$$

with

mean value

$$\mu_M = a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}$$

and

variance

$$\sigma_M^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \rho_{ij} a_i a_j \sigma_i \sigma_j$$

Basics of Structural Reliability Methods

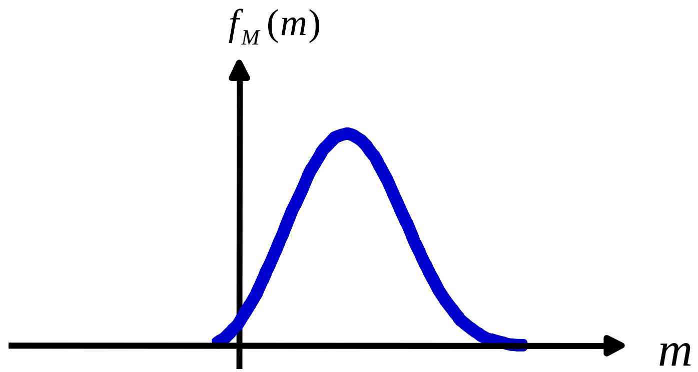
The failure probability can then be written as

$$P_F = P(g(\mathbf{X}) \leq 0) = P(M \leq 0)$$

The reliability index is defined as

$$\beta = \frac{\mu_M}{\sigma_M}$$

(Basler and Cornell)



Provided that the safety margin is Normal distributed

the failure probability is determined as

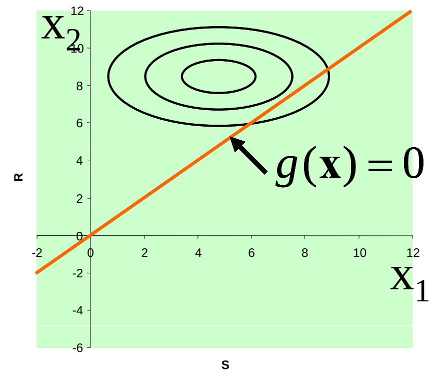
$$P_F = \Phi(-\beta)$$

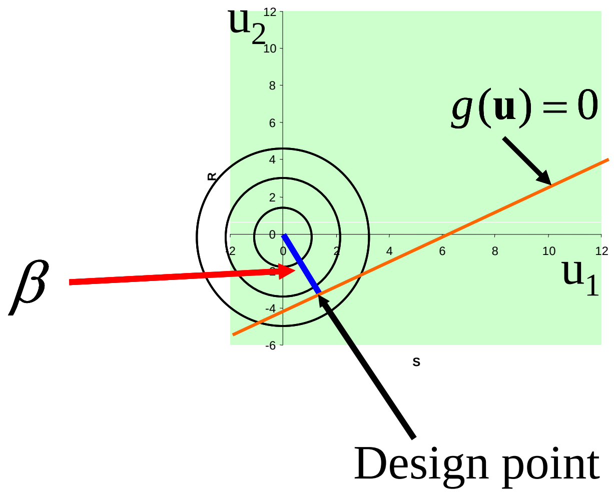
Basics of Structural Reliability Methods

The reliability index β has the geometrical interpretation of being the shortest distance between the failure surface and the origin in standard Normal distributed space U

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}$$

in which case the components of U have zero means and variances equal to 1





Basics of Structural Reliability Methods

Example:

Consider a steel rod with resistance *r* subjected to a tension force *s*

$$g(\mathbf{X}) = R - S$$

r* and *s* are modeled as uncorrelated Normal distributed random variables *R* and *S

$$\mu_R = 350, \sigma_R = 35$$
$$\mu_S = 200, \sigma_S = 40$$

The probability of failure is required

$$P(R - S \leq 0)$$

Basics of Structural Reliability Methods

Example:

Consider a steel rod with resistance *r* subjected to a tension force *s*

$$g(\mathbf{X}) = R - S$$

r* and *s* are modeled by the random variables *R* and *S

$$\mu_R = 350, \sigma_R = 35$$
$$\mu_S = 200, \sigma_S = 40$$

The probability of failure is wanted

$$P(R - S \leq 0)$$

The safety margin is

$$M = R - S \begin{cases} \mu_M = 350 - 200 = 150 \\ \sigma_M = \sqrt{35^2 + 40^2} = 53.15 \end{cases}$$

The reliability index is then

$$\beta = \frac{150}{53.15} = 2.84$$

and the probability of failure

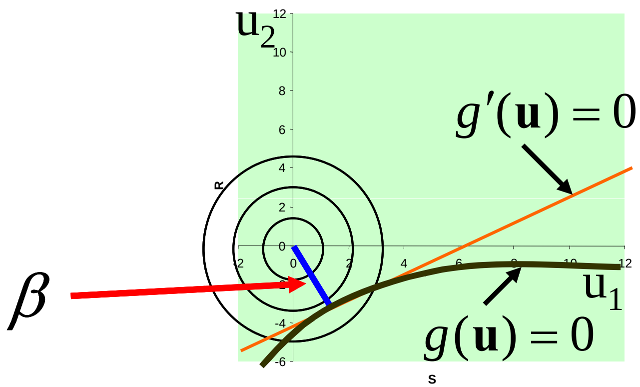
$$P_F = \Phi(-2.84) = 2.4 \cdot 10^{-3}$$

Basics of Structural Reliability Methods

Usually the limit state function is non-linear

- this small phenomenon caused the so-called invariance problem

Can however easily be linearized !

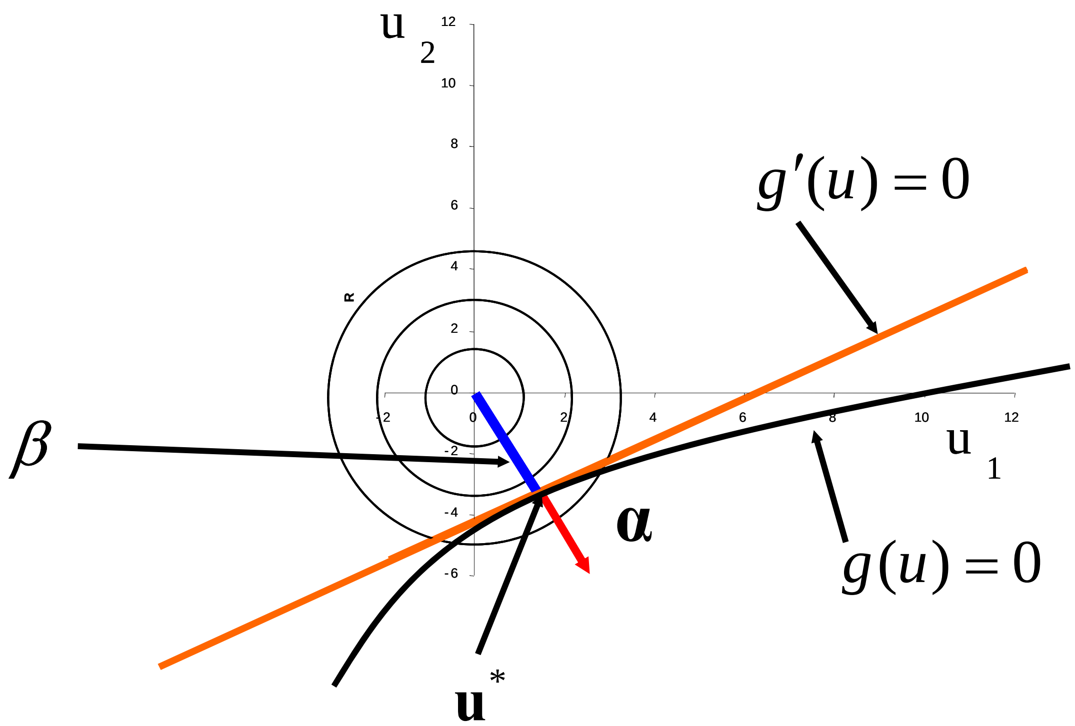


Hasofer & Lind suggested to linearize the limit state function in the design point

