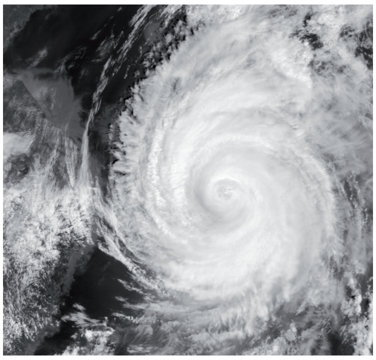
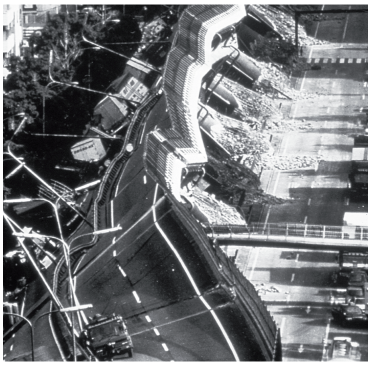
Risk and Safety in Civil Engineering






Exercise Solutions

Prof. Dr. M. H. Faber

2007

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Exercise 1:

Exercise 1.1- Multiple choice questions:

In the following multiple choice questions it should be noted that for some of the questions several answers may be correct. Tick the correct alternatives in every question.

1.1 In probability theory the probability, $P(A)$, of an event A can take any value within the following boundaries:

$0 \leq P(A) \leq 1$ ☑

$-1 \leq P(A) \leq 1$ ☐

$-\infty \leq P(A) \leq \infty$ ☐

1.2 Which one(s) of the following expressions is(are) correct?

The probability of the union of two events A and B is equal to the sum of the probability of event A and the probability of event B, given that the two events are mutually exclusive. ☑

The probability of the union of two events A and B is equal to the probability of the sum of event A and event B, given that the two events are mutually exclusive. ☐

The probability of the intersection of two events A and B is equal to the product of the probability of event A and the probability of event B, given that the two events are mutually exclusive. ☐

The probability of the intersection of two events A and B is equal to the product of the probability of event A and the probability of event B, given that the two events are independent. ☑

1.3 Within the theory of sample spaces and events, which one(s) of the following statements is(are) correct?

An event A is defined as a subset of a sample space Ω. ☑

A sample space Ω is defined as a subset of an event A. ☐

1.4 If the intersection of two events, A and B corresponds to the empty set $\varnothing$, i.e. $A \cap B = \varnothing$, the two events are:

Mutually exclusive. ☑

Independent. ☐

Empty events. ☐

1.5 The probability of the intersection of two mutually exclusive events is equal to:

The product of the probabilities of the individual events. ☐

The sum of the probabilities of the individual events. ☐

The difference between the probabilities of the individual events. ☐

One (1). ☐

Zero (0). ☑

None of the above. ☐

1.6 The probability of the union of two not mutually exclusive events A and B is given as: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. It is provided that the probability of event A is equal to 0.1, the probability of event B is 0.1 and the probability of event B given event A, i.e. $P(B|A)$ is 0.8. Which result is correct?

$P(A \cup B) = -0.6$ ☐

$P(A \cup B) = 0.12$ ☑

$P(A \cup B) = 0.04$ ☐

1.7 For an event A in the sample space Ω, event $\overline{A}$ represents the complementary event of event A. Which one(s) of the following hold?

$A \cup \overline{A} = \Omega$ ☑

$A \cap \overline{A} = \Omega$ ☐

$A \cup \overline{A} = \varnothing$ ☐

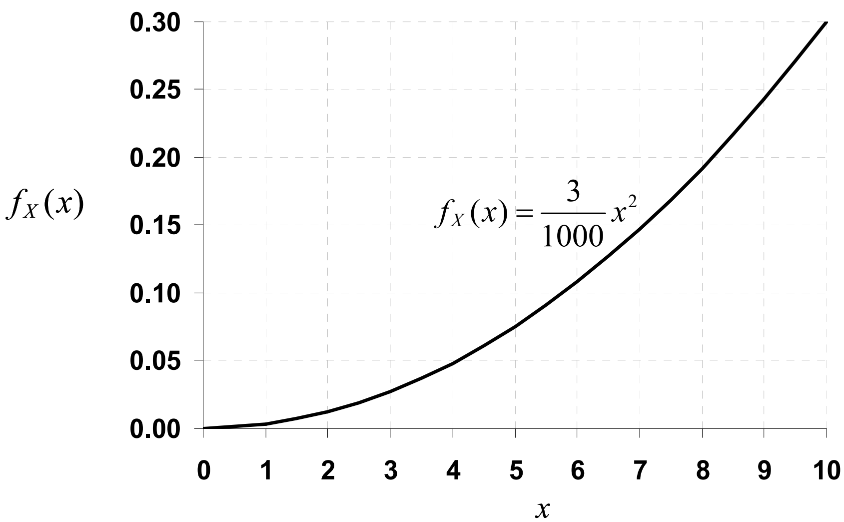
1.8 Probability distribution functions may be defined in terms of their moments. If X is a continuous random variable which one(s) of the following is(are) correct?

The first moment of X corresponds to its mean value, μ_X. ☑

The second moment of X corresponds to its mean value, μ_X. ☐

The second central moment of X corresponds to its variance, σ_X^2. ☑

1.9 The probability density function of a continuous random variable X is illustrated in the following diagram.



The probability of X exceeding the value of 5 is equal to:

$P(X > 5) = 0.875$ ☑

$P(X > 5) = 0.055$ ☐

$P(X > 5) = 0.125$ ☐

1.10 At a given location in Switzerland it has been observed that on average 4 avalanches occur per year. The annual probability of a house being hit by an avalanche on this location is thus:

Equal to one (1). ☐

Larger than one (1). ☐

None of the above. ☑

1.11 The variance of a continuous random variable X can be expressed as: $Var(X) = E\left[(X-\mu_X)^2\right]$, where μ_X is the mean value of X and $E[\cdot]$ is the expectation operator. Based on this expression which one(s) of the following expressions is(are) correct?

- $Var(X) = E\left[X^2\right] - \mu_X^{\ 2}$ ☑
- $Var(X) = \mu_X - E\left[X^2\right]$ ☐
- $Var(X) = X^2 - \mu_X$ ☐

1.12 Imagine that you have thrown a dice and that the dice is still hidden by a cup. What kind(s) of uncertainty is(are) associated with the outcome of the dice?

- Aleatory uncertainty. ☐
- Statistical uncertainty. ☐
- Inherent random variability. ☐
- None of the above. ☑

1.13 The convolution integral in probability describes how the probability density function for the sum of two random variables can be established. However, assumption(s) for its derivation is(are) that:

- The random variables are normally distributed. ☐
- The random variables are independent. ☑
- The random variables are continuous. ☑
- None of the above. ☐

1.14 Which one(s) of the following statements is(are) meaningful:

- The probability of a big earthquake for the region around Zurich is close to 0.02. ☐
- Strong winds occur in Ireland with a probability of 0.7. ☐
- The probability of getting struck by lighting is equal to 0.1, if you stand under a tree. ☐
- None of the above. ☑

1.15 A given random variable is assumed to follow a normal distribution. Which parameter(s) is(are) sufficient to define the probability distribution function of the random variable:

The variance and the standard deviation. ☐

The standard deviation and the mean value. ☑

The mode and the coefficient of variation. ☐

None of the above. ☐

1.16 Which one(s) of the following features is(are) characteristics of a normal distribution function?

The variance is equal to the coefficient of variation. ☐

The mode is equal to the median. ☑

The skewness is equal to zero. ☑

None of the above. ☐

1.17 The median of a data set corresponds to:

The lower quartile of the data set. ☐

The 0.5 quantile of the data set. ☑

The upper quartile of the data set. ☐

1.18 The commutative, associative and distributive laws describe how to:

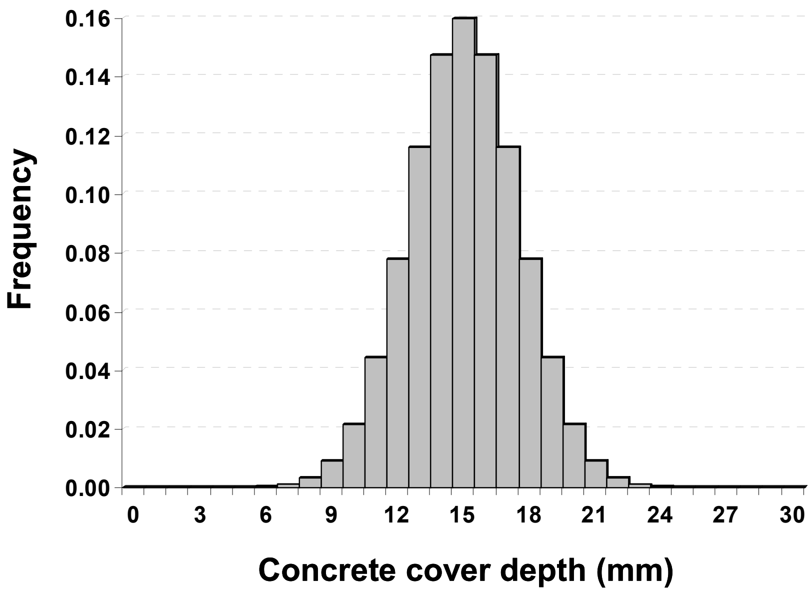
Operate with probabilities. ☑

Operate with intersections of sets. ☑

Operate with unions of sets. ☑

None of the above. ☐

1.19 Measurements were taken of the concrete cover depth of a bridge column. The following symmetrical histogram results from the plot of the measured values:



If X represents the random variable for the concrete cover depth, which one(s) of the following statements is(are) correct?

The sample mean, $\bar{x}$, is equal to 0.16 mm. ☐

The sample mean, $\bar{x}$, is equal to 15 mm. ☑

The mode of the data set is equal to 15 mm. ☑

1.20 After the completion of a concrete structure an engineer tests the null hypothesis that the mean value of the concrete cover depth corresponds to design assumptions. Measurements of the concrete cover depth are taken and after performing the hypothesis test the engineer accepts the null hypothesis. In its early years in service the structure shows signs of deterioration that can be explained only in the case that the design assumptions are not fulfilled. Which of the following statement(s) is(are) correct?

The engineer has performed a Type I error. ☐

The engineer has performed a Type II error. ☑

The engineer has performed a Type I and a Type II error. ☐

None of the above. ☐

1.21 Which one(s) of the following statements is(are) correct for a uniformly distributed random variable?

The expected value of the random variable is equal to 1. ☐

The probability distribution function is constant over the definition space. ☐

The probability density function is constant over the definition space. ☑

None of the above. ☐

1.22 According to the central limit theorem which of the following statement(s) hold?

The probability distribution function of the sum of a number of independent random variables approaches the normal distribution as the number of the variables increases. ☑

The probability distribution function of the product of a number of independent random variables approaches the normal distribution as the number of the variables increases. ☐

None of the above. ☐

1.23 The maximum likelihood method enables engineers to estimate the distribution parameters of a random variable on the basis of data. Which of the following statement(s) is(are) correct for the maximum likelihood method?

It provides point estimates of the distribution parameters. ☑

It provides information about the uncertainty associated with the estimated parameters. ☑

It provides no information about the uncertainty associated with the estimated parameters. ☐

None of the above. ☐

1.24 Consider a number of log-normally distributed and independent random variables. Which of the following statement(s) hold?

The probability distribution function of the sum of the random variables approaches the log-normal distribution as the number of the variables increases. ☐

The probability distribution function of the sum of the random variables approaches the normal distribution as the number of the variables increases. ☑

None of the above. ☐

1.25 It is given that the operational life (until breakdown) T of a diesel engine has an exponential distribution, $F_T(t) = 1 - e^{-\lambda t}$, with parameter λ and mean value, $\mu_T = 1/\lambda$, equal to 10 years. The engine is inspected every 2 years and if a problem is observed it is fully repaired. The probability that the engine breaks down before the first inspection is equal to:

$P(T \leq 2 years) = 0.181$. ☑

$P(T \leq 2 years) = 0.819$. ☐

$P(T \leq 2 years) = 0.0067$. ☐

None of the above. ☐

1.26 From past experience it is known that the shear strength of soil can be described by a log-normal distribution. 15 samples of soil are taken from a site and an engineer wants to use the data in order to estimate the parameters of the log-normal distribution. The engineer:

may use a probability paper to estimate the parameters of the log-normal distribution. ☑

may use the maximum likelihood method to estimate the parameters of the log-normal distribution. ☑

may use the method of moments to estimate the parameters of the log-normal distribution. ☑

None of the above. ☐

1.27 Based on experience it is known that the concrete compressive strength may be modeled by a normally distributed random variable X, with mean value $\mu_X = 30MPa$ and standard deviation $\sigma_X = 5MPa$. The compressive strength of 20 concrete cylinders is measured. An engineer wants to test the null hypothesis H_o that X follows a normal distribution with the above given parameters. He/she carries out a Chi-square test by dividing the sample into $k = 4$ intervals. He/she calculates a Chi-square sample statistic equal to $\varepsilon_m^2 = 0.5$. Which of the following statement(s) is(are) correct?

The engineer cannot reject the null hypothesis H_o at the 5% significance level. ☑

The engineer can reject the null hypothesis H_o at the 5% significance level. ☐

The engineer can accept the null hypothesis H_o at the 10% significance level. ☑

None of the above. ☐

1.28 Consider a simply supported timber beam. The beam will fail if the applied central moment exceeds the bending strength of the beam. The bending strength R of the beam and the annual maximum of the applied central moment L are modeled by uncorrelated normally distributed variables with parameters: $\mu_R = 30kNm$, $\sigma_R = 5kNm$, $\mu_L = 10kNm$, $\sigma_L = 2kNm$. Which of the following statement(s) is(are) correct? (***HINT***: *If M represents the linear safety margin then the probability of failure is given by: $P_F = P(M \leq 0) = \Phi(-\beta)$. β is the so-called reliability index given as: $\beta = \frac{\mu_M}{\sigma_M}$, where μ_M and σ_M are the mean and standard deviation of the safety margin respectively.)*

The reliability index of the timber beam corresponding to a one year reference period is equal to 3.71. ☑

The probability of failure of the timber beam in a year is equal to $1.04 \cdot 10^{-4}$. ☑

The reliability index of the timber beam corresponding to a one year reference period is equal to 4.08. ☐

The probability of failure of the timber beam in a year is equal to $2.25 \cdot 10^{-5}$. ☐

1.29 In a mediterranean city there are on average 5 snowfalls a year. Assume that the occurrence of snowfalls X follows a Poisson process with distribution function $F_X(x) = \frac{(\nu)^x e^{-\nu}}{x!}$ and with mean ν. Which of the following statement(s) is(are) correct?