- this solved the invariance problem

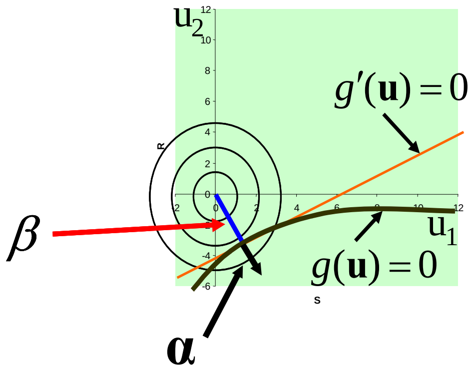
The reliability index may then be determined by the following optimization problem

$$\beta = \min_{\mathbf{u} \in \{g(\mathbf{u})=0\}} \sqrt{\sum_{i=1}^{n} u_i^2}$$



u_2
12
10
8
6
4
2
0
-2
-4
-6
R
-2
0
2
4
6
8
10
12
u_1
$g'(u)=0$
$g(u)=0$
β
$\boldsymbol{\alpha}$
$\mathbf{u}^*$

Basics of Structural Reliability Methods



The optimization problem can be formulated as an iteration problem

1) the design point is determined as $\mathbf{u}^* = \beta \cdot \boldsymbol{\alpha}$

2) the normal vector to the limit state function is determined as

$$\alpha_i = \frac{-\dfrac{\partial g}{du_i}(\beta \cdot \boldsymbol{\alpha})}{\left[\sum_{j=1}^{n} \dfrac{\partial g}{du_i}(\beta \cdot \boldsymbol{\alpha})^2\right]^{1/2}}, \quad i = 1,2,..n$$

3) the safety index is determined as $g(\beta \cdot \alpha_1, \beta \cdot \alpha_2, ... \beta \cdot \alpha_n) = 0$

4) a new design point is determined as $u^* = \left(\beta \cdot \alpha_1, \beta \cdot \alpha_2, ... \beta \cdot \alpha_n\right)^T$

5) the above steps are continued until convergence in β is attained

Basics of Structural Reliability Methods

Example :

Consider the steel rod with cross-sectional area *a* and yield stress *r*

$$h = r \cdot a$$

The rod is loaded with the tension force s

The limit state function can then be written as

$$g(\mathbf{x}) = r \cdot a - s$$

***r*, *a* and *s* are uncertain and modeled by normal distributed random variables**

$$\mu_R = 350, \sigma_R = 35 \quad \mu_S = 1500, \sigma_S = 300$$
$$\mu_A = 10, \sigma_A = 1$$

we would like to calculate the probability of failure

Basics of Structural Reliability Methods

The first step is to transform the basic random variables into standardized Normal distributed space

$$U_R = \frac{R - \mu_R}{\sigma_R}$$

$$U_A = \frac{A - \mu_A}{\sigma_A}$$

$$U_S = \frac{S - \mu_S}{\sigma_S}$$

Then we write the limit state function in terms of the realizations of the standardized Normal distributed random variables

$$g(u) = (u_R \sigma_R + \mu_R)(u_A \sigma_A + \mu_A) - (u_S \sigma_S + \mu_S)$$

$$= (35u_R + 350)(u_A + 10) - (300u_S + 1500)$$
$$= 350\mathrm{u}_\mathrm{R} + 350u_A - 300u_S + 35u_R u_A + 2000$$

Basics of Structural Reliability Methods

The reliability index is calculated as

$$\beta = \frac{-2000}{350\alpha_R + 350\alpha_A - 300\alpha_S + 35\beta\alpha_R\alpha_A}$$

the components of the α-vector are then calculated as

$$\begin{cases} \alpha_R = -\frac{1}{k}(350 + 35\beta\alpha_A) \\ \alpha_A = -\frac{1}{k}(350 + 35\beta\alpha_R) \\ \alpha_S = \frac{300}{k} \end{cases}$$

where

$$k = \sqrt{\alpha_R^2 + \alpha_A^2 + \alpha_S^2}$$

Basics of Structural Reliability Methods

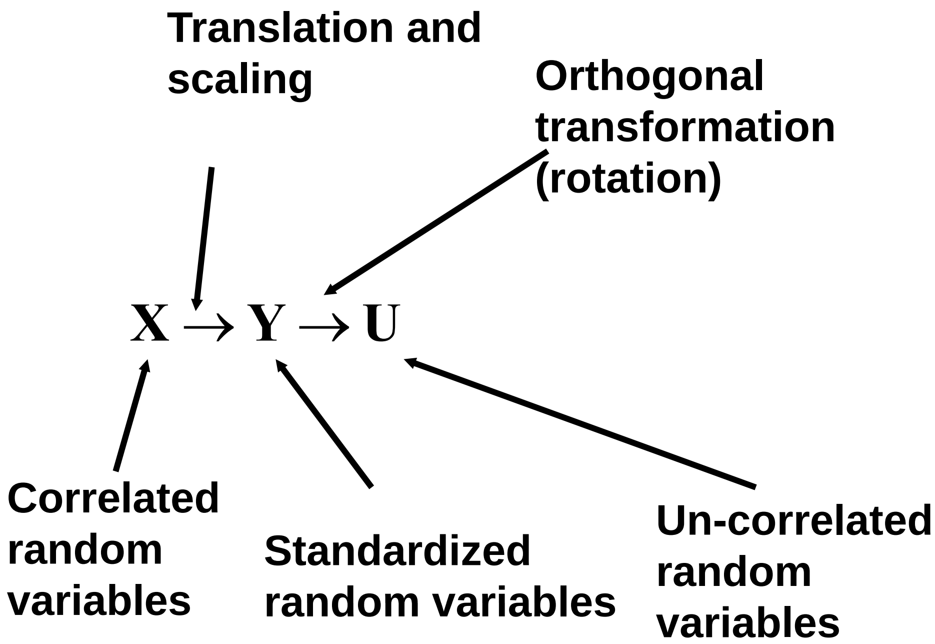
Following the iteration scheme we get the following iteration history

Iteration	Start	1	2	3	4	5
β	3.0000	3.6719	3.7399	3.7444	3.7448	3.7448
α_R	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_A	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_S	0.5800	0.5916	0.6084	0.6086	0.6087	0.6087

Basics of Structural Reliability Methods

The procedure can be extended to consider

Correlated random variables



Basics of Structural Reliability Methods

Correlated random variables

The covariance matrix for the random variables is given as

$$\mathbf{C_X} = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] .. & Cov[X_1, X_n] \\ \vdots & \vdots & \vdots \\ Cov[X_n, X_1] & \cdots & Var[X_n] \end{bmatrix}$$

and the correlation coefficient matrix is

$$\boldsymbol{\rho}_{\mathbf{X}} = \begin{bmatrix} 1 & \cdots & \rho_{1n} \\ \vdots & 1 & \vdots \\ \rho_{n1} & \cdots & 1 \end{bmatrix}$$

The first step is the standardization

$$Y_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}, i = 1,2,..n$$

Basics of Structural Reliability Methods

Correlated random variables

The transformation of the correlated random variables into non-correlated random variables can be written as

$$\mathbf{Y} = \mathbf{T}\mathbf{U}$$

where $\mathbf{T}$ is a lower triangular matrix

then we can write

$$\mathbf{C}_{\mathbf{Y}} = E\left[\mathbf{Y}\cdot\mathbf{Y}^{T}\right] = E\left[\mathbf{T}\cdot\mathbf{U}\cdot\mathbf{U}^{T}\cdot\mathbf{T}^{T}\right] = \mathbf{T}\cdot E\left[\mathbf{U}\cdot\mathbf{U}^{T}\right]\cdot\mathbf{T}^{T} = \mathbf{T}\times\mathbf{T}^{T} = \boldsymbol{\rho}_{\mathbf{X}}$$

with *T* ***standing for transpose matrix***

Basics of Structural Reliability Methods

Correlated random variables

In the case of 3 random variables we have

$$\mathbf{T}\cdot\mathbf{T}^T=\boldsymbol{\rho}_{\mathbf{X}}=\begin{bmatrix}\rho_{11} & \rho_{12} & \rho_{13}\\ & \rho_{22} & \rho_{23}\\ sym. & & \rho_{33}\end{bmatrix}$$

As $\mathbf{T}$ is a lower triangular matrix we have

$$\mathbf{T}\cdot\mathbf{T}^T=\begin{bmatrix}T_{11} & 0 & 0\\ T_{21} & T_{22} & 0\\ T_{31} & T_{32} & T_{33}\end{bmatrix}\begin{bmatrix}T_{11} & T_{12} & T_{13}\\ 0 & T_{22} & T_{23}\\ 0 & 0 & T_{33}\end{bmatrix}=\begin{bmatrix}\rho_{11} & \rho_{12} & \rho_{13}\\ & \rho_{22} & \rho_{23}\\ sym. & & \rho_{33}\end{bmatrix}$$

$$T_{11}=\sqrt{1}$$
$$T_{21}=\rho_{12}$$
$$T_{31}=\rho_{13}$$
$$T_{22}=\sqrt{1-T_{21}^2}$$
$$T_{32}=\frac{\rho_{23}-T_{31}\cdot T_{21}}{T_{22}}$$
$$T_{33}=\sqrt{1-T_{31}^2-T_{32}^2}$$
$$\vdots$$

Basics of Structural Reliability Methods

The normal-tail approximation

$$F_{X_{ii}}(x_i^*) = \Phi\left(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}\right)$$

$$f_{X_{ii}}(x_i^*) = \frac{1}{\sigma_{X_i}}\varphi\left(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}\right)$$

$$\sigma'_{X_i} = \frac{\varphi(\Phi^{-1}(F_{X_i}(x_i^*)))}{f_{X_i}(x_i^*)}$$

$$\mu'_{X_i} = x_i^* - \Phi^{-1}(F_{X_i}(x_i^*))\sigma'_{X_i}$$

Basics of Structural Reliability Methods

Non-normal distributed random variables

$$F_X(x) = F_{X_n}(x_n | x_1, x_2, \ldots x_{n-1}) \cdot F_{X_{n-1}}(x_{n-1} | x_1, x_2, \ldots x_{n-2}) \ldots F_{X_1}(x_1)$$