The probability of exactly 5 snowfalls in the next year is equal to 0.175. ☑

The probability of exactly 5 snowfalls in the next year is equal to 1. ☐

The probability of no snowfall in the next year is equal to 0.774. ☐

The probability of no snowfall in the next year is equal to 0.0067. ☑

Exercise 2-Solution:

Exercise 2.1:

The risk associated with an event can be estimated as:

$$R = PC$$

Where P is the probability of the event and C represents the consequences associated with this event.

So it is:

$$R_1 = P_1C_1 = \frac{10}{100}100 = 10 \text{ CHF}$$

$$R_2 = P_2C_2 = \frac{1}{100}500 = 5 \text{ CHF}$$

$$R_3 = P_3C_3 = \frac{20}{100}100 = 20 \text{ CHF}$$

So it is seen that event 3 is the event associated with a higher risk. Here event 3 also had the higher probability of occurrence. That however does not necessarily imply a high risk in cases where the consequences associated with the event are low (are associated with a low cost).

Exercise 2.2:

Case	Comment	Correlation coefficient
A	If a straight line would fit relatively well so a relatively high positive correlation exists	$\rho_{XY} > 0.8$
B	Medium positive correlation. The data would fit on a straight line but they are not absolutely close to it.	$0.6 < \rho_{XY} < 0.8$
C	Absolutely no correlation between X and Y	$\rho_{XY} = 0$
D	Medium negative correlation since the data would fit on a straight line but they are not absolutely close to it	$-0.8 < \rho_{XY} < -0.6$

Exercise 2.3:

It is given that the probability of corrosion initiation is: $P(CI) = 0.06$. Hence the probability of no corrosion initiation is: $P(\overline{CI}) = 1 - P(CI) = 1 - 0.06 = 0.94$. The probability that the test will indicate corrosion initiation provided that corrosion initiation is present is: $P(I|CI) = 1$ while the probability of false indication is: $P(I|\overline{CI}) = 0.14$. Using the theorem of Bayes the probability of corrosion initiation given a positive indication from the test can be calculated as:

$$P(CI|I) = \frac{P(I|CI)P(CI)}{P(I|CI)P(CI) + P(I|\overline{CI})P(\overline{CI})} = \frac{1 \cdot 0.06}{1 \cdot 0.06 + 0.14 \cdot 0.94} = 0.313$$

Exercise 2.4:

Case 1.

There are 6 possible outcomes when we throw a dice. These are numbers ranging from 1 to 6. Hence the even numbers are 2, 4 and 6, making 3 out of 6 possible outcomes. The probability that the outcome is an even number is: $P(A) = 1/2$. The numbers which can be divided by 3 are 2 i.e. the numbers 3 and 6. So the probability that the outcome is a number dividable by 3 is: $P(B) = 1/3$.

The probability of both events occurring simultaneously is:

$P(A \cap B) = 1/6$ which can also be seen by looking the common numbers in the two sets. There is actually one common number, that is number 6 (it is an even number and dividable by 3).

Case 2.

Thinking in the same way as before the probability that the outcome is a prime number is: $P(B) = 1/2$. That is because the possible outcomes being a prime number are numbers 2, 3 and 5. So the probability that events *A* and *B* occur simultaneously is:

$P(A \cap B) = 1/6$, since there is one outcome that can be both an even number and a prime number, i.e. number 2.

Exercise 2.5:

It is given that:

D_1: the proposal is accepted and the project will be funded.
D_2: the proposal should be revised by the Professor and resubmitted to SNF.
D_3: the proposal is not accepted and hence no funding is provided.

$P(D_1) = 0.45$, $P(D_2) = 0.35$, $P(D_3) = 0.2$.

a. The table is completed as follows:

SNF final decision D_i	**Dr. Beispiel's indicative assessment,** I_j		
	$I_j = D_1$	$I_j = D_2$	$I_j = D_3$
D_1	**0.86**	**0.1**	***0.04***
D_2	**0.2**	***0.74***	**0.06**
D_3	***0***	**0.1**	**0.9**

Observe that the horizontal sum is equal to 1:
$P(D_1|I_1) + P(D_1|I_2) + P(D_1|I_3) = 1$.

b. Using the Bayes' Theorem the probability that the final decision made by SNF is the same with the indicative assessment of Dr. Beispiel is:

$$P\left(D_2 \middle| I = D_2\right) = \frac{P(I = D_2 | D_2)P(D_2)}{\sum_{i=1}^{3} P(I = D_2 | D_i)P(D_i)} = \frac{P(I = D_2 | D_2)P(D_2)}{P(I = D_2 | D_1)P(D_1) + P(I = D_2 | D_2)P(D_2) + P(I = D_2 | D_3)P(D_3)}$$

$$\frac{0.74 \cdot 0.35}{(0.1 \cdot 0.45) + (0.74 \cdot 0.35) + (0.1 \cdot 0.2)} = 0.799$$

Exercise 2.6:

a. In order to plot the Tukey box plot five main features are required:

- the lower quartile
- the lower adjacent value
- the median
- the upper adjacent value
- the upper quartile

Consider the Zurich data. A value v is required such that

$$v = nQ + Q$$

Therefore for the lower quartile (i.e. the 0.25 quantile) it is:

$$v = 20 * 0.25 + 0.25 = 5.25$$

v has a non integer value. The value is split to its integer part $k = 5$ and the fractional part $p = 0.25$. Then the 0.25 quantile is given by:

$$Q_{0.25} = (1 - p)x_k + px_{k+1} = (1 - 0.25) * 9.11 + 0.25 * 9.24 = 9.1425$$

In the same way for the upper quartile it is:

$$v = 20 * 0.75 + 0.75 = 15.75$$

v has a non integer value. The value is splitted to its integer part $k = 15$ and the fractional part $p = 0.75$. Then the 0.25 quantile is given by:

$$Q_{0.75} = (1 - p)x_k + px_{k+1} = (1 - 0.5) * 9.63 + 0.5 * 9.7 = 9.6825$$

The median is given as following:

$$v = 20 * 0.5 + 0.5 = 10.5$$

In this case we deal with a non integer value. The value is splitted to its integer part $k = 10$ and the fractional part $p = 0.5$. Then the 0.5 quantile (i.e. the median) is given by:

$$median = (1-p)x_k + px_{k+1} = (1-0.5)*9.45 + 0.5*9.48 = 9.465$$

To evaluate the adjacent values the interquartile range is required:

$$r = Q_{0.75} - Q_{0.25} = 9.6825 - 9.1425 = 0.54$$

The lower adjacent value is the smallest observation that is greater than or equal to the lower quartile minus *1.5r.* It is:

$$Q_{0.25} - 1.5r = 9.1425 - 1.5*0.54 = 8.3325$$

Thus the lower adjacent value is 8.39.
In the same way the upper adjacent value is found as:

$$Q_{0.75} + 1.5r = 9.6825 + 1.5*0.54 = 10.4925$$

Therefore the upper adjacent value is a value less than or equal to 10.4925, that is 10.47 which actually coincides with the higher value of the data set.

The values necessary to plot the Tukey box plots are given in the following table.

Zurich Data:

Median			
9.465			
Lower quartile			
v	**k**	**p**	**0.25 quantile**
5.25	5	0.25	9.1425
Upper quartile			
v	**k**	**p**	**0.75 quantile**
15.75	15	0.75	9.6825
Interquartile			
r	**1.5*r**	**0.25 quantile -1.5r**	**0.75 quantile +1.5r**
0.54	0.81	8.3325	10.4925
Adjacent values			
Lower	**Upper**		
8.39	10.47		
Statistic		**Outside values**	
q0.25	9.1425	8.23	
min	8.39	8.33	
median	9.465		
max	10.47		
q0.75	9.6825		

Global Data:

Median			
14.22			
Lower quartile			
v	**k**	**p**	**0.25 quantile**
5.5	5	0.25	14.1125
Upper quartile			
v	**k**	**p**	**0.75 quantile**
15.5	15	0.75	14.38
Interquartile			
r	**1.5*r**	**0.25 quantile - 1.5r**	**0.75 quantile +1.5r**
0.2675	0.40125	13.71125	14.78125
Adjacent values			
Lower	**Upper**		
13.99	14.47		
Statistic		**Outside values**	
q0.25	14.1125		
min	13.99		
median	14.22		
max	14.47		
q0.75	14.38		

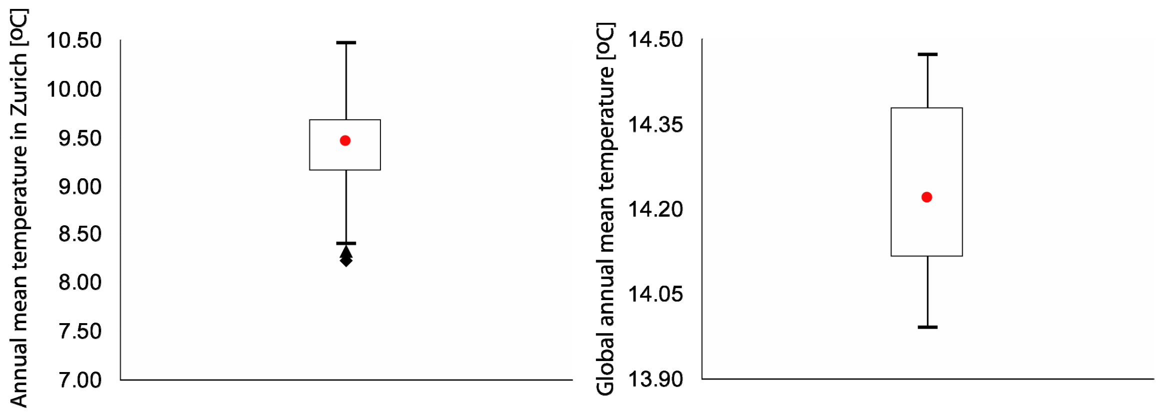


Figure 2.1: Tukey box plots.

b. The histogram and cumulative distribution function for both data sets are shown in the following.

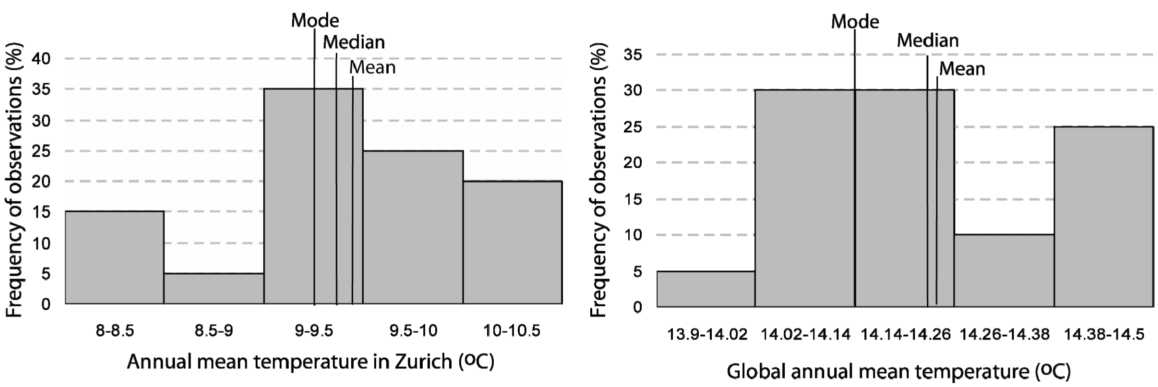


Figure 2.2: Histograms for both data sets.

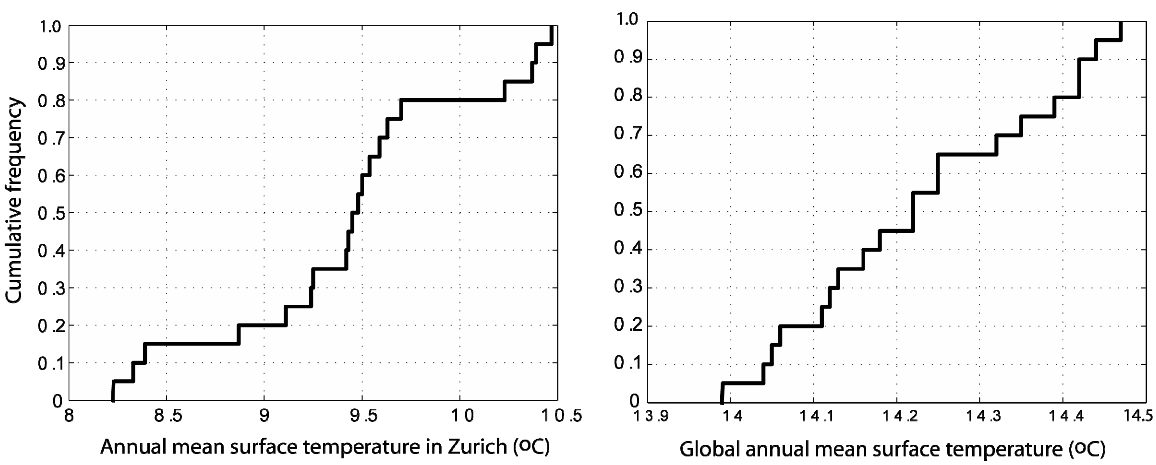


Figure 2.3: Cumulative frequency distributions for both data sets.

The detailed calculations on which the above figures are based on may be found in the excel file with name “Exercise 2” provided in:

http://www.ibk.ethz.ch/fa/education/ws_safety
You may find that there are actually two ways of presenting the cumulative frequency distribution. Both ways are provided in the above mentioned excel file. Figure 2.3 has been produced using the second way in Matlab.