Rosenblatt Transformation

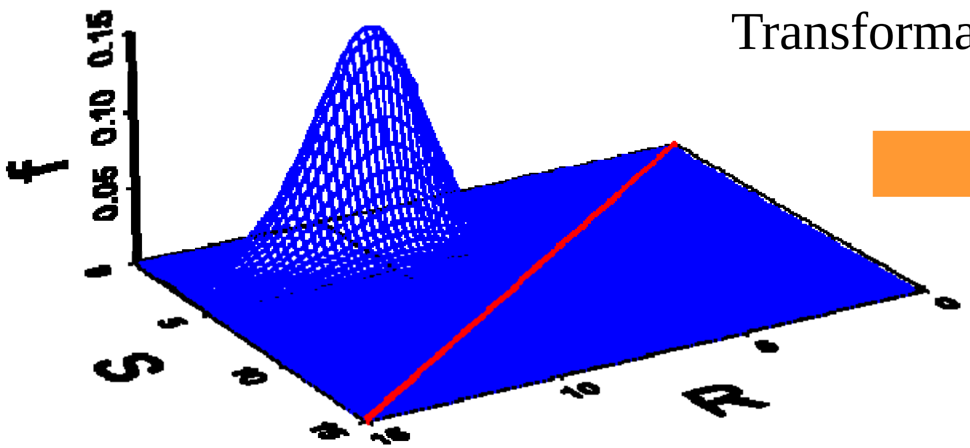
$$\Phi(u_1) = F_{X_1}(x_1)$$

$$\Phi(u_2) = F_{X_2}(x_2 | x_1)$$

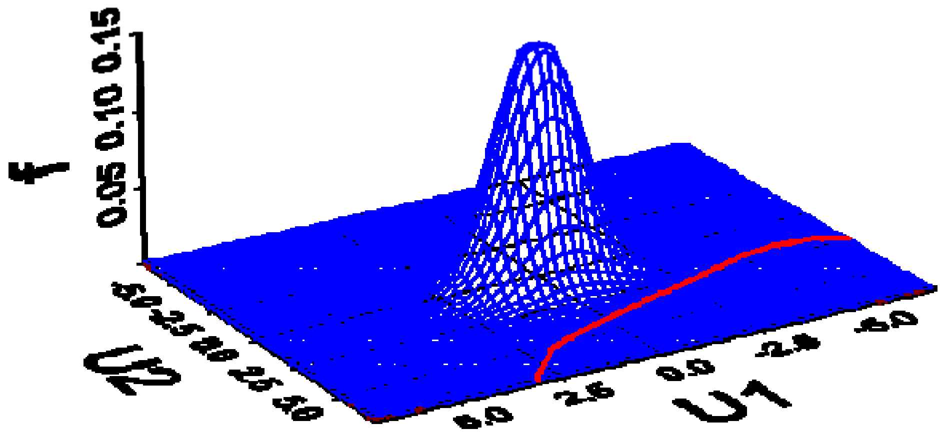
$$\vdots$$

$$\Phi(u_n) = F_{X_n}(x_n | x_1, x_2, \ldots x_{n-1})$$

Basics of Structural Reliability Methods



Transformation



g(Z): linear

$\mu_{Z1}, \mu_{Z2} \quad \in \mathbb{R}$

$\sigma_{Z1}, \sigma_{Z2} \quad \in \mathbb{R}$

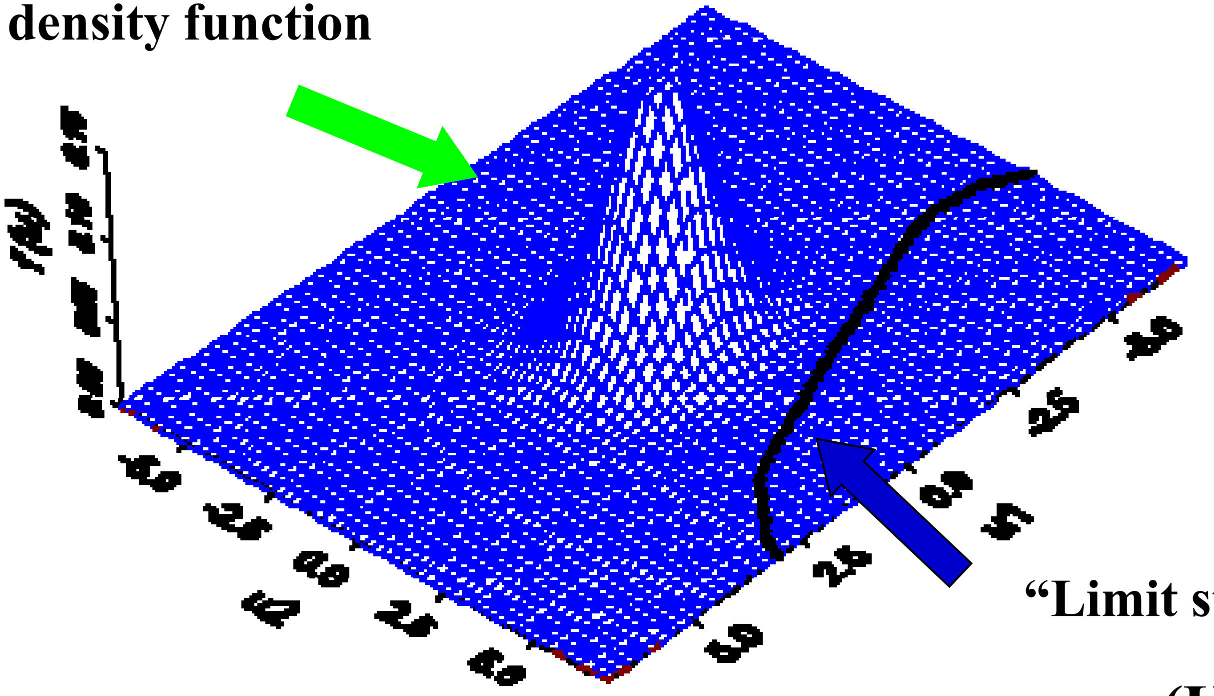
g(U): non linear

$\mu_{U1} = \mu_{U2} = 0$

$\sigma_{U1} = \sigma_{U2} = 1$

Basics of Structural Reliability Methods

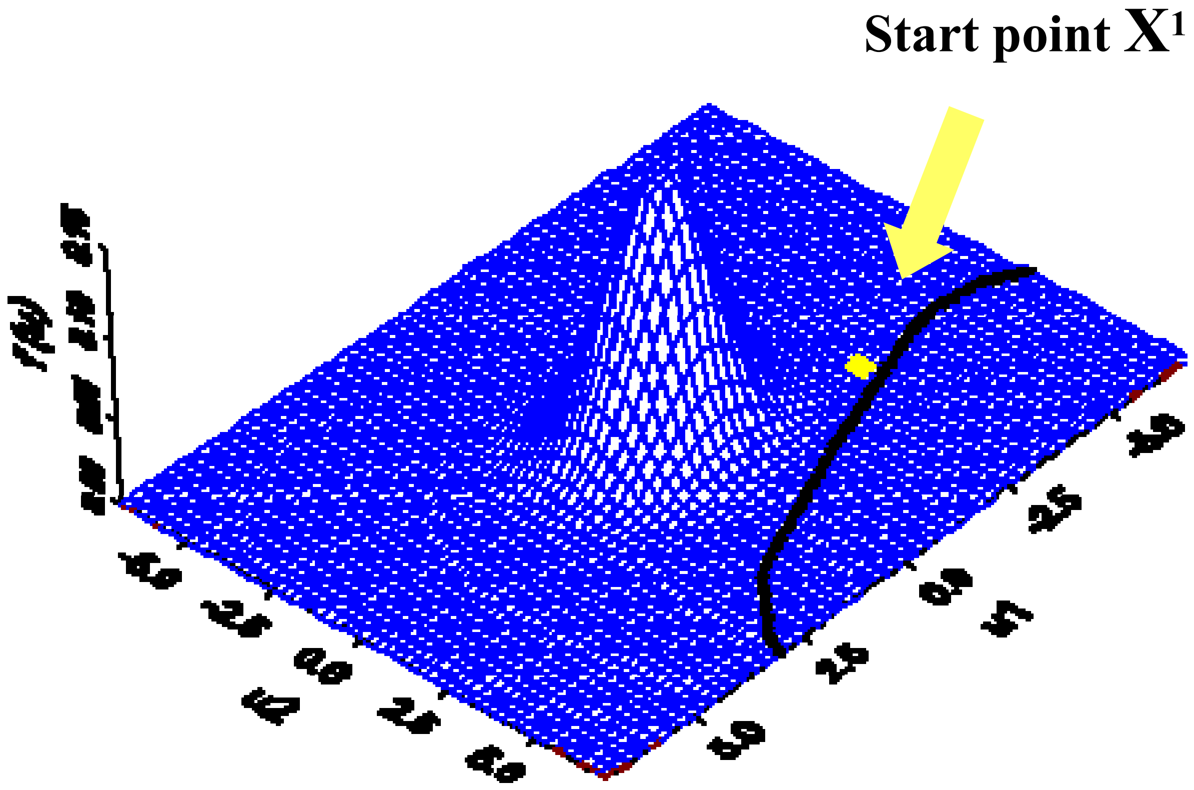
joint probability density function



“Limit state function”