Exercise 2.7:

a. The first sample moment of the considered variable is calculated as:

$$m_1 = \frac{1}{n}\sum x_i = \frac{1}{9}923 = 102.56 \text{ mm/hour}$$

The second sample moment is calculated as:

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{1}{9}97237 = 10804 \text{ mm}^2/\text{hour}^2$$

Using the expressions for the first two moments, provided in Table 2.3 of Exercise 2 it is:

$$m_1 = u + \frac{0.5772}{\alpha} \Rightarrow 102.56 = u + \frac{0.5772}{\alpha} \Rightarrow u = 102.56 - \frac{0.5772}{\alpha} \qquad (1)$$

And

$$m_2 = \left(u + \frac{0.5772}{\alpha}\right)^2 + \frac{\pi^2}{6\alpha^2} \Rightarrow 10804 = \left(102.56 - \frac{0.5772}{\alpha} + \frac{0.5772}{\alpha}\right)^2 + \frac{\pi^2}{6\alpha^2} \Rightarrow \alpha = 0.076$$

And from Equation (1) it is:

$$u = 102.56 - \frac{0.5772}{\alpha} = 102.56 - \frac{0.5772}{0.076} = 94.94$$

b. In the following table the necessary values for plotting the data on the probability paper are provided. An example of calculation is given in the following:

$F_{X,o}(x_1) = \frac{1}{9+1} = 0.1$ and $-\ln(-\ln(F_{X,o}(x_1))) = -\ln(-\ln(0.1)) = -0.834$

i	Annual maximum precipitation per hour x_i (mm/hour)	$F_{X,o}(x_i) = \frac{i}{n+1}$	$-\ln(-\ln(F_{X,o}(x_i)))$
1	73	0.1	-0.834
2	85	0.2	-0.476
3	93	0.3	-0.186
4	98	0.4	0.087
5	100	0.5	0.367
6	108	0.6	0.672
7	115	0.7	1.031
8	121	0.8	1.500
9	130	0.9	2.250

The values of the above table are now plotted:

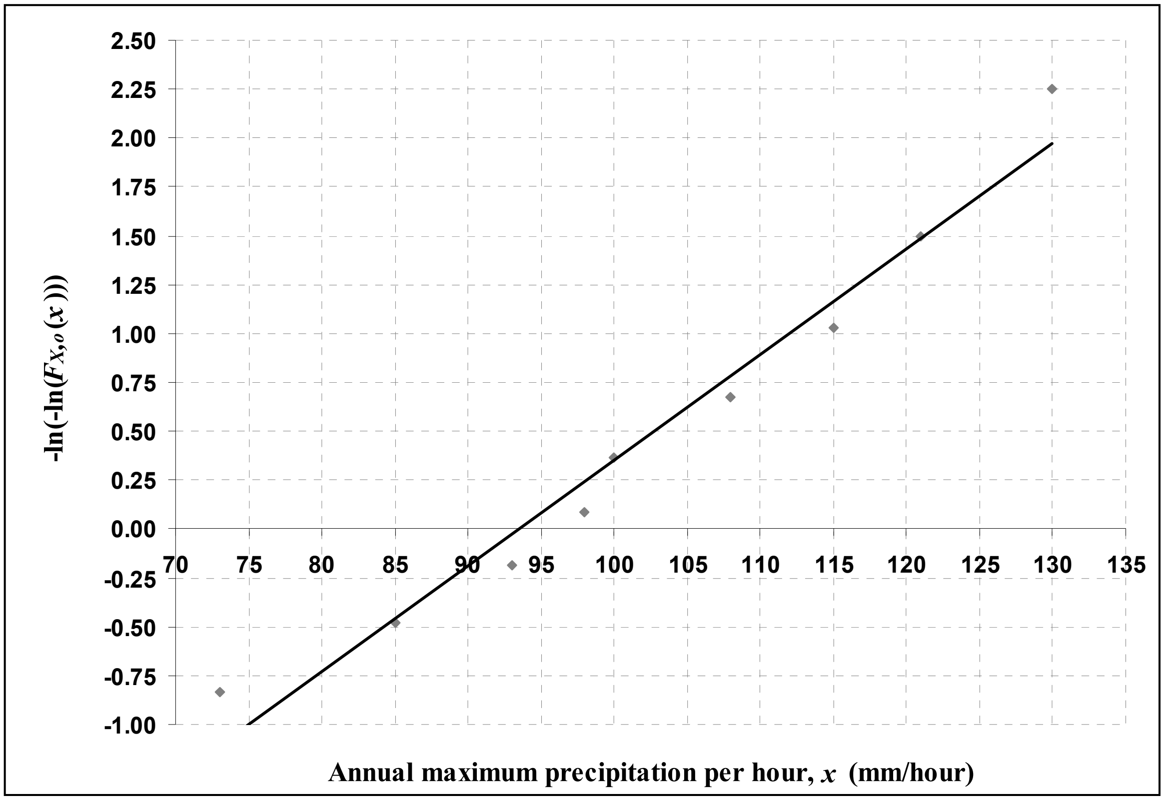


Figure 2.4: Probability paper for testing the suitability of the Gumbel distribution.

c. Yes, the belief of the engineer is correct. Figure 2.4 shows the plotted data on the probability paper for the Gumbel distribution. A "best-fit" line is plotted. It is seen that the data fit quiet well to the straight "best-fit" line. Hence the belief of the engineer is correct.

d. From the probability distribution function of the Gumbel distribution, $F_X(x) = \exp\left(-\exp\left(-\alpha\left(x-u\right)\right)\right)$ it is seen that after a small transformation we get:

$-\ln(-\ln(F_X(x))) = \alpha x - \alpha u$ which is a linear relationship between:

$y = -\ln(-\ln(F_X(x)))$ and x with slope α and y-intercept αu.

Using the "best-fit" line two equations may be formed with two unknowns from where we can calculate the parameters of the distribution. It is:
Hence the two linear equations can be written using the general form: $y = \kappa x - \lambda$, where $\kappa = \alpha$ and $\lambda = \alpha u$.
For $x = 121$ it is $y = 1.5$, and for $x = 93$ it is $y = 0$.
It is:
$y = \kappa x - \lambda \Rightarrow 1.5 = \kappa 121 - \lambda$
and
$y = \kappa x - \lambda \Rightarrow 0 = \kappa 93 - \lambda$

Subtracting the above two equations we get:
$1.5 - 0 = \kappa 121 - \lambda - (\kappa 93 - \lambda) \Rightarrow 1.5 = 28\kappa \Rightarrow \kappa = 0.0536$

And then from one of the two equations it is: $y = \kappa x - \lambda \Rightarrow 0 = 0.0536 \cdot 93 - \lambda \Rightarrow \lambda = 4.9848$

Hence the parameters of the Gumbel distribution are:

$\alpha = \kappa = 0.0536$ and $\alpha u = \lambda \Rightarrow 0.0536u = 4.9848 \Rightarrow u = \dfrac{4.9848}{0.0536} = 93$.

It is observed that the values of the parameters calculated using the probability paper differ from the ones calculated with the method of moments. This can be explained from the fact that with the probability paper we use a “best-fit” line for the calculation of the parameters.

Note also that if you would use other combinations for the x *and* y *values to form the set of the two linear relationships with the two unknowns the result may be slightly different.*

Exercise 3- Solution:

Exercise 3.1:

Using the information provided it is:

$$P(SW) = 0.6$$
$$P(SD) = 0.4$$
$$P(I_{SD} \mid SD) = 0.75$$
$$P(I_{SW} \mid SW) = 0.75$$
$$P(I_{SD} \mid SW) = 1 - P(I_{SD} \mid SD) = 1 - 0.75 = 0.25$$
$$P(I_{SW} \mid SD) = 1 - P(I_{SW} \mid SW) = 1 - 0.75 = 0.25$$

Using the Bayes' Theorem it is:

$$P(SD \mid I_{SD}) = 0.6667 = \frac{P(I_{SD} \mid SD) \cdot P(SD)}{P(I_{SD} \mid SD) \cdot P(SD) + P(I_{SD} \mid SW) \cdot P(SW)} = \frac{0.75 \cdot 0.4}{0.75 \cdot 0.4 + 0.25 \cdot 0.6} = \frac{2}{3} = 0.6667$$

$$P(SD \mid I_{sw}) = \frac{P(I_{SW} \mid SD) \cdot P(SD)}{P(I_{SW} \mid SD) \cdot P(SD) + P(I_{SW} \mid SW) \cdot P(SW)} = \frac{0.25 \cdot 0.4}{0.25 \cdot 0.4 + 0.75 \cdot 0.6} = \frac{2}{11} = 0.18182$$

$$P(SW \mid I_{SD}) = \frac{P(I_{SD} \mid SW) \cdot P(SW)}{P(I_{SD} \mid SW) \cdot P(SW) + P(I_{SD} \mid SD) \cdot P(SD)} = \frac{0.25 \cdot 0.6}{0.25 \cdot 0.6 + 0.75 \cdot 0.4} = \frac{1}{3} = 0.3333$$

$$P(SW \mid I_{SW}) = \frac{P(I_{SW} \mid SW) \cdot P(SW)}{P(I_{SW} \mid SW) \cdot P(SW) + P(I_{SW} \mid SD) \cdot P(SD)} = \frac{0.75 \cdot 0.6}{0.75 \cdot 0.6 + 0.25 \cdot 0.4} = \frac{9}{11} = 0.8181$$

And:

$$P(I_{SW}) = P(I_{SW} \mid SW) \cdot P\left(SW\right) + P(I_{SW} \mid SW) \cdot P\left(SW\right) = 0.75 \cdot 0.6 + 0.25 \cdot 0.4 = 0.55$$

$$P(I_{SD}) = P(I_{SD} \mid SW) \cdot P\left(SW\right) + P(I_{SD} \mid SD) \cdot P\left(SD\right) = 0.25 \cdot 0.6 + 0.75 \cdot 0.4 = 0.45$$

The event tree can now be filled in. An example of calculation is provided in the following.

Consider the branch associated with the activity "clean up the roof". If the roof is cleaned up there are two events that may occur according to our problem:

a. the roof may collapse (due to various reasons)
b. the roof will not collapse (survival of the roof)

These events are associated with some probability as shown in the event tree branches:

a. $P_f(SN) = 5 \cdot 10^{-4}$ and b. $P_s(SN) = 1 - 5 \cdot 10^{-4} = 0.9995$. Hence the expected cost of this action is:

$$E[C_{clean\ up}] = P_f(SN) \cdot 1000004 + P_s(SN) \cdot 4000 = 5 \cdot 10^{-4} \cdot 1004000 + 0.9995 \cdot 4000$$
$$= 4500\ CHF$$

In a similar way the rest of the event tree may be completed.

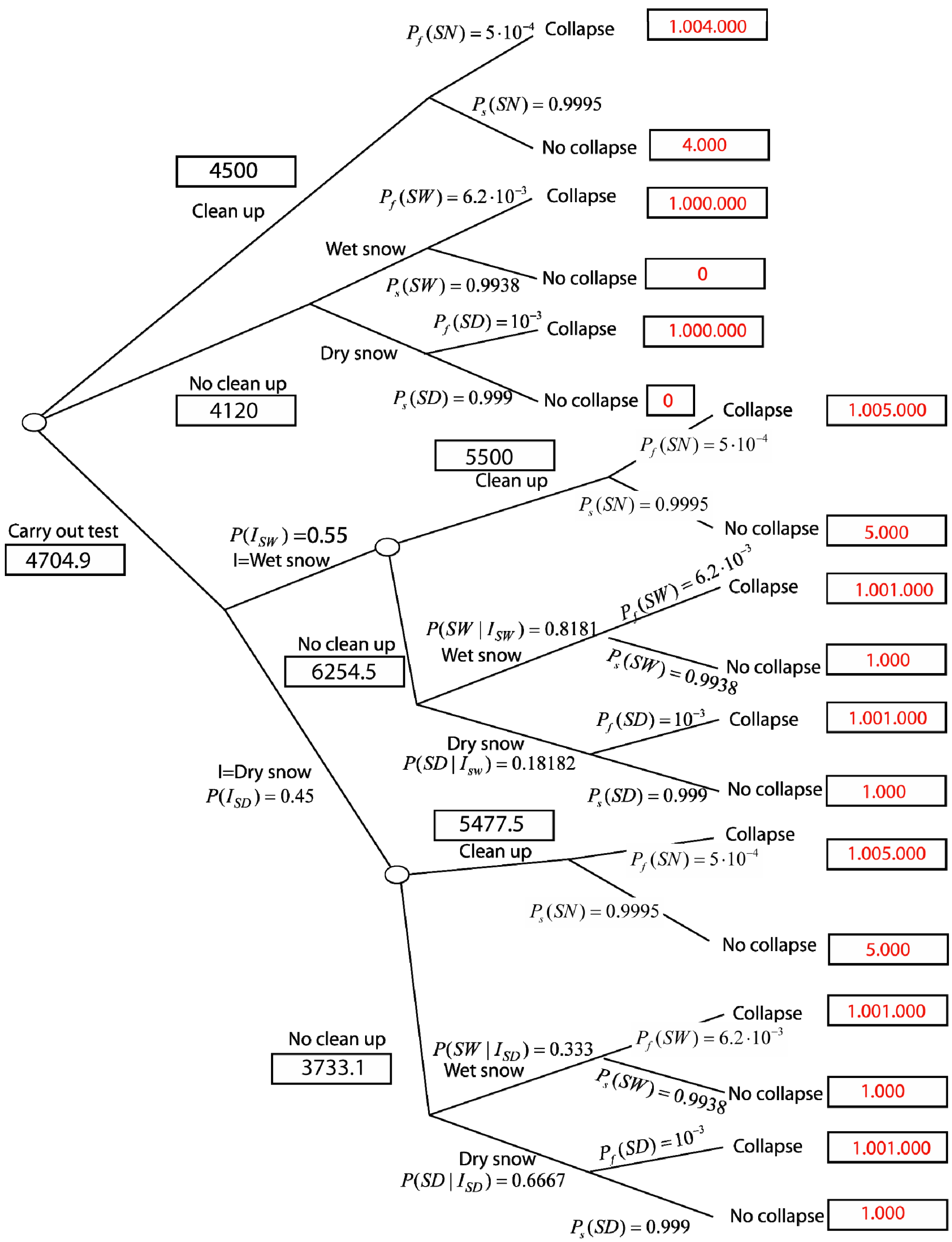


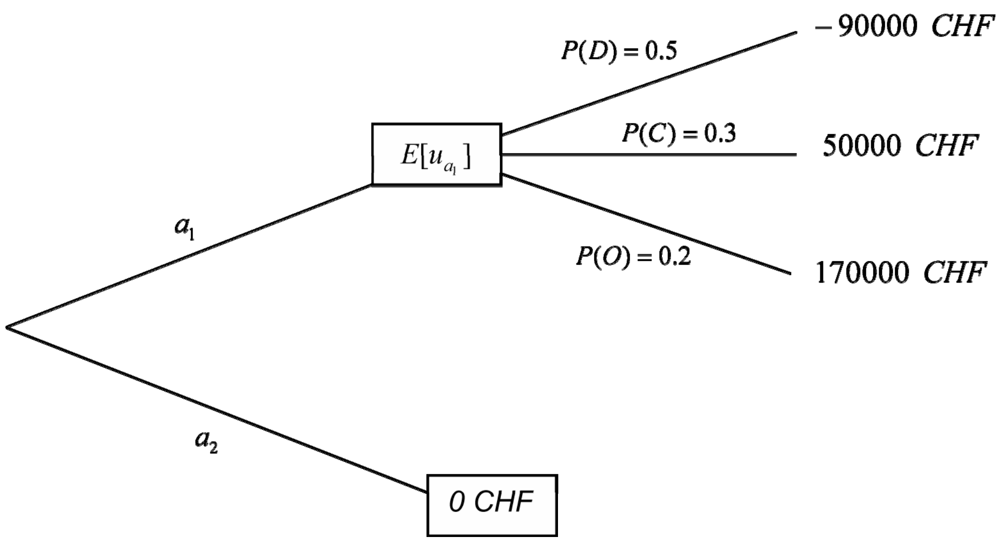
Figure 3.1. Event tree.