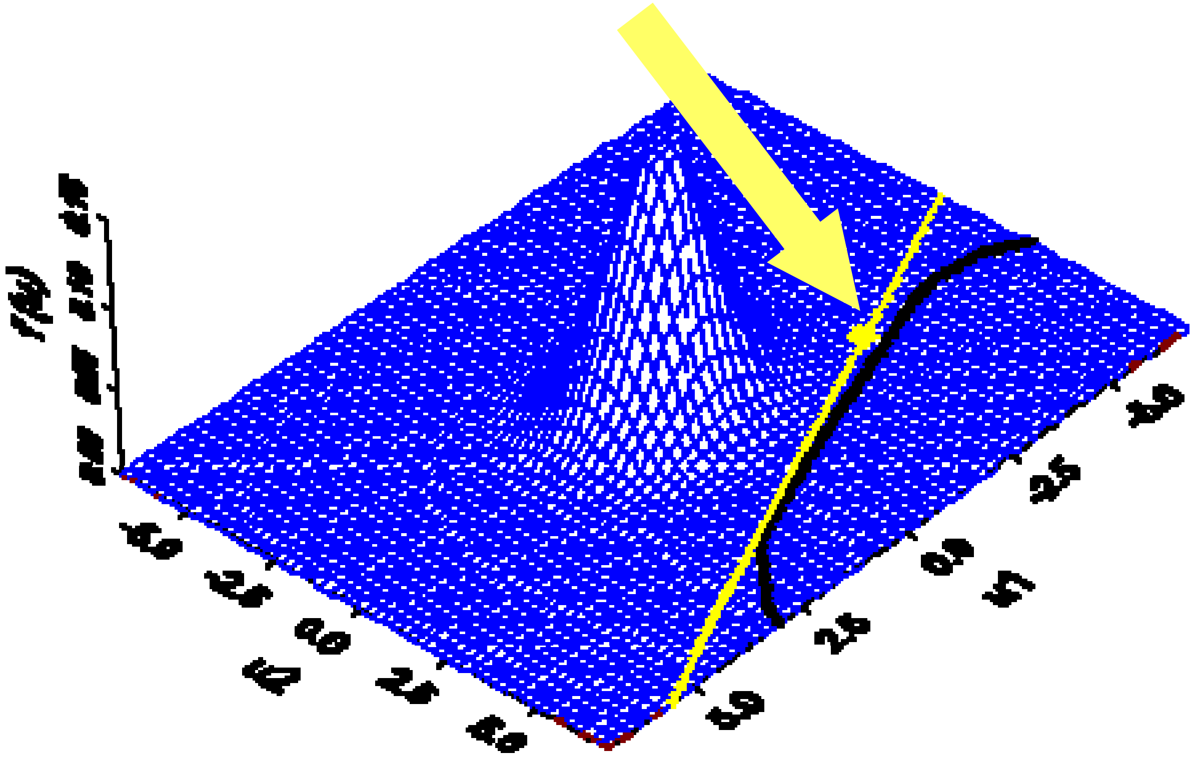
g(U) = R-S

Basics of Structural Reliability Methods



Basics of Structural Reliability Methods

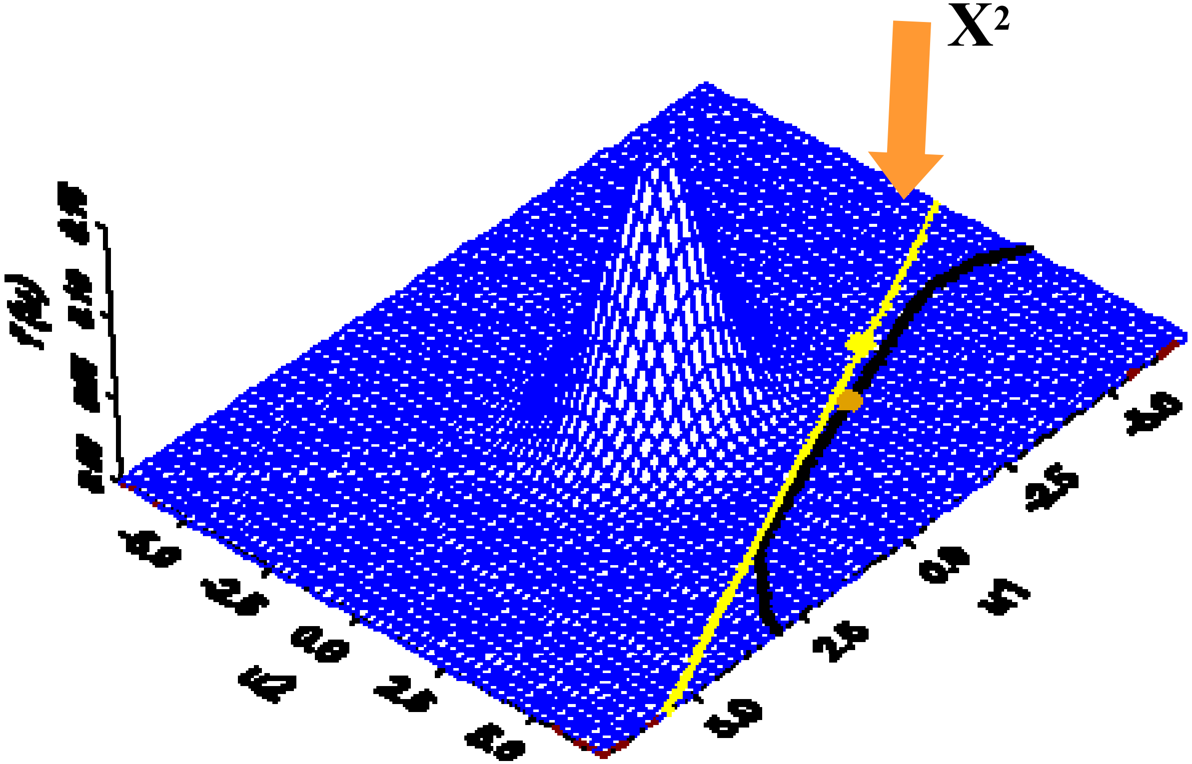
Linearization of Limit state function in starting point