It can be seen that the action associated with the smaller cost is not to clean up the roof.

Exercise 3.2:

Note: **Please correct the question 3.2.a such as to read:**
"Carry out a ***prior*** decision analysis (using an appropriate event tree) and determine whether it is beneficial to open up the borehole."
I am sorry if this has caused any inconvenience. The pre-posterior analysis is requested in 3.2.b.

a. Based on the information provided the following event tree is constructed for carrying out the prior analysis:



The benefit associated with the opening of the borehole, a-priori, is estimated as follows:

$$\begin{aligned}E'[u_{a_1}] &= P'[D]\cdot(-90000)+P'[D]\cdot(15000)+P'[O]\cdot(170000)\\ &= 0.5\cdot(-90000)+0.3\cdot(15000)+0.2\cdot(170000)\\ &= 4000\ CHF\end{aligned}$$

Hence the action that gives the larger utility (larger expected benefit in terms of cost) is action a_1,

$$E'[u]=\max\left\{E'\left[u_{a_1}\right];\ E'\left[u_{a_2}\right]\right\}=\max\left\{4000;\ 0\right\}=4000\ CHF$$

and hence a-priori the engineer would decide to open up the borehole.

b. The event tree is now extended to include the cases of performing a test, a_{11}, or not performing a test, a_{12}. The following probabilities can readily be estimated:

In case that the test is carried out the probability of receiving the indication that the well is dry is:

$$\begin{aligned}P'(I_D) &= P(I_D\,|\,D)\cdot P'(D)+P(I_C\,|\,C)\cdot P'(C)+P(I_O\,|\,O)\cdot P'(O)\\ &= 0.6\cdot 0.5+0.3\cdot 0.3+0.1\cdot 0.2=0.41\end{aligned}$$

The probabilities of the states of the well are updated given the above indication:

$$P''(D \mid I_D) = \frac{P(I_D \mid D) \cdot P'(D)}{P'(I_D)} = \frac{0.6 \cdot 0.5}{0.41} = \frac{0.3}{0.41} = 0.732$$

$$P''(C \mid I_D) = \frac{P(I_D \mid C) \cdot P'(C)}{P'(I_D)} = \frac{0.3 \cdot 0.3}{0.41} = \frac{0.09}{0.41} = 0.220$$

$$P''(O \mid I_D) = \frac{P(I_D \mid O) \cdot P'(O)}{P'(I_D)} = \frac{0.1 \cdot 0.2}{0.41} = \frac{0.02}{0.41} = 0.048$$

Similarly for the other two possible outcomes of the test it is:

$$P'(I_C) = P(I_C \mid D) \cdot P'(D) + P(I_C \mid C) \cdot P'(C) + P(I_C \mid O) \cdot P'(O)$$
$$= 0.1 \cdot 0.5 + 0.3 \cdot 0.3 + 0.5 \cdot 0.2 = 0.24$$

$$P''(D \mid I_C) = \frac{P(I_C \mid D) \cdot P'(D)}{P'(I_C)} = \frac{0.1 \cdot 0.5}{0.24} = \frac{0.05}{0.24} = 0.208$$

$$P''(C \mid I_C) = \frac{P(I_C \mid C) \cdot P'(C)}{P'(I_C)} = \frac{0.3 \cdot 0.3}{0.24} = \frac{0.09}{0.24} = 0.375$$

$$P''(O \mid I_C) = \frac{P(I_C \mid O) \cdot P'(O)}{P'(I_C)} = \frac{0.5 \cdot 0.2}{0.24} = \frac{0.1}{0.24} = 0.417$$

$$P'(I_O) = P(I_O \mid D) \cdot P'(D) + P(I_O \mid C) \cdot P'(C) + P(I_O \mid O) \cdot P'(O)$$
$$= 0.3 \cdot 0.5 + 0.4 \cdot 0.3 + 0.4 \cdot 0.2 = 0.35$$

$$P''(D \mid I_O) = \frac{P(I_O \mid D) \cdot P'(D)}{P'(I_O)} = \frac{0.3 \cdot 0.5}{0.35} = \frac{0.15}{0.35} = 0.429$$

$$P''(C \mid I_O) = \frac{P(I_O \mid C) \cdot P'(C)}{P'(I_O)} = \frac{0.4 \cdot 0.3}{0.35} = \frac{0.12}{0.35} = 0.343$$

$$P''(O \mid I_O) = \frac{P(I_O \mid O) \cdot P'(O)}{P'(I_O)} = \frac{0.4 \cdot 0.2}{0.35} = \frac{0.08}{0.35} = 0.228$$

The expected utility can be written:

$$E[u] = \sum_{i=1}^{n} P'[I_i] E''[u \mid I_i] = \sum_{i=1}^{n} P'[I_i] \max_{j=1,\ldots,m} \left\{ E''[u(a_j) \mid I_i] \right\}$$

Where n is the number of different possible experiment findings and m is the number of different decision alternatives. So it is:

$$E''[u \mid I_D] = \max\left\{ P''[D \mid I_D](-90000) + P''[C \mid I_D](50000) + P''[O \mid I_D](170000); 0 \right\} =$$
$$= \max\left\{ 0.732(\text{-}90000) + 0.220(50000) + 0.048(170000); 0 \right\} = \max\left\{ -46720; 0 \right\} =$$
$$= 0 \; CHF$$

Similarly:

$$E''[u \mid I_C] = \max\left\{ P''[D \mid I_C](-90000) + P''[C \mid I_C](50000) + P''[O \mid I_C](170000); 0 \right\} =$$
$$= \max\left\{ 0.208(\text{-}90000) + 0.375(50000) + 0.417(170000); 0 \right\} = \max\left\{ -46720; 0 \right\} =$$
$$= 70920 \; CHF$$

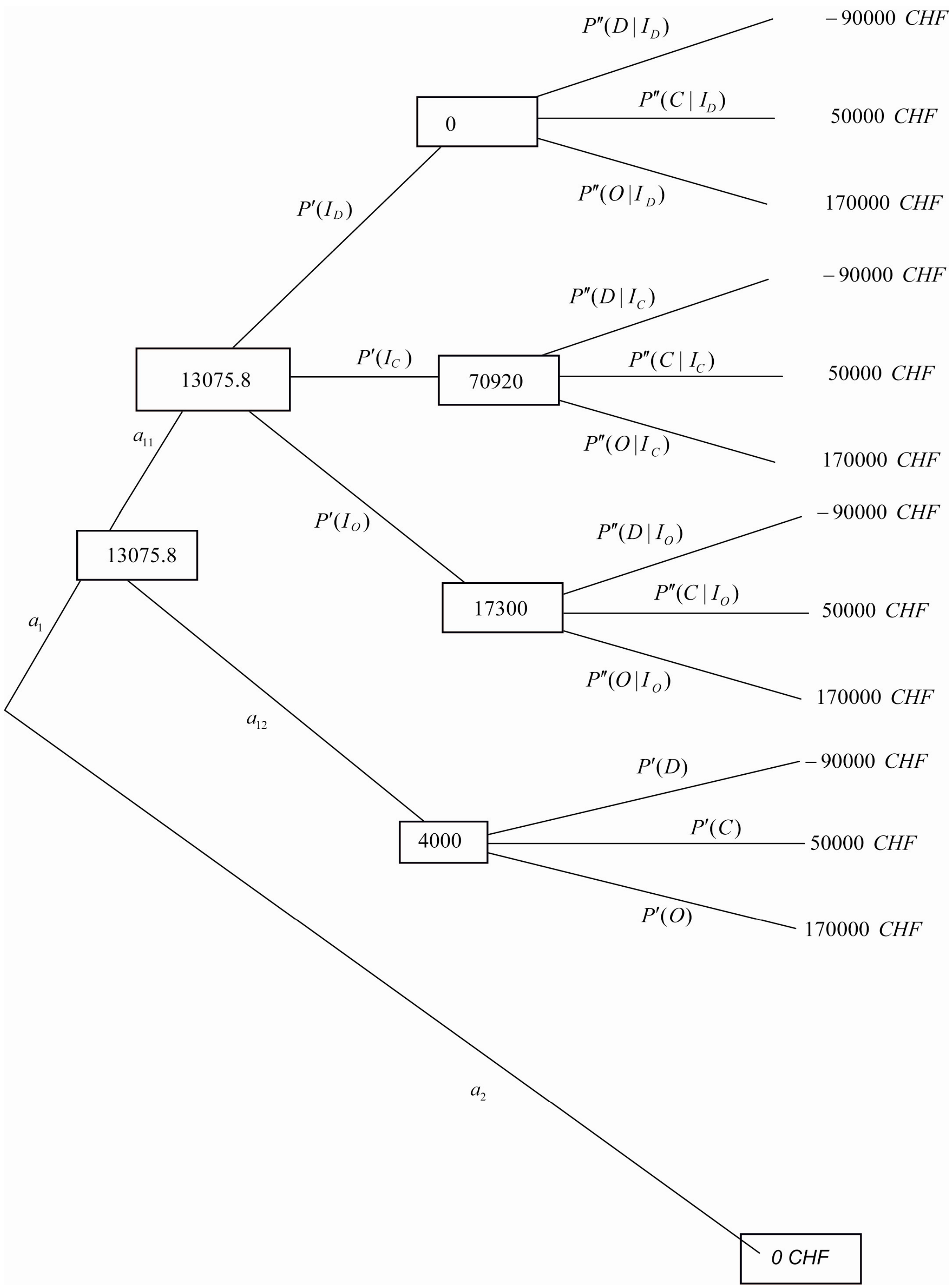
$$E''[u|I_C] = \max\left\{P''[D|I_O](-90000) + P''[C|I_O](50000) + P''[O|I_O](170000); 0\right\} =$$
$$= \max\left\{0.429(-90000)+0.343(50000)+0.228(170000); 0\right\} = \max\left\{17300; 0\right\} =$$
$$=17300\ CHF$$

And the expected utility considering the costs of the test is:

$$E\left[u\right] = \left\{E''\left[u \mid I_D\right] \cdot P'(I_D) + E''\left[u \mid I_C\right] \cdot P'(I_C) + E''\left[u \mid I_O\right] \cdot P'(I_O)\right\} - 10000 =$$
$$= \left\{(0) \cdot 0.41 + (70920) \cdot 0.24 + (17300) \cdot 0.35\right\} - 10000 =$$
$$= 23076 - 10000 = 13076\ CHF$$

Hence if this is compared to the case of not carrying out the test it can be seen that the utility is higher in the case that the test is carried out.

The benefit associated from opening up the borehole is then equal to 13076 CHF.


$P''(D|I_D)$
$-90000\ CHF$
$P''(C|I_D)$
0
$50000\ CHF$
$P''(O|I_D)$
$170000\ CHF$
$P'(I_D)$
$-90000\ CHF$
$P''(D|I_C)$
$P'(I_C)$
$P''(C|I_C)$
13075.8
70920
$50000\ CHF$
a_{11}
$P''(O|I_C)$
$170000\ CHF$
$P'(I_O)$
$-90000\ CHF$
$P''(D|I_O)$
13075.8
$P''(C|I_O)$
17300
$50000\ CHF$
a_1
$P''(O|I_O)$
$170000\ CHF$
a_{12}
$P'(D)$
$-90000\ CHF$
$P'(C)$
4000
$50000\ CHF$
$P'(O)$
$170000\ CHF$
a_2
0 CHF

Exercise 4- Solution:

Exercise 4.1:

For convenience we name the events that may cause a delay as shown in the last column of the following table:

	Works	Representation	Probability of delay
Phase 1	Form working	*A*	0.2
	Bending of Reinforcement	*B*	0.1
Phase 2	Installation of reinforcement	*C*	0.05
Phase 3	Concrete delivering and concreting	*D*	0.15

In order to construct the fault tree, the top event and the basic events are identified. In this case the top event is the delay of the construction project and the various works that may lead to a delay are the basic events. Further more the different phases involved in construction should be taken into account. The fault tree is shown in the following figure:

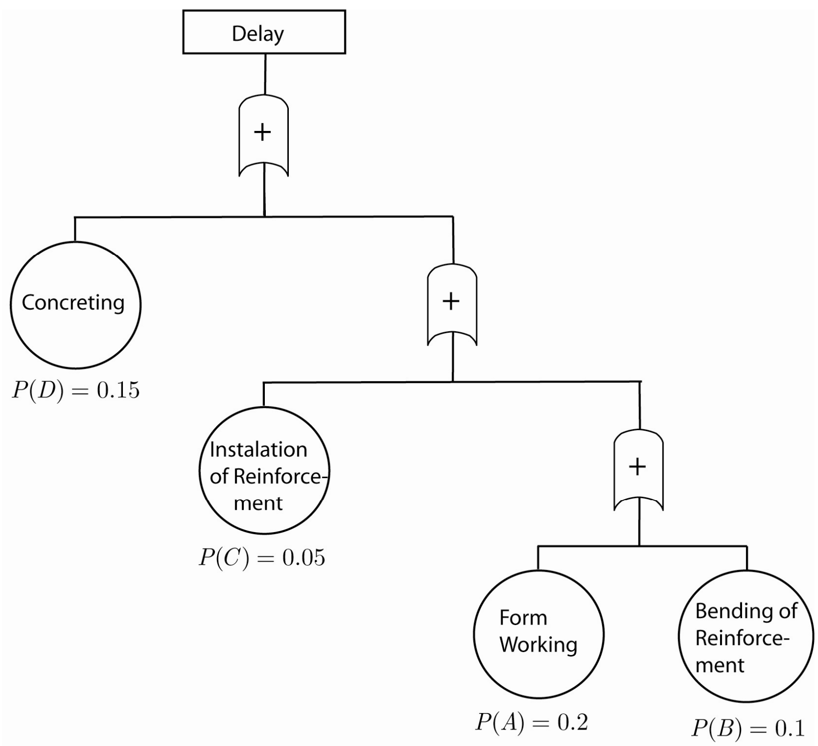


Figure 4.1: Fault tree for the delay of a construction project.

The basic events are connected with OR gates and that is due to the fact that a delay of the construction may result due to either *A, B, C*, **or** *D* occurring.