Basics of Structural Reliability Methods

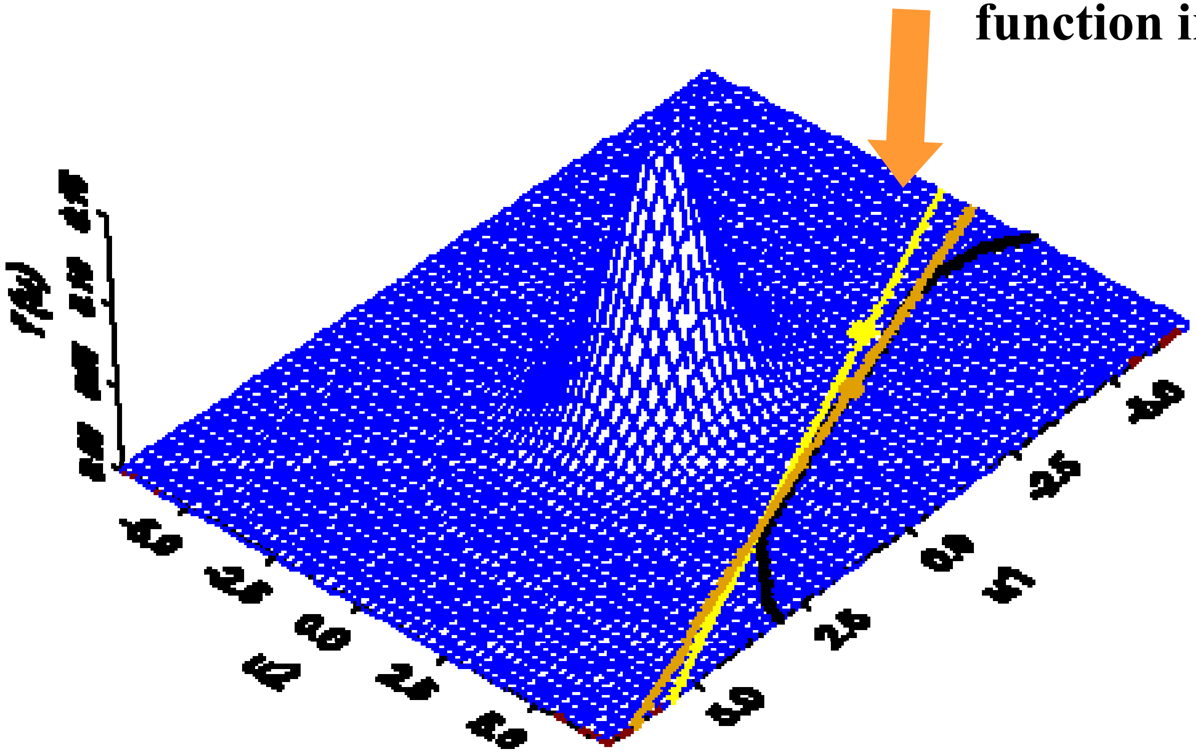
Calculation of new design point

X^2



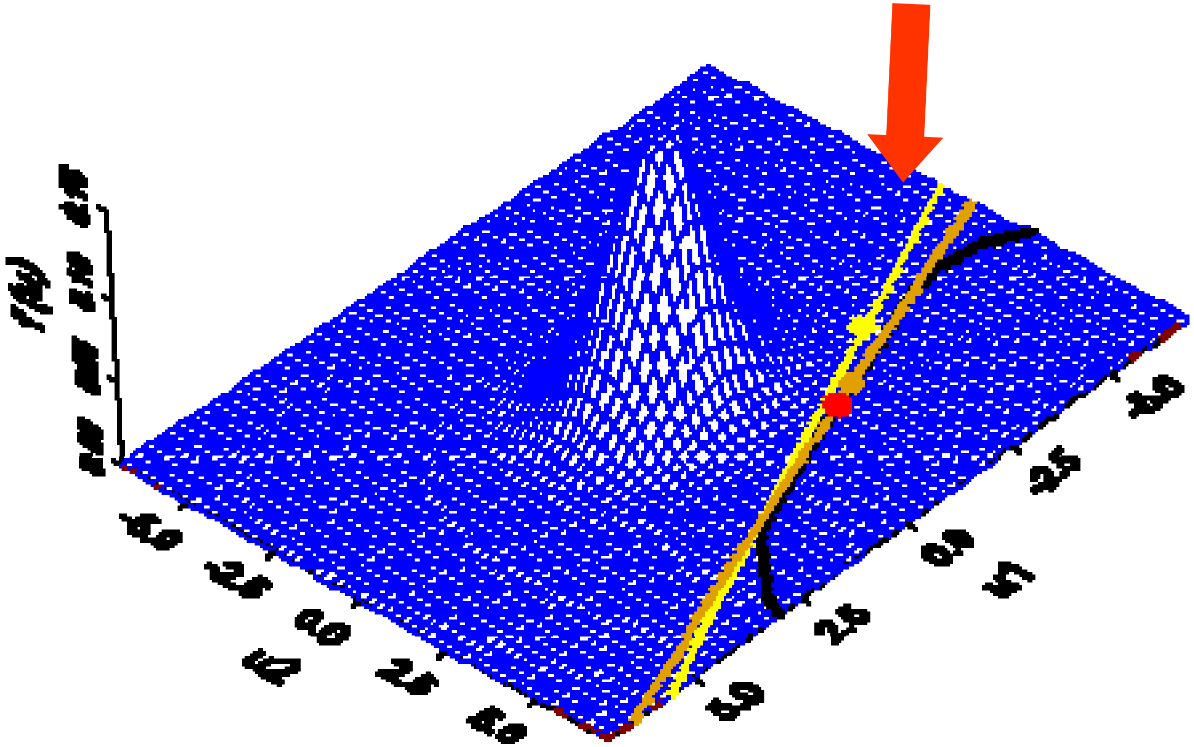
Basics of Structural Reliability Methods

Linearisation of Limit state function in X^2

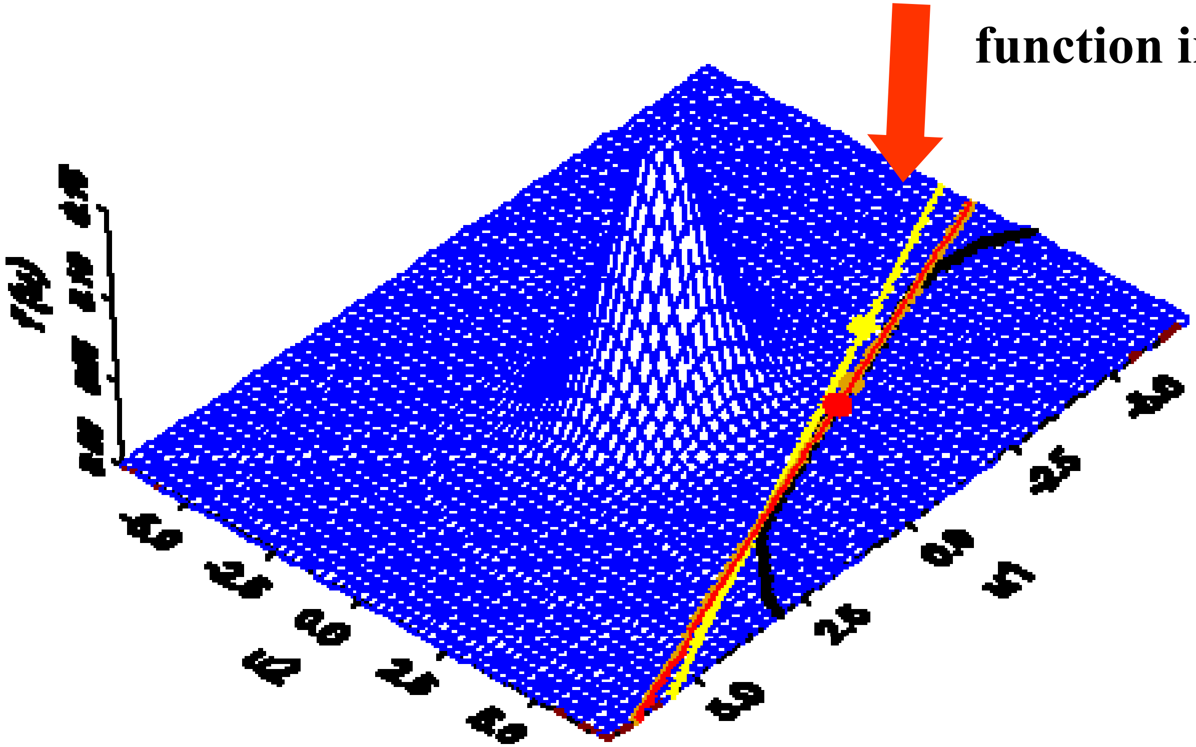


Basics of Structural Reliability Methods

Calculation of new design point X^3

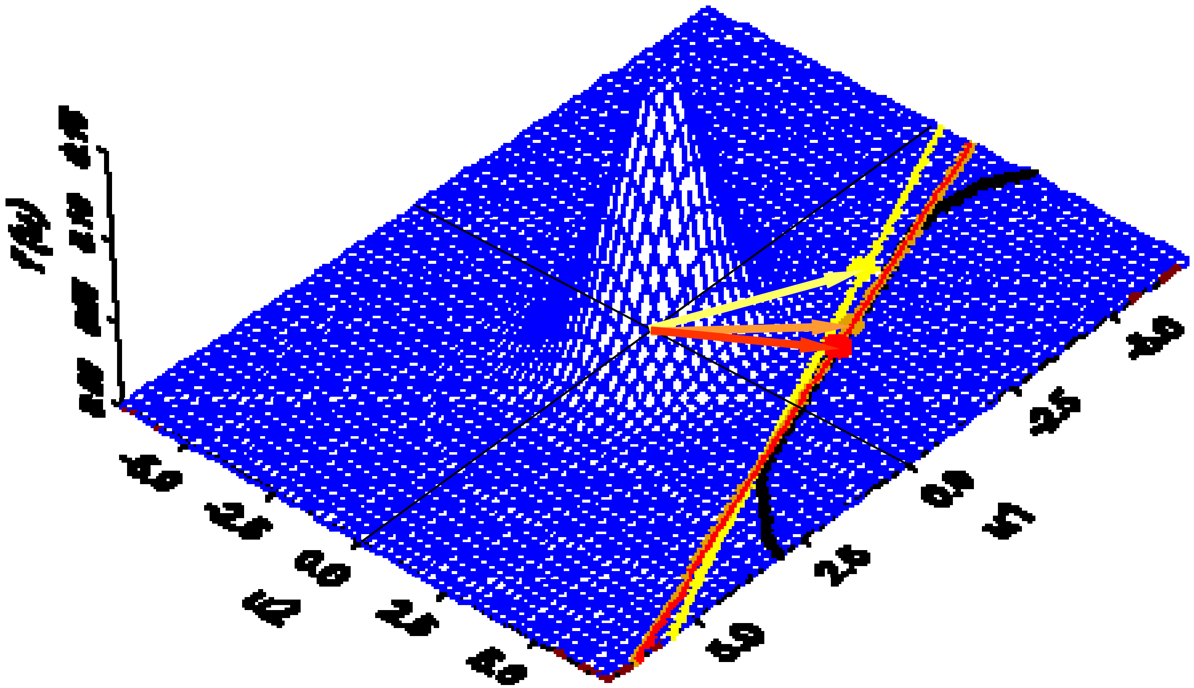


Basics of Structural Reliability Methods



Linearization of Limit state function in X^3

Basics of Structural Reliability Methods



β^1=3.556

β^2=3.607

β^3=3.608

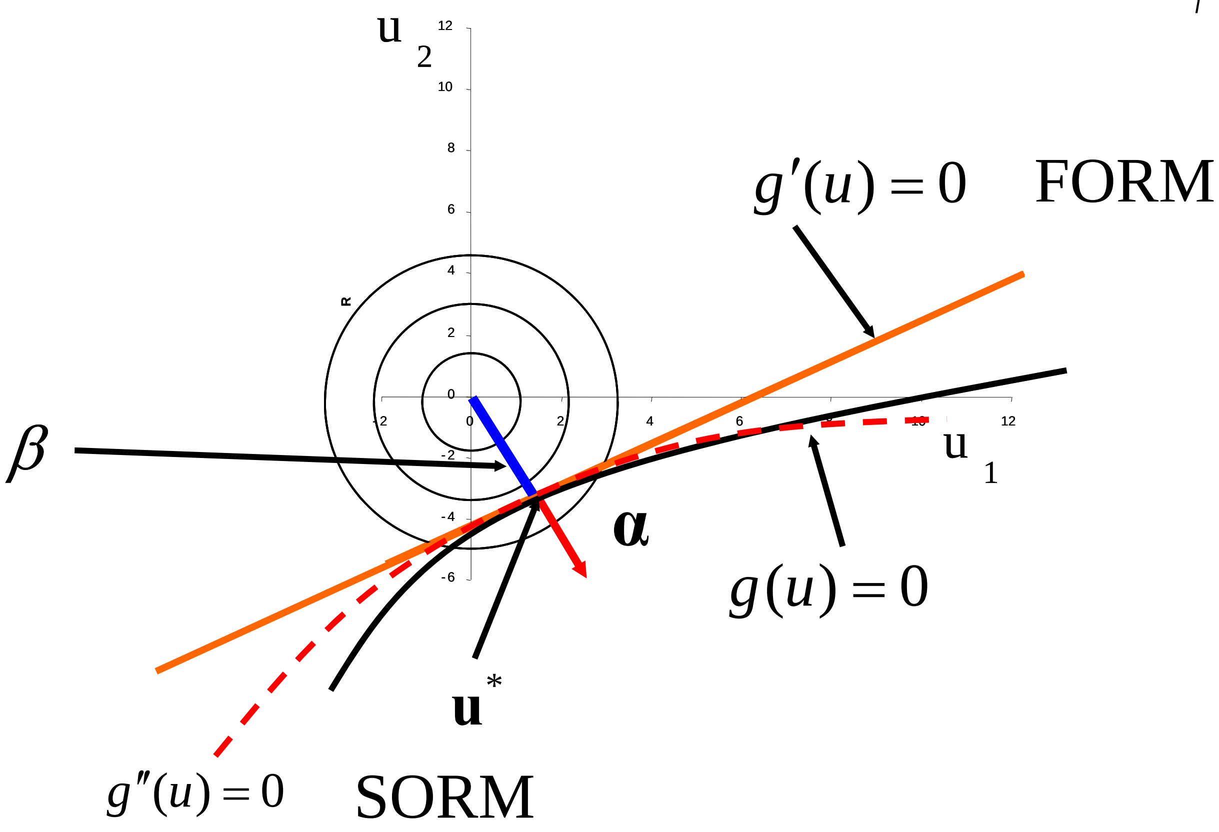
β^4=3.608

Convergency Criteria: $$\Delta\beta = \left|\beta^{n+1} - \beta^{n}\right| \leq \varepsilon$$

Basics of Structural Reliability Methods

SORM Improvements

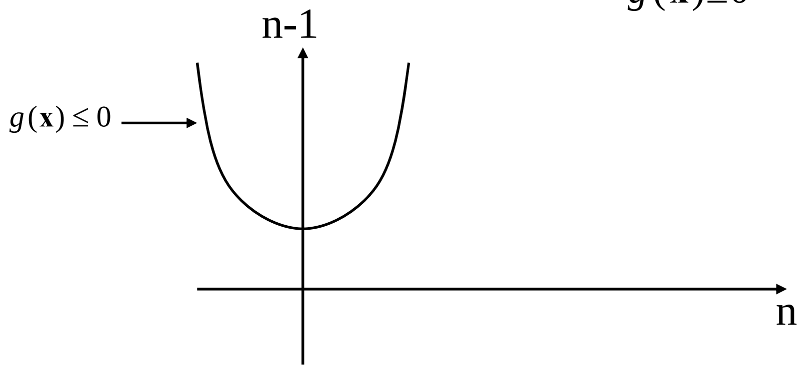
$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \le 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$



Basics of Structural Reliability Methods

$$I = \int_{g(\mathbf{x})\le 0} e^{\lambda h(\mathbf{x})} d\mathbf{x}$$

SORM Improvements



Asymptotic Laplace integral solutions

$$I = \int_{g(\mathbf{x})\le 0} e^{\lambda h(\mathbf{x})} d\mathbf{x} \approx \frac{(2\pi)^{(n-1)/2}\lambda^{-(n+1)}}{\lambda^{(n+1)}\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}}$$

Main curvatures

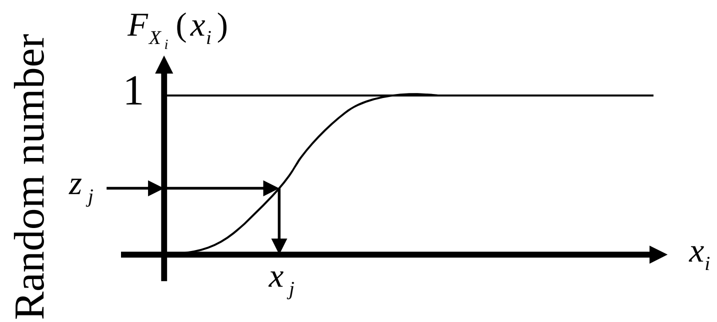
$$P_f = \int_{g(\mathbf{x})\le 0} \frac{e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}}{\left(\sqrt{2\pi}\right)^n} d\mathbf{x} \approx \frac{e^{-\beta^2/2}}{\sqrt{2\pi}\beta\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}} = \frac{\varphi(-\beta)}{\beta\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}} \cong \frac{\Phi(-\beta)}{\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}}$$

Basics of Structural Reliability Methods

Simulation methods may also be used to solve the integration problem

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \leq 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