Now the probability of delay of the construction project can be estimated:
The probability of delay in the first phase is:

$$P_{delay\_Phase\_1} = 1-(1-P(A))\cdot(1-P(B)) = 1-(1-0.2)\cdot(1-0.1) = 0.28$$

The probability of delay in the second phase is due to event *C* or a delay during the first phase. So it is:

$$P_{delay\_Phase\_2} = 1-(1-P_{delay\_Phase\_1})\cdot(1-P(C)) = 1-(1-0.28)\cdot(1-0.05) = 0.316$$

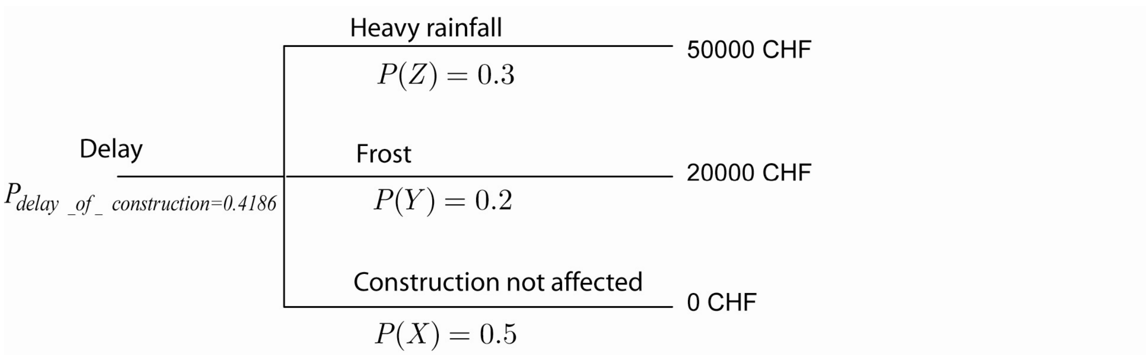
And eventually the probability of delay of the construction project, taken into account all phases of the construction, is:

$$P_{delay\_of\_construction} = 1-(1-P_{delay\_Phase\_2})\cdot(1-P(D)) = 1-(1-0.316)\cdot(1-0.15) = 0.4186$$

b. The events associated with the weather effect (if any) are referred to as in the following:

- the construction work will not be affected: *X*
- there may be problems in concreting due to frost: *Y*
- the construction work may stop due to heavy rainfall: *Z*

An event tree is constructed to compute the costs associated with a delay of the construction project:



The cost associated with a delay in the project is:

$$E[C_{delay}] = E[\sum p\cdot E[C_i]] = 0.4186\cdot100000 + 0.4186\cdot0.3\cdot50000 + 0.4186\cdot0.2\cdot20000 + 0 =$$
$$= 49813.4CHF$$

c. The fault tree of figure 4.1 remains the same but now it is:

$P(A) = 0.05$ and $P_{delay\_Phase\_1} = 1-(1-P(A))\cdot(1-P(B)) = 1-(1-0.05)\cdot(1-0.1) = 0.145$ and:
$P_{delay\_Phase\_2} = 1-(1-P_{delay\_Phase\_1})\cdot(1-P(C)) = 1-(1-0.145)\cdot(1-0.05) = 0.188$ and:
$P_{delay\_of\_construction} = 1-(1-P_{delay\_Phase\_2})\cdot(1-P(D)) = 1-(1-0.188)\cdot(1-0.15) = 0.3098$

And finally using a tree similar to the one in the previous question (4.1.b) it is:

$$E[C_{delay}] = E[\sum p\cdot E[C_i]] + E[C_{Team}] = (0.3098\cdot100000 + 0.3098\cdot0.3\cdot50000 + 0.3098\cdot0.2\cdot20000 + 0) + 12000 =$$
$$= 48866.2CHF$$

As it can be seen the expected cost of the delay reduces if the extra team for the form working is used.

Exercise 5-Solution:

Exercise 5.1:

Note:Please correct the units of the mean value and standard deviation of the yield stress in this exercise:
$\mu_{f_y} = 425 \cdot 10^{-3} KN/mm^2$ and $\sigma_{f_y} = 25 \cdot 10^{-3} KN/mm^2$

The Safety margin is simply:
$M = R - S = A \cdot f_y - S = 100 \cdot f_y - 35$

Since the yield stress f_y is normal distributed, M is also normal distributed and we may estimate its mean and standard deviation as follows:
$\mu_M = E[M] = E[100 \cdot f_y - 35] = 100 \cdot \mu_{f_y} - 35 = 100 \cdot 425 \cdot 10^{-3} - 35 = 7.5KN$
And the variance is:
$\sigma_M^2 = VAR[M] = VAR[100 \cdot f_y - 35] = VAR[100 \cdot f_y] - VAR[35] =$
$= 100^2 \cdot \sigma_{f_y}^2 - 0 = 100^2 \cdot (25 \cdot 10^{-3})^2 = 6.25KN^2$
And the standard deviation is then:
$\sigma_M = \sqrt{\sigma_M^2} = \sqrt{6.25} = 2.5KN$

(Note: The mean and standard deviation are estimated using the properties of operators provided in the Lecture Notes-Lecture 2)

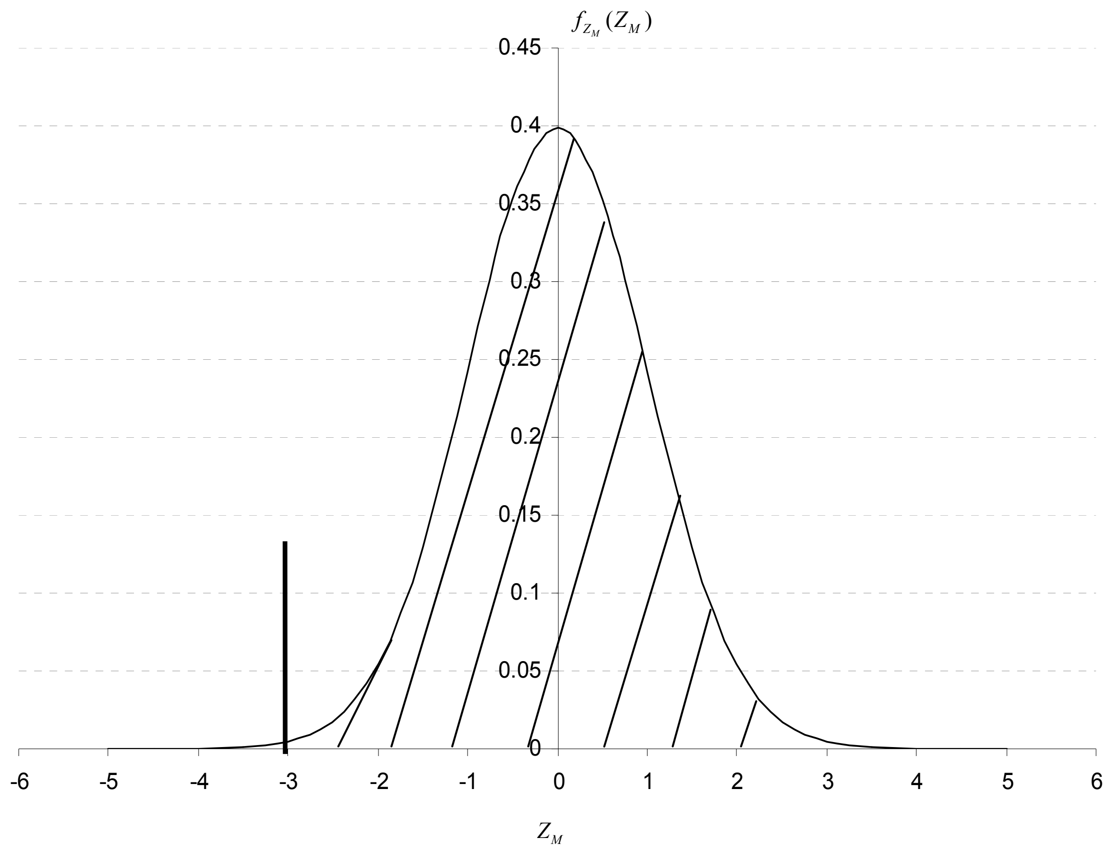
The probability of failure of the rod is then (following Equations 5.24 and 5.25 of the lecture notes:

$$P_f = P(M \le 0) = P(Z_M \le \frac{0 - \mu_M}{\sigma_M}) = \Phi(\frac{0 - \mu_M}{\sigma_M}) = \Phi(\frac{0 - 7.5}{2.5}) =$$
$$= \Phi(-3) = 0.00135$$

Whereas the reliability of the rod is simply:
$\text{Reliability} = 1 - P_f = 1 - 0.00135 = 0.99865$

(**<u>Note:</u>** You can estimate the standard normal distribution value corresponding to -3 either using standard normal distribution tables (available in any statistics book) or using Excel's direct function: **normsdist(z)**)

It is easier to draw the probability density function of the standardised safety margin i.e. of Z_M. The area under the density function to the right of -3 in the x-axis represents the safe region.


$f_{Z_M}(Z_M)$
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
Z_M

Exercise 6-Solution:

Exercise 6.1:

Using the provided Figure and basic principles of geometry it is:

$$\frac{b}{\sin(\beta)} = \frac{c}{\sin(\pi-\alpha-\beta)} \quad \Rightarrow \quad b = f(c,\,\alpha,\,\beta) = c\cdot\frac{\sin(\beta)}{\sin(\alpha+\beta)}$$

Using the properties of the expectators it is:

$$E[b] = E\left[c\cdot\frac{\sin(\beta)}{\sin(\alpha+\beta)}\right] = E[c]\cdot\frac{\sin(\beta)}{\sin(\alpha+\beta)} = 6\cdot\frac{\sin(1.225)}{\sin(1.225+0.813)} = 6.32\ km$$

While the estimation of the error associated with the measurement of side b is represented by the standard deviation $\sigma[b]$ and is estimated as in the following :

$$V[b] = \left[\frac{\partial f}{\partial c}\right]^2\cdot\sigma_c^{\,2} + \left[\frac{\partial f}{\partial \alpha}\right]^2\cdot\sigma_\alpha^{\,2} + \left[\frac{\partial f}{\partial \beta}\right]^2\cdot\sigma_\beta^{\,2}$$

$$\frac{\partial f}{\partial c} = \frac{\sin(\beta)}{\sin(\alpha+\beta)}$$

$$\frac{\partial f}{\partial \alpha} = \left(c\cdot\frac{\sin(\beta)}{\sin(\alpha+\beta)}\right)\frac{\partial}{\partial\alpha} = \left(c\cdot\sin(\beta)\cdot\sin(\alpha+\beta)^{-1}\right)\frac{\partial}{\partial\alpha} = -c\cdot\frac{\sin(\beta)\cdot\cos(\alpha+\beta)}{\left(\sin(\alpha+\beta)\right)^2}$$

$$\frac{\partial f}{\partial \beta} = \left(c\cdot\frac{\sin(\beta)}{\sin(\alpha+\beta)}\right)\frac{\partial}{\partial\beta} = \left(c\cdot\sin(\beta)\cdot\left(\sin(\alpha+\beta)\right)^{-1}\right)\frac{\partial}{\partial\beta}$$

$$= \left(c\cdot\cos(\beta)\cdot\left(\sin(\alpha+\beta)\right)^{-1} + (-1)\cdot\left(\sin(\alpha+\beta)\right)^{-2}\cdot\cos(\alpha+\beta)\cdot(1)\cdot c\cdot\sin(\beta)\right)$$

$$= c\cdot\left(\frac{\cos(\beta)}{\sin(\alpha+\beta)} - \frac{\sin(\beta)\cdot\cos(\alpha+\beta)}{\left(\sin(\alpha+\beta)\right)^2}\right) = c\cdot\left(\frac{\cos(\beta)\cdot\sin(\alpha+\beta)-\sin(\beta)\cdot\cos(\alpha+\beta)}{\left(\sin(\alpha+\beta)\right)^2}\right)$$

$$= c\cdot\left(\frac{\sin(\alpha+\beta-\beta)}{\left(\sin(\alpha+\beta)\right)^2}\right) = c\cdot\left(\frac{\sin(\alpha)}{\left(\sin(\alpha+\beta)\right)^2}\right)$$

And eventually it is:

$$V[b]=\left[\frac{\partial f}{\partial c}\right]^2\cdot\sigma_c{}^2+\left[\frac{\partial f}{\partial \alpha}\right]^2\cdot\sigma_\alpha{}^2+\left[\frac{\partial f}{\partial \beta}\right]^2\cdot\sigma_\beta{}^2$$

$$=\left[\frac{\sin\beta}{\sin(\alpha+\beta)}\right]^2\cdot\sigma_c{}^2+\left[\frac{c\cdot\sin\beta\cdot\cos(\alpha+\beta)}{(\sin(\alpha+\beta))^2}\right]^2\cdot\sigma_\alpha{}^2+\left[\frac{c\cdot\sin\alpha}{(\sin(\alpha+\beta))^2}\right]^2\sigma_\beta{}^2$$

$$=1.0537^2\cdot 0.005^2+3.1894^2\cdot 0.011^2+5.4671^2\cdot 0.011^2=0.004875km^2$$

The error in *b* is calculated by:

$$\sigma[b]=\sqrt{0.004875}=0.0698km$$

Exercise 6.2:

The price of the TV game is a normal distributed random variable $A\sim N(50,10^2)$.
The money that he expects to receive from his parents are represented by the normal distributed random variable $B\sim N(20,5^2)$.

The safety margin can be formulated as follows:

$$M=20+B-A$$

If $M>0$, the boy can buy the TV game, whereas if $M\leq 0$ he cannot.

Therefore the probability that he cannot buy the TV game can be expressed as:

$$\mathrm{P}[M<0]=P[(20+B-A)<0]$$

The probability can be obtained by integrating the probability distribution function over the corresponding area.

But before carrying out the integration, the random variables in the limit state function have to be standardized.

$$A\sim N(50,10^2)\quad ----->\ U_A=\frac{A-\mu_A}{\sigma_A}\Leftrightarrow A=\mu_A+U_A\cdot\sigma_A\Leftrightarrow A=50+U_A\cdot 10$$

$$B\sim N(20,5^2)\quad ----->\ U_B=\frac{B-\mu_B}{\sigma_B}\Leftrightarrow B=\mu_B+U_B\cdot\sigma_B\Leftrightarrow B=20+U_B\cdot 5$$

The limit state function in the standard normal space is given by:

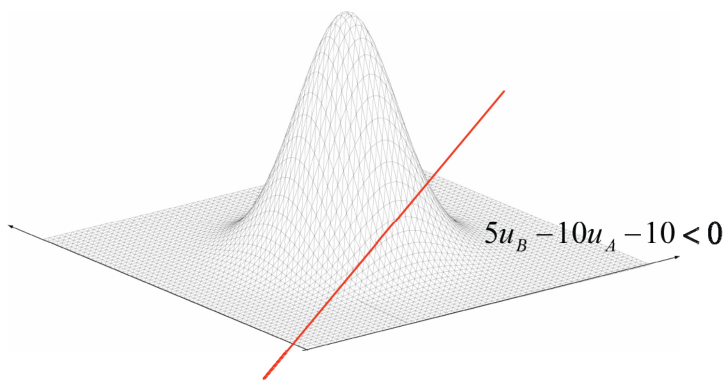
$M = 20 + (20 + U_B \cdot 5) - (50 + U_A \cdot 10) = 5 \cdot U_B - 10 \cdot U_A - 10$

The probability can be calculated by solving the following integration

$$P[M = 5U_B - 10U_A - 10 < 0] = \int_{5u_B - 10u_A - 10 < 0} \phi(u_A)\phi(u_B) du_A du_B$$

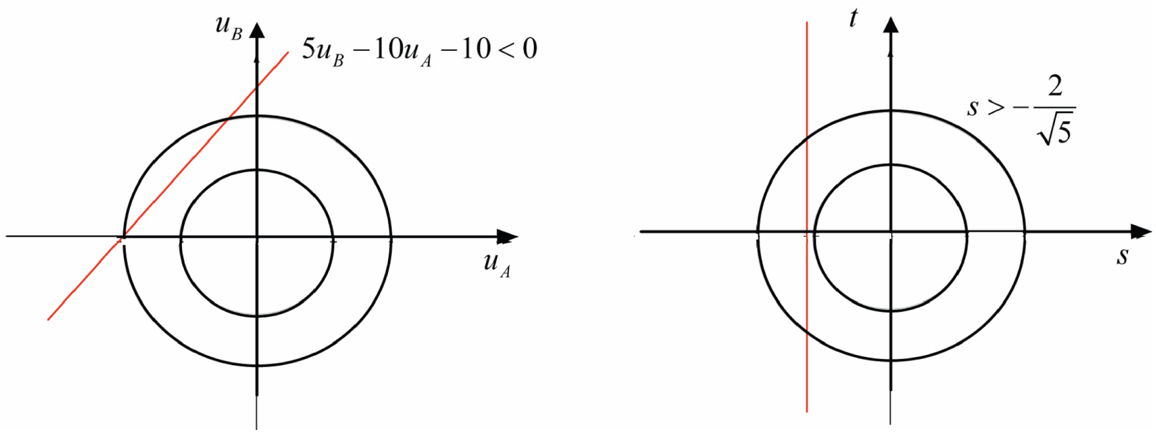
$$s > -\frac{2}{\sqrt{5}}$$

$\phi(\bullet)$ denotes the standard normal density function of u_A and u_B.



Visualization of the integration area

Since the joint density probability function is symmetric, we can rotate the limit state function. Due to the rotation the two dimensional integration is converted into a one dimensional integration.



2D visualization of the original and the rotated limit state function.

$u_B = 2 \cdot u_A + 2$

$u_B^* = -\frac{1}{2} \cdot u_A$

$v = v^*$

$$2 \cdot u_A + 2 = -\frac{1}{2} \cdot u_A \Leftrightarrow u_A = -\frac{4}{5}$$

$$u_B = \frac{2}{5}$$

$$|s| = \sqrt{\left(-\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \frac{2}{\sqrt{5}}$$

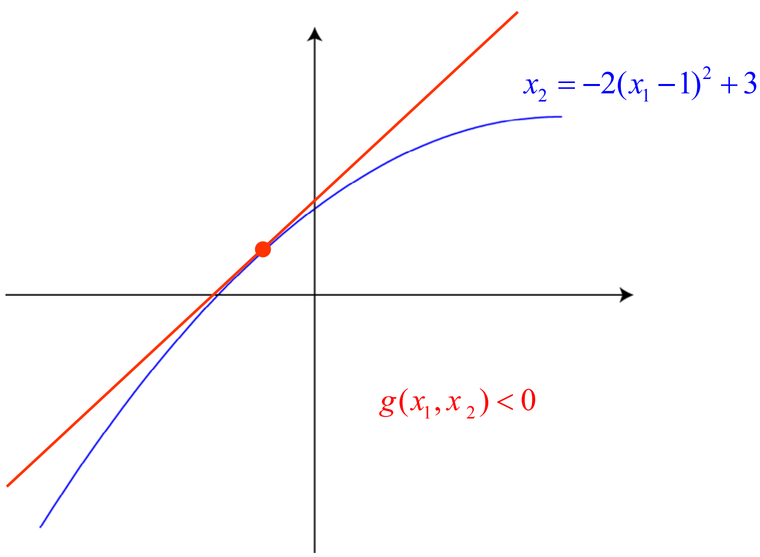
The minimal distance s between the straight line and the origin is:

$$s = -\frac{2}{\sqrt{5}}$$

$$\begin{aligned} P[M = 5U_B - 10U_A - 10 < 0] &= \int_{-2/\sqrt{5}}^{-\infty} \phi(s)ds \\ &= 1 - \left(1 - \Phi(2/\sqrt{5})\right) = 1 - \left(1 - \Phi(0.8944)\right) \\ &= 0.89 \end{aligned}$$

The probability that the boy cannot buy the game is 0.89.

Exercise 6.3:



The reliability index can be estimated by:

$$\beta = \min_{\mathbf{u} \in \{g(\mathbf{u})=0\}} \sqrt{\sum_{i=1}^{n} u_i^2}$$

$$\alpha_i = \frac{-\frac{\partial g}{\partial u_i}(\beta \cdot \boldsymbol{\alpha})}{\left[\sum_{i=1}^{n}\left(\frac{\partial g}{\partial u_i} \cdot (\beta \cdot \boldsymbol{\alpha})\right)^2\right]^{0.5}} \quad i = 1, 2, ...n$$

The limit state function is given by:

$$g(X_1, X_2) = 2(X_1 - 1)^2 + X_2 - 3$$

Since the random variables X_1 and X_2 are standard normal distributed we do not need to standardize them.

$$g(u) = 2(u_{X_1} - 1)^2 + u_{X_2} - 3$$

We substitute

$$u_{X_1} = \beta \cdot \alpha_{X_1}$$

$$u_{X_2} = \beta \cdot \alpha_{X_2}$$

$$0 = 2 \cdot (\beta \cdot \alpha_{X_1} - 1)^2 + \beta \cdot \alpha_{X_2} - 3$$
$$0 = 2 \cdot \beta^2 \cdot \alpha_{X_1}^2 - 4 \cdot \beta \cdot \alpha_{X_1} + \beta \cdot \alpha_{X_2} - 1$$

The reformulation of this equation and solving in respect to β gives:

$$\beta = \frac{1}{2 \cdot \beta \cdot \alpha_{X_1}^2 - 4 \cdot \alpha_{X_1} + \alpha_{X_2}}$$
$$\beta_{new} = \frac{1}{2 \cdot \beta_{old} \cdot \alpha_{X_1}^2 - 4 \cdot \alpha_{X_1} + \alpha_{X_2}}$$

To calculate the α values we need to calculate the derivatives:

$$-\frac{\partial g}{\partial u_{X_1}} = -\frac{\partial g}{\partial u_{X_1}}\left(2 \cdot (u_{X_1} - 1)^2 + u_{X_2} - 3\right) = -4 \cdot (u_{X_1} - 1)$$
$$-\frac{\partial g}{\partial u_{X_1}}\left(\beta \cdot \alpha_{X_1}\right) = -4 \cdot (\beta \cdot \alpha_{X_1} - 1)$$

$$-\frac{\partial g}{\partial u_{X_2}} = -\frac{\partial g}{\partial u_{X_2}}\left(2 \cdot (u_{X_1} - 1)^2 + u_{X_2} - 3\right) = -1$$
$$-\frac{\partial g}{\partial u_{X_2}}\left(\beta \cdot \alpha_{X_2}\right) = -1$$

$$\sum_{i=1}^{n}\left(\frac{\partial g}{\partial u_i}\cdot(\beta\cdot\boldsymbol{\alpha})\right)^2=\left(4\cdot(\beta\cdot\alpha_{X_1}-1)\right)^2+(1)^2$$

$$\alpha_{X_1}=\frac{-\frac{\partial g}{\partial u_{X_1}}(\beta\cdot\boldsymbol{\alpha})}{\left[\sum_{i=1}^{n}\left(\frac{\partial g}{\partial u_{Xi}}\cdot(\beta\cdot\boldsymbol{\alpha})\right)^2\right]^{0.5}}=\frac{-4\cdot(\beta\cdot\alpha_{X_1}-1)}{\sqrt{\left(4\cdot(\beta\cdot\alpha_{X_1}-1)\right)^2+(1)^2}}$$

$$\alpha_{X_2}=\frac{-\frac{\partial g}{\partial u_{X_2}}(\beta\cdot\boldsymbol{\alpha})}{\left[\sum_{i=1}^{n}\left(\frac{\partial g}{\partial u_{Xi}}\cdot(\beta\cdot\boldsymbol{\alpha})\right)^2\right]^{0.5}}=\frac{-1}{\sqrt{\left(4\cdot(\beta\cdot\alpha_{X_1}-1)\right)^2+(1)^2}}$$

The starting values for the iteration are chosen:

$$\alpha_{X_1}=0.6$$
$$\alpha_{X_2}=-0.6$$
$$\beta=-1$$

1st. Iteration:

Calculation of a new β value:

$$\beta_{new}=\frac{1}{2\cdot\beta_{old}\cdot\alpha_{X_1}^2-4\cdot\alpha_{X_1}+\alpha_{X_2}}=\frac{1}{2\cdot(-1)\cdot 0.36-4\cdot 0.6-0.6}=-0.268$$

Calculation of the new α values:

$$\alpha_{X_1}=\frac{-4\cdot(\beta\cdot\alpha_{X_1}-1)}{\sqrt{\left(4\cdot(\beta\cdot\alpha_{X_1}-1)\right)^2+(1)^2}}=\frac{-4\cdot(-0.268\cdot 0.6-1)}{\sqrt{\left(4\cdot(-0.268\cdot 0.6-1)\right)^2+(1)^2}}=0.988$$

$$\alpha_{X_2}=\frac{-1}{\sqrt{\left(4\cdot(\beta\cdot\alpha_{X_1}-1)\right)^2+(1)^2}}=\frac{-1}{\sqrt{\left(4\cdot(-0.268\cdot 0.6-1)\right)^2+(1)^2}}=-0.154$$

2nd. Iteration:

With these values one can calculate again a new β value

$$\beta_{new}=\frac{1}{2\cdot\beta_{old}\cdot\alpha_{X_1}^2-4\cdot\alpha_{X_1}+\alpha_{X_2}}=\frac{1}{2\cdot(-0.268)\cdot 0.988^2-4\cdot 0.988-0.154}=-0.216$$

And new alpha values:

$$\alpha_{X_1} = \frac{-4\cdot(\beta\cdot\alpha_{X_1}-1)}{\sqrt{\left(4\cdot(\beta\cdot\alpha_{X_1}-1)\right)^2+\left(1\right)^2}} = \frac{-4\cdot(-0.216\cdot 0.988-1)}{\sqrt{\left(4\cdot(-0.216\cdot 0.988-1)\right)^2+\left(1\right)^2}}$$
$$= 0.981$$

$$\alpha_{X_2} = \frac{-1}{\sqrt{\left(4\cdot(-0.216\cdot 0.988-1)\right)^2+\left(1\right)^2}} = -0.194$$

3rd. Iteration:

$$\beta_{new} = \frac{1}{2\cdot\beta_{old}\cdot\alpha_{X_1}^2-4\cdot\alpha_{X_1}+\alpha_{X_2}} = \frac{1}{2\cdot\left(-0.216\right)\cdot 0.981^2-4\cdot 0.981-0.194} = -0.221$$

$$\alpha_{X_1} = \frac{-4\cdot(\beta\cdot\alpha_{X_1}-1)}{\sqrt{\left(4\cdot(\beta\cdot\alpha_{X_1}-1)\right)^2+\left(1\right)^2}} = \frac{-4\cdot(-0.221\cdot 0.981-1)}{\sqrt{\left(4\cdot(-0.221\cdot 0.981-1)\right)^2+\left(1\right)^2}}$$
$$= 0.979$$

$$\alpha_{X_2} = \frac{-1}{\sqrt{\left(4\cdot(-0.221\cdot 0.981-1)\right)^2+\left(1\right)^2}} = -0.202$$

4th. Iteration:

$$\beta_{new} = \frac{1}{2\cdot\beta_{old}\cdot\alpha_{X_1}^2-4\cdot\alpha_{X_1}+\alpha_{X_2}} = \frac{1}{2\cdot\left(-0.221\right)\cdot 0.979^2-4\cdot 0.979-0.202} = -0.22$$

The results of this iteration are given in the following table.

Iteration i	Start	1	2	3	4	5	6
β	-1	-0.269	-0.216	-0.221	-0.22	-0.22	-
α_{X1}	0.6	0.988	0.981	0.980	0.980	-	-
α_{X2}	-0.6	-0.154	-0.194	-0.201	-0.201	-	-

$$P[2(X_1-1)^2+X_2-3<0] = \Phi(-(-0.22)) = \Phi(0.22) = 0.587$$

Exercise 6.4:

The limit state function describing the event of failure may be written as:

$$g(x) = 8 - w$$

where w is the central deflection.

By replacing with the relevant values we get:

$$g(x) = 8 - w = 8 - \frac{5 \cdot q \cdot l^4}{384 \cdot E \cdot I} = 8 - 1.235 \cdot 10^6 \frac{q}{h^3}$$

where q in [kN/m] and h in [mm].