1) *m* realizations of the vector X are generated



2) for each realization the value of the limit state function is evaluated

3) the realizations where the limit state function is zero or negative are counted

$$n_f$$

4) The failure probability is estimated as

$$p_f = \frac{n_f}{m}$$

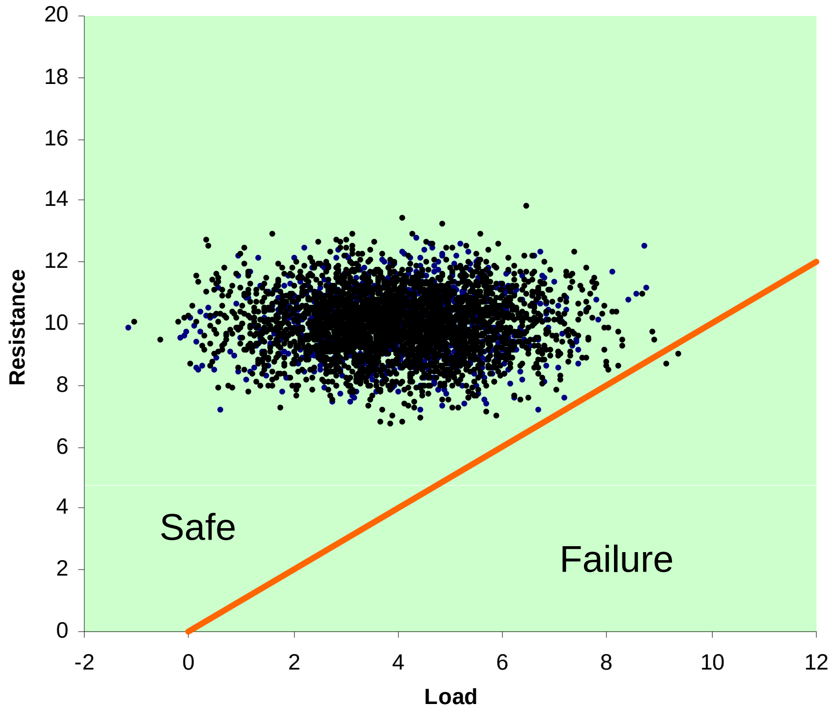
Basics of Structural Reliability Methods

- **Estimation of failure probabilities using Monte Carlo Simulation**

  - ***m* random outcomes of R und S are generated and the number of outcomes n_f in the failure domain are recorded and summed**

  - **The failure probability p_f is then**

$$p_f = \frac{n_f}{m}$$



Basics of Structural Reliability Methods

Partial safety factors

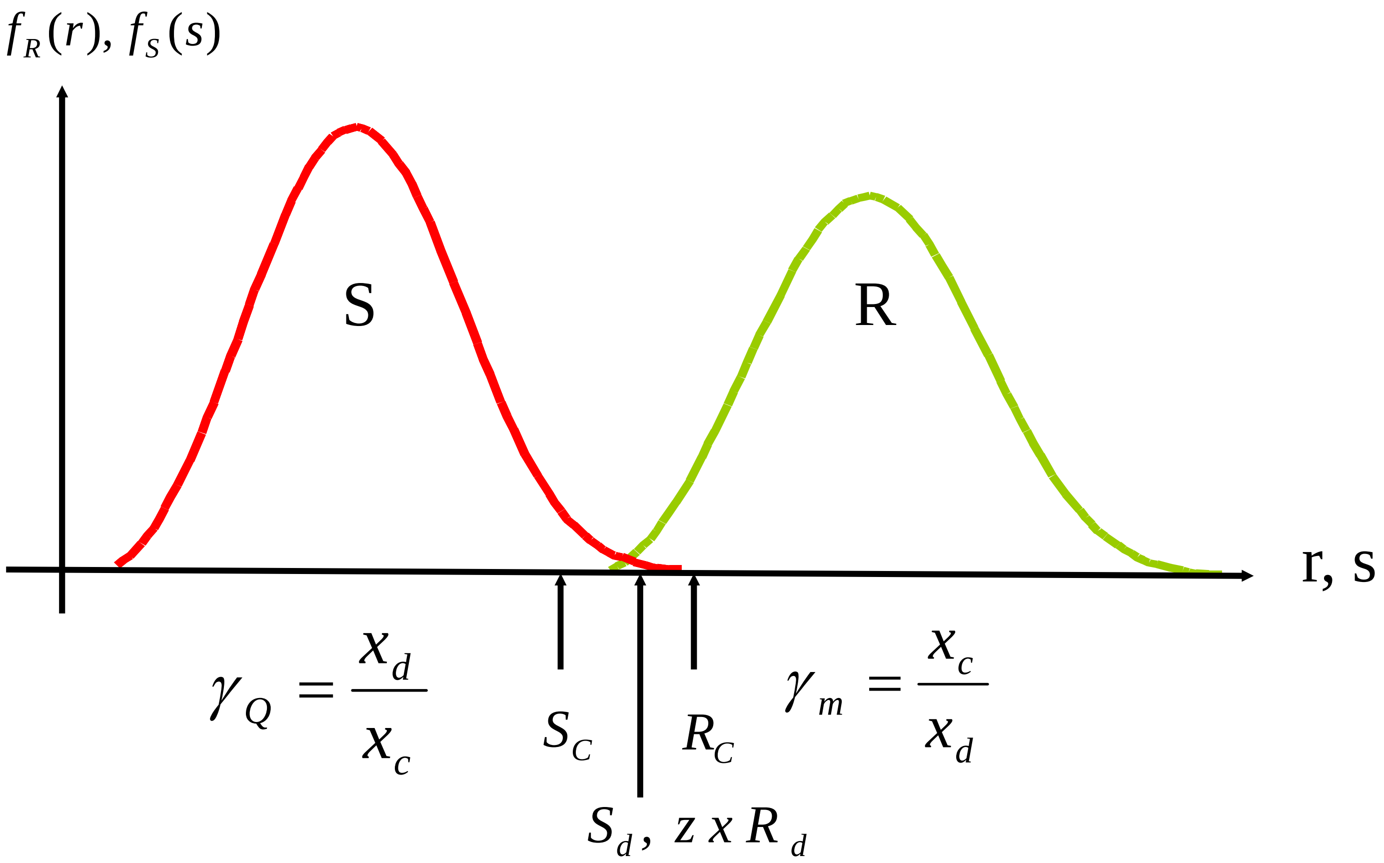
Design codes prescribe design equations where the design variables (e.g. cross-sections) are to be determined as a function of

$$zR_c / \gamma_m - \left(\gamma_{G_a} G_c + \gamma_Q Q_C\right) = 0$$

- Characteristic values $R_C \quad G_C \quad Q_C$
- Partial safety factors $\gamma_m \quad \gamma_G \quad \gamma_Q$

The design variables are selected such that the design equation is close to zero

Basics of Structural Reliability Methods



Basics of Structural Reliability Methods

Example

Iteration	Start	1	2	3	4	5
β	3.0000	3.6719	3.7399	3.7444	3.7448	3.7448
α_R	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_A	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_S	0.5800	0.5916	0.6084	0.6086	0.6087	0.6087

$$\mu_R = 350, \sigma_R = 35$$

$$\mu_A = 10, \sigma_A = 1$$

$$\mu_S = 1500, \sigma_R = 300$$

Design value for r

$$r_d = u_R^* \cdot \sigma_R + \mu_R = -0.561 \cdot 3.7448 \cdot 35 + 350.0 = 276.56$$

Characteristic value for r

$$r_c = -1.64 \cdot \sigma_R + \mu_R = -1.64 \cdot 35 + 350 = 292.60$$

Partial safety factor

$$\gamma_R = \frac{292.60}{276.56} = 1.06$$