Reforming gives:

$$g(x) = 8 \cdot h^3 - 1.235 \cdot 10^6 \cdot q$$

Now the normally distributed variables are transformed into standard normal distributed variables as follows:

$$U_Q = \frac{Q - \mu_Q}{\sigma_Q} \quad \text{and} \quad U_H = \frac{H - \mu_H}{\sigma_H}$$

Thus the limit state function may now be written in the space of the standardized normal distributed random variables as:

$$\begin{aligned} g(u) &= 8 \cdot (\mu_H + u_H \cdot \sigma_H)^3 - 1.235 \cdot 10^6 \cdot (\mu_Q + u_Q \cdot \sigma_Q) \\ &= 8 \cdot (100 + u_H \cdot 5)^3 - 1.235 \cdot 10^6 \cdot (5 + u_Q \cdot 1) \end{aligned}$$

$$g(u) = u_H^3 + 40 \cdot u_H^2 + 800 \cdot u_H - 1235 \cdot u_Q + 1825$$

The design point values are:

$$u_H = \alpha_H \beta \quad u_Q = \alpha_Q \beta$$

Thus:

$$g(u) = (\alpha_H \cdot \beta)^3 + 40 \cdot (\alpha_H \cdot \beta)^2 + 800 \cdot \alpha_H \cdot \beta - 1235 \cdot \alpha_Q \cdot \beta + 1825$$

The reliability index may be estimated by:

$$\beta = \min_{z \in (g(z)=0)} \sqrt{\sum_i^n u_i^2}$$

And reforming gives:

$$\beta_i = \frac{-1825}{\alpha_H^3 \cdot \beta^2 + 40 \cdot \alpha_H^2 \cdot \beta_{i-1} + 800 \cdot \alpha_H - 1235 \cdot \alpha_Q}$$

Now the following iteration scheme is followed:

$$\alpha_i = \frac{-\dfrac{\partial g}{\partial u_i}(\beta \cdot \alpha)}{\left[\sum_i^n \left(\dfrac{\partial g}{\partial u_i}(\beta \cdot \alpha)\right)^2\right]^{1/2}}$$

$$\alpha_H = -\frac{1}{k}(3 \cdot u_H^2 + 80 \cdot u_H + 800)$$

$$k = \sqrt{(-1235)^2 + (3 \cdot u_H^2 + 80 \cdot u_H + 800)^2}$$

$$\alpha_Q = -\frac{1}{k}(-1235)$$

Iteration i	Start	1	2	3	4	5	6
β	3	1.380346	1.266017	1.264043	1.263972	1.263974	1.263974
α_H	-0.7071	-0.462254	-0.519157	-0.518418	-0.518494	-0.518491	-0.518492
α_Q	0.7071	0.886747	0.854679	0.855127	0.855081	0.855083	0.855083

Iteration.

The probability of failure may now be estimated as:

$$P_F = \Phi(-\beta) = \Phi(-1.263974) = 0.103 \approx 10.3\%$$

Exercise 7-Solution:

Exercise 7.1:

1. Open Matlab

2. Copy the routine provided in the last pages of this file and paste it in the "EDITOR"

3. Save the file with the name: "Exercise_7.mat"

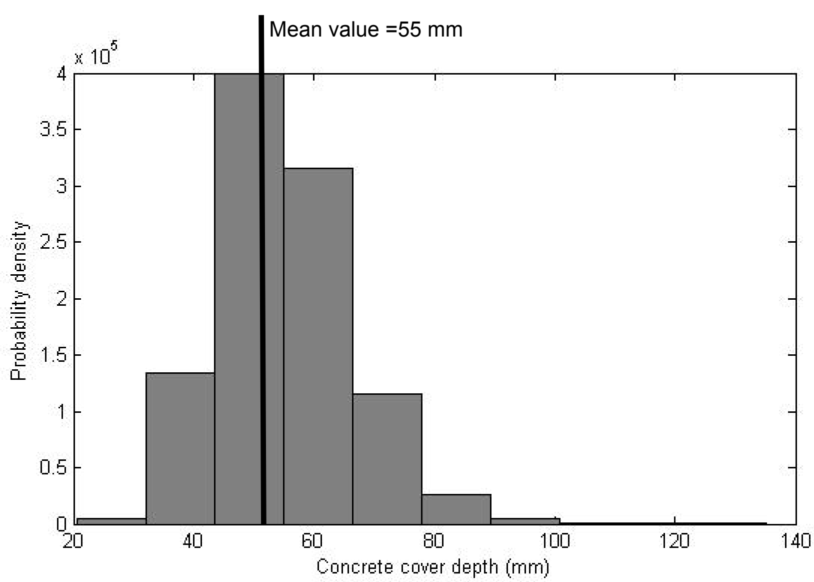
4. In the COMMAND WINDOW write "Exercise_7" and click enter

5. Wait until you see the results in the workspace. (The time required for the simulation depends on the number of simulations. In the provided routine the number of simulations, ilast, is set to 1 million. If you want to have a view of the results faster decrease this number initially to e.g. 10000- however the accuracy of the result is smaller for smaller number of simulations).

6. The probability of corrosion initiation after 50 years is equal to: 0.0219.

7. The probability of visual corrosion after 50 years is equal to: 0.0112.

8. You can use Matlab to plot and check the distributions of the variables in the model. For example right click "cd" in the workspace.
   Choose "hist" from the available plots to plot the histogram of the simulated values of the concrete cover depth. The plots should agree with the input values for every variable.



Routine for Monte Carlo Simulation in Matlab

```
%mean__ stands for the mean value of a variable

%stdev__ stands for the standard deviation of a variable

%other statistical parameters are provided according to the relevant
%distribution of a variable

%for the relation between moments and parameters of a distribution please
%use Tables 2.7 and/or 2.8 in the chapter 2 of the script

%the variables are defined as provided in Table 7.1 of Exercise 7.1

%concrete cover depth, cd
meancd=55;
stdevcd=11;
lamdacd = log(meancd^2)-log(sqrt(stdevcd^2+meancd^2));
zitacd = sqrt(log((stdevcd/meancd)^2+1));

% diffusion coefficient, D
meanD=40;
stdevD=10;
lamdaD = log(meanD^2)-log(sqrt(stdevD^2+meanD^2));
zitaD = sqrt(log((stdevD/meanD)^2+1));

%surface concentration, Cs
meanCs=0.4;
stdevCs=0.08;
lamdaCs = log(meanCs^2)-log(sqrt(stdevCs^2+meanCs^2));
zitaCs = sqrt(log((stdevCs/meanCs)^2+1));

%critical concentration, Ccr
meanCcr=0.15;
stdevCcr=0.05;;
lamdaCcr = log(meanCcr^2)-log(sqrt(stdevCcr^2+meanCcr^2));
zitaCcr = sqrt(log((stdevCcr/meanCcr)^2+1));



%propagation time
meanTp=7.5;
stdevTp=1.88;
lamdaTp = log(meanTp^2)-log(sqrt(stdevTp^2+meanTp^2));
zitaTp = sqrt(log((stdevTp/meanTp)^2+1));



%model uncertainty, Xi
meanXi=1;
stdevXi=0.05;
lamdaXi = log(meanXi^2)-log(sqrt(stdevXi^2+meanXi^2));
zitaXi = sqrt(log((stdevXi/meanXi)^2+1));

%reference time,t
```

```
t=50;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%
%%%%Monte Carlo Simulation%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%number of simulations
ilast=1000000;

%create memory space in Matlab to enable faster performance
cd=zeros(ilast,1);
Cs=zeros(ilast,1);
Ccr=zeros(ilast,1);
Xi=zeros(ilast,1);
Tp=zeros(ilast,1);
D=zeros(ilast,1);

  %random numbers for the concrete cover depth, cd
        cd=lognrnd(lamdacd,zitacd,ilast,1);

   %random numbers for the diffusion coefficient, D
        D=lognrnd(lamdaD,zitaD,ilast,1);

  %random numbers for the surface chloride concentration, Cs
        Cs=lognrnd(lamdaCs,zitaCs,ilast,1);

  %random numbers for the critical chloride concentration, Ccr
        Ccr=lognrnd(lamdaCcr,zitaCcr,ilast,1);

  %random numbers for the model uncertainty,Xi
        Xi=lognrnd(lamdaXi,zitaXi,ilast,1);

  %random numbers for the propagation time, Tp
        Tp=lognrnd(lamdaTp,zitaTp,ilast,1);

save Random_numbers
clear


%%INSERTION OF THE MODEL FUNCTION%%%%%%%%%%%%%%%%%%%%

load Random_numbers;

   %inverse of standard normal disribution
     normstandardinv=norminv((1 - (Ccr./(2 * Cs))),0,1);

   %estimation of the error function
      errorfunction=normstandardinv/sqrt(2);
      partA=errorfunction.^(-2);

   %time till corrosion initiation, CI
      Ti_mod = ((cd.^2)./D).*partA;
```

```
  %time till visual corrosion
    T_CV=Xi.*Ti_mod+Tp;

  %margin for corrosion initiation
    M_CI=Xi.*Ti_mod-t;

  %margin for visual corrosion,CV
    M_CV=T_CV-t;


%%LOOPS FOR THE ESTIMATION OF THE PROBABILTIY OF FAILURE%%%%%%%%%%

%corrosion initiation
        for i=1:ilast
            if M_CI(i)<=0
                ni(i)=1;
            else
                ni(i)=0;
            end
          end
   %nitotal is the total number of cases where M_CI<=0
            sum_nitotal=sum(ni);
   %Probability of corrosion initiation after 50 years
            Pf_CI=sum_nitotal/ilast;


 %visual corrosion
        for i=1:ilast
            if M_CV(i)<=0
                nv(i)=1;
            else
                nv(i)=0;
            end
          end
   %nvtotal is the total number of cases where M_CV<=0
            sum_nvtotal=sum(nv);
   %Probability of corrosion initiation after 50 years
    Pf_CV=sum_nvtotal/ilast;
```

Exercise 8- Solution:

Exercise 8.1:

a. The deterministic values of r_o is estimated as follows:

$$M = R - S \qquad \begin{matrix}\mu_s = 16 \\ \sigma_s = 8\end{matrix} \quad \beta = \frac{\mu_M}{\sigma_M} = \frac{r_o - 16}{8} = 7$$

$$\Rightarrow \qquad r_o = 72 \text{ MPa}$$

From the resistance function it is:

$$r(t) = r_o \cdot g(t)$$

$$g(t) = \frac{a_{cr} - a(t)}{a_{cr} - a_o} = \frac{20 - a(t)}{20 - 1} = \frac{20}{19} - \frac{a(t)}{19}$$

where $a(t) = \left(a_o^{(2-m)/2} + \frac{2-m}{2} \cdot C \cdot \pi^{m/2} \Delta\sigma^m \cdot \nu \cdot t\right)^{2/(2-m)}$

and inserting the known parameters it is:

$$a(t) = \left(1 - 0.8 \cdot 10^{-8} \cdot \pi^{1.8} \cdot 30^{3.6} \cdot t\right)^{-1.25}$$

And so:

$$g(t) = \frac{20}{19} - \frac{a(t)}{19} = \frac{20}{19} - \frac{\left(1 - 0.8 \cdot 10^{-8} \cdot \pi^{1.8} \cdot 30^{3.6} \cdot t\right)^{-1.25}}{19}$$

$$r(t) = 72 \cdot \left[\frac{20}{19} - \frac{\left(1 - 0.8 \cdot 10^{-8} \cdot \pi^{1.8} \cdot 30^{3.6} \cdot t\right)^{-1.25}}{19}\right]$$

The out-crossing rate is estimated as follows:

$$v^+(t) = \omega_o \cdot \varphi(\eta) \cdot \left[\varphi(\frac{\dot{\eta}}{\omega_o}) - \frac{\dot{\eta}}{\omega_o} \cdot \Phi(\frac{\dot{\eta}}{\omega_o})\right]$$

where: $\eta(t) = \dfrac{r(t) - \mu_s}{\sigma_s}$

and hence $\dot{\eta}(t) = \dfrac{\dot{r}(t)}{\sigma_s}$

In the following the out-crossing rate is estimated for years 1 to 50.

Time	a(t)	g(t)	r(t)	η(t)	η' (t)/ωₒ	φ(η)	φ(η'/ω₀)	ϕ(-η'/ωₒ)	$\nu^+(t)$
0	1	1	72	7	-7.7E-09	9.13E-12	0.398942	0.5	3.64E-06
1	1.016555	0.999129	71.93727	6.992158	-8E-09	9.65E-12	0.398942	0.5	3.85E-06
2	1.03361	0.998231	71.87264	6.98408	-8.2E-09	1.02E-11	0.398942	0.5	4.07E-06
3	1.051187	0.997306	71.80603	6.975754	-8.5E-09	1.08E-11	0.398942	0.5	4.32E-06
4	1.069309	0.996352	71.73736	6.96717	-8.7E-09	1.15E-11	0.398942	0.5	4.58E-06
5	1.088001	0.995368	71.66652	6.958315	-9E-09	1.22E-11	0.398942	0.5	4.87E-06
6	1.10729	0.994353	71.59343	6.949178	-9.3E-09	1.3E-11	0.398942	0.5	5.19E-06
7	1.127204	0.993305	71.51796	6.939746	-9.6E-09	1.39E-11	0.398942	0.5	5.55E-06
8	1.147771	0.992223	71.44002	6.930003	-9.9E-09	1.49E-11	0.398942	0.5	5.93E-06
9	1.169024	0.991104	71.35949	6.919936	-1E-08	1.59E-11	0.398942	0.5	6.36E-06
10	1.190996	0.989948	71.27623	6.909528	-1.1E-08	1.71E-11	0.398942	0.5	6.84E-06
11	1.213723	0.988751	71.1901	6.898763	-1.1E-08	1.85E-11	0.398942	0.5	7.36E-06
12	1.237242	0.987514	71.10098	6.887622	-1.1E-08	1.99E-11	0.398942	0.5	7.95E-06
13	1.261595	0.986232	71.00869	6.876087	-1.2E-08	2.16E-11	0.398942	0.5	8.61E-06
14	1.286824	0.984904	70.91309	6.864136	-1.2E-08	2.34E-11	0.398942	0.5	9.35E-06
15	1.312976	0.983528	70.81399	6.851748	-1.3E-08	2.55E-11	0.398942	0.5	1.02E-05
16	1.340101	0.9821	70.7112	6.8389	-1.3E-08	2.78E-11	0.398942	0.5	1.11E-05
17	1.368251	0.980618	70.60452	6.825565	-1.4E-08	3.05E-11	0.398942	0.5	1.22E-05
18	1.397484	0.97908	70.49375	6.811718	-1.4E-08	3.35E-11	0.398942	0.5	1.34E-05
19	1.42786	0.977481	70.37863	6.797329	-1.5E-08	3.7E-11	0.398942	0.5	1.48E-05
20	1.459446	0.975819	70.25894	6.782367	-1.5E-08	4.09E-11	0.398942	0.5	1.63E-05
21	1.492313	0.974089	70.13439	6.766799	-1.6E-08	4.55E-11	0.398942	0.5	1.81E-05
22	1.526536	0.972288	70.00471	6.750588	-1.7E-08	5.07E-11	0.398942	0.5	2.02E-05
23	1.562198	0.970411	69.86956	6.733696	-1.7E-08	5.69E-11	0.398942	0.5	2.27E-05
24	1.599389	0.968453	69.72863	6.716079	-1.8E-08	6.4E-11	0.398942	0.5	2.55E-05
25	1.638204	0.96641	69.58154	6.697693	-1.9E-08	7.24E-11	0.398942	0.5	2.89E-05
26	1.678748	0.964276	69.4279	6.678488	-2E-08	8.24E-11	0.398942	0.5	3.29E-05
27	1.721135	0.962046	69.26728	6.65841	-2.1E-08	9.41E-11	0.398942	0.5	3.76E-05
28	1.765488	0.959711	69.0992	6.637401	-2.1E-08	1.08E-10	0.398942	0.5	4.32E-05
29	1.811941	0.957266	68.92317	6.615396	-2.3E-08	1.25E-10	0.398942	0.5	5E-05
30	1.860642	0.954703	68.73862	6.592327	-2.4E-08	1.46E-10	0.398942	0.5	5.82E-05
31	1.911751	0.952013	68.54494	6.568118	-2.5E-08	1.71E-10	0.398942	0.5	6.82E-05
32	1.965443	0.949187	68.34148	6.542685	-2.6E-08	2.02E-10	0.398942	0.5	8.06E-05
33	2.021913	0.946215	68.12749	6.515936	-2.7E-08	2.41E-10	0.398942	0.5	9.6E-05
34	2.081371	0.943086	67.90217	6.487772	-2.9E-08	2.89E-10	0.398942	0.5	0.000115
35	2.144052	0.939787	67.66464	6.458081	-3E-08	3.5E-10	0.398942	0.5	0.00014
36	2.210215	0.936304	67.41392	6.42674	-3.2E-08	4.29E-10	0.398942	0.5	0.000171
37	2.280146	0.932624	67.14892	6.393615	-3.4E-08	5.3E-10	0.398942	0.5	0.000211
38	2.354164	0.928728	66.86843	6.358554	-3.6E-08	6.63E-10	0.398942	0.5	0.000264
39	2.432621	0.924599	66.57112	6.32139	-3.8E-08	8.39E-10	0.398942	0.5	0.000335
40	2.515913	0.920215	66.25549	6.281936	-4.1E-08	1.08E-09	0.398942	0.5	0.000429
41	2.604484	0.915553	65.91985	6.239981	-4.3E-08	1.4E-09	0.398942	0.5	0.000558
42	2.69883	0.910588	65.56233	6.195291	-4.6E-08	1.85E-09	0.398942	0.5	0.000737
43	2.799511	0.905289	65.1808	6.1476	-4.9E-08	2.48E-09	0.398942	0.5	0.000989
44	2.90716	0.899623	64.77287	6.096608	-5.3E-08	3.39E-09	0.398942	0.5	0.001351
45	3.022497	0.893553	64.3358	6.041975	-5.7E-08	4.72E-09	0.398942	0.5	0.001883
46	3.14634	0.887035	63.8665	5.983313	-6.1E-08	6.71E-09	0.398942	0.5	0.002679
47	3.279625	0.88002	63.36142	5.920177	-6.6E-08	9.78E-09	0.398942	0.5	0.003901
48	3.423431	0.872451	62.81647	5.852059	-7.1E-08	1.46E-08	0.398942	0.5	0.005825
49	3.578999	0.864263	62.22695	5.778369	-7.7E-08	2.24E-08	0.398942	0.5	0.008941
50	3.747771	0.85538	61.58739	5.698424	-8.3E-08	3.55E-08	0.398942	0.5	0.014145

The probability of failure at time *t* is given by: $P_f(t) = 1 - \exp(-\int_0^t \nu^+(t) \cdot dt)$

Where for *t* =50 years: $\int_0^{50} \nu^+(t) \cdot dt = 0.0347$.

And the probability of failure is then: $P_f(t) = 1 - \exp(-0.0347) = 3.41\%$

b. The probability of failure is now given by:

$$P_f(t) = 1 - \int_{-\infty}^{\infty} \exp(-\int_0^t \nu^+(t) \cdot dt) \cdot f_R(r_o) \cdot dr$$

Since r_o is a random variable for random values around its mean values we get the following table:

r_o	$f_R(r_o)$	$\int_0^{50} \nu^+(t)\cdot dt$	$exp(-\int_0^{50} \nu^+(t)\cdot dt\ )$	$exp(-\int_0^{50} \nu^+(t)\cdot dt\ ) f_R(r_o)$
60	0.0004284	40.788933	1.93015E-18	8.26898E-22
61	0.0010405	23.943161	3.99592E-11	4.15764E-14
62	0.0023393	13.912772	9.07319E-07	2.1225E-09
63	0.004869	8.0026024	0.000334591	1.62911E-06
64	0.0093816	4.556417	0.010499612	9.85028E-05
65	0.0167341	2.567893	0.076696977	0.001283458
66	0.0276325	1.4324386	0.238726044	0.006596607
67	0.0422404	0.790867	0.453451472	0.019153964
68	0.0597757	0.4321561	0.64910807	0.038800879
69	0.0783089	0.2337049	0.79159535	0.061988925
70	0.0949701	0.1250735	0.88243207	0.083804646
71	0.1066234	0.0662387	0.935907394	0.099789594
72	0.1108173	0.0347127	0.96588289	0.107036534
73	0.1066234	0.018	0.982161002	0.104721309
74	0.0949701	0.0092352	0.99080729	0.094097049
75	0.0783089	0.004688	0.995322931	0.077942598
76	0.0597757	0.0023544	0.997648336	0.059635111
77	0.0422404	0.0011698	0.998830876	0.042191
78	0.0276325	0.000575	0.999425178	0.027616655
79	0.0167341	0.0002796	0.999720462	0.01672946
80	0.0093816	0.0001345	0.999865538	0.009380308
81	0.004869	6.398E-05	0.999936025	0.004868661
82	0.0023393	3.011E-05	0.999969893	0.002339245
83	0.0010405	1.401E-05	0.999985986	0.001040456
84	0.0004284	6.452E-06	0.999993548	0.000428408

And through numerical integration (essentially the sum of all the values estimated in the last column of the above table) it is:

$$\int_{-\infty}^{\infty} \exp(-\int_0^t \nu^+(t)\cdot dt)\cdot f_R(r_o)\cdot dr = 0.859$$

And the probability of failure is:

$$P_f(t) = 1-0.859 = 0.141 = 14.1\%$$

Exercise 10- Solution:

Exercise 10.1:

a.
The failure probability for a specific time range [0,t] is determined as follows,

$$P_f(t) = 1 - \exp(-\int_0^t \nu^+(t) \cdot dt)$$

Solving the integral for each year from 0 to 50 years and plotting over the time yields Figure 10.1.

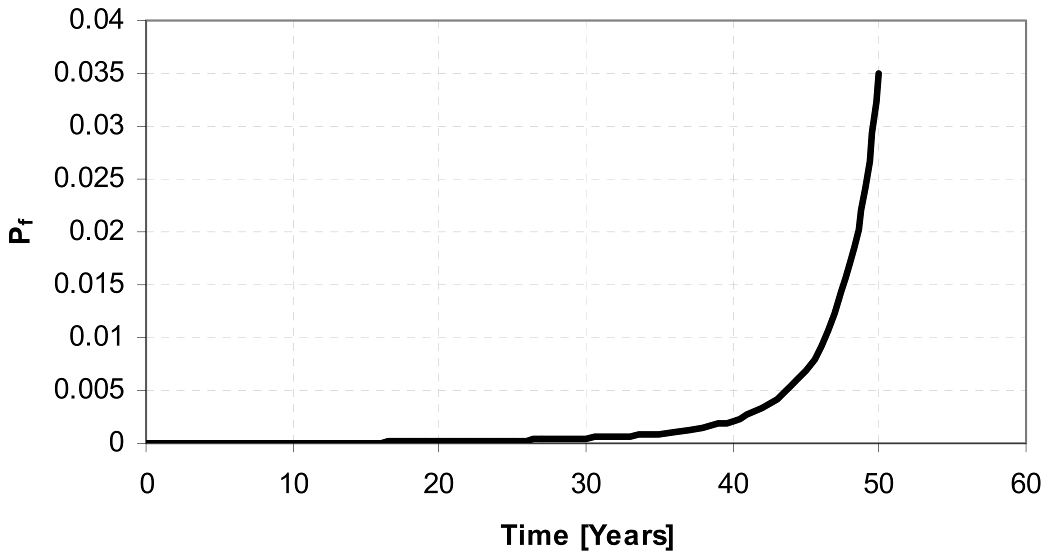


Figure 10.1: Cumulative probability of failure

A series system with 50 components, where the components are one-year periods is assumed.

The failure probabilities for the one-year periods are computed with $P_f(t) = P_f(t) - P_f(t-1)$ from Figure 10.1 (see also Table 10.1).

Time (Years)	P_f (cumulative)	P_f (yearly)	1- P_f (yearly)	Time (Years)	P_f (cumulative)	P_f (yearly)	1- P_f (yearly)
1	3.74537E-06	3.74537E-06	0.999996255	26	0.00029608	3.08014E-05	0.999969199
2	7.7051E-06	3.95972E-06	0.99999604	27	0.0003312	3.51178E-05	0.999964882
3	1.18983E-05	4.19316E-06	0.999995807	28	0.00037147	4.02644E-05	0.999959736
4	1.63461E-05	4.44788E-06	0.999995552	29	0.00041791	4.64409E-05	0.999953559
5	2.10726E-05	4.72643E-06	0.999995274	30	0.00047181	5.39053E-05	0.999946095
6	2.61043E-05	5.0317E-06	0.999994968	31	0.00053481	6.29928E-05	0.999937007
7	3.14713E-05	5.36703E-06	0.999994633	32	0.00060895	7.41431E-05	0.999925857
8	3.72075E-05	5.73623E-06	0.999994264	33	0.00069689	8.79386E-05	0.999912061
9	4.33513E-05	6.14375E-06	0.999993856	34	0.00080205	0.000105159	0.999894841
10	4.9946E-05	6.59473E-06	0.999993405	35	0.0009289	0.000126856	0.999873144
11	5.70411E-05	7.09514E-06	0.999992905	36	0.00108338	0.000154473	0.999845527
12	6.46931E-05	7.65197E-06	0.999992348	37	0.00127338	0.000190002	0.999809998
13	7.29665E-05	8.27341E-06	0.999991727	38	0.00150962	0.000236238	0.999763762
14	8.19356E-05	8.96908E-06	0.999991031	39	0.00180677	0.000297153	0.999702847
15	9.1686E-05	9.75037E-06	0.99999025	40	0.00218524	0.00037847	0.99962153
16	0.000102317	1.06307E-05	0.999989369	41	0.00267381	0.000488567	0.999511433
17	0.000113943	1.16263E-05	0.999988374	42	0.00331371	0.000639908	0.999360092
18	0.000126699	1.27563E-05	0.999987244	43	0.00416508	0.000851364	0.999148636
19	0.000140743	1.40437E-05	0.999985956	44	0.00531711	0.001152033	0.998847967
20	0.00015626	1.55167E-05	0.999984483	45	0.00690478	0.00158767	0.99841233
21	0.000173469	1.7209E-05	0.999982791	46	0.00913651	0.002231725	0.997768275
22	0.000192631	1.91622E-05	0.999980838	47	0.01234116	0.003204657	0.996795343
23	0.000214058	2.14271E-05	0.999978573	48	0.01704959	0.004708427	0.995291573
24	0.000238124	2.40665E-05	0.999975933	49	0.02413871	0.007089116	0.992910884
25	0.000265283	2.71587E-05	0.999972841	50	0.03509059	0.010951887	0.989048113

Table 10.1 Annual probability of failure and reliability

The probability of failure of the series system is then:

$$P_F = 1 - P_S = 1 - \prod_{i=1}^{50}(1 - P(F_i)) = 1 - 0.9654 = 3.46\%$$

b. The resistance parameter r_o is normal distributed with mean value 72 and standard deviation 3.6. The failure probability was computed in Exercise 8.1a for a single deterministic value of $r_o = 72$. The failure probabilities for the normal distributed r_o are computed analogous for each integer r_o from e.g. $r_o = 60$ to $r_o = 84$.

The probability of survival for each r_o is obtained by factorizing with its frequency of occurrence:

$$\overline{P}_f(t) = \prod_{i=1}^{50}(1 - P_{f_i}) \cdot f_R(r_o)$$

The probability of failure of the series system is obtained through summation of the probabilities of survival for each r_o value. $P_f(t) = 1 - \int_{60}^{84}\prod_{i=1}^{50}(1 - P_{f_i}) \cdot f_R(r_o) \cdot dr$

r_o	$f_R(r_o)$	$P_f(r_o)$	$\prod_{i=1}^{50}(1-P_{f_i})$	$\prod_{i=1}^{50}(1-P_{f_i})\cdot f_R(r_o)$	$P_f(r_o)$ from $v^+(t)$
60	0.0004284	0.6383	0.3617	0.000154956	1
61	0.0010405	0.6369	0.3631	0.000377795	1
62	0.0023393	0.6368	0.3632	0.000849639	0.9999
63	0.004869	0.6377	0.3623	0.001764029	0.9996
64	0.0093816	0.6359	0.3641	0.00341583	0.9895
65	0.0167341	0.6139	0.3861	0.006461051	0.9233
66	0.0276325	0.5471	0.4529	0.012514777	0.7613
67	0.0422404	0.4347	0.5653	0.023878489	0.5465
68	0.0597757	0.3059	0.6941	0.041490302	0.3509
69	0.0783089	0.1942	0.8058	0.063101275	0.2084
70	0.0949701	0.1144	0.8856	0.084105504	0.1176
71	0.1066234	0.064	0.936	0.099799468	0.0641
72	0.1108173	0.0346	0.9654	0.106983022	0.0341
73	0.1066234	0.0183	0.9817	0.104672156	0.0178
74	0.0949701	0.0095	0.9905	0.094067866	0.0092
75	0.0783089	0.0048	0.9952	0.077932972	0.0047
76	0.0597757	0.0024	0.9976	0.059632222	0.0024
77	0.0422404	0.0012	0.9988	0.042189695	0.0012
78	0.0276325	0.0006	0.9994	0.027615959	0.0006
79	0.0167341	0.000293	0.999707	0.016729235	0.000280
80	0.0093816	0.000141	0.999859	0.009380247	0.000134
81	0.004869	0.000067	0.999933	0.004868646	0.000064
82	0.0023393	0.000032	0.999968	0.00233924	0.000030
83	0.0010405	0.000015	0.999985	0.001040455	0.000014
84	0.0004284	0.000007	0.999993	0.000428408	0.0000064

Table 10.2 Probability of failure of the series system

Numerical integration yields:

$$\int_{60}^{84}\prod_{i=1}^{50}(1-P_{f_i})\cdot f_R(r_o)\cdot dr = 0.886$$

Thus the probability of failure of the series system is:

$$P_f(t) = 1-0.886 = 0.114 = 11.4\%$$

Comments: In Exercise 8 we had obtained a failure probability of 14.1%. By comparing columns 3 and 6 it can be seen that with smaller r_o-values the differences in the failure probabilities are bigger. An explanation for this difference may be the fact that out-crossings before the considered time are neglected in computing the out-crossing rate.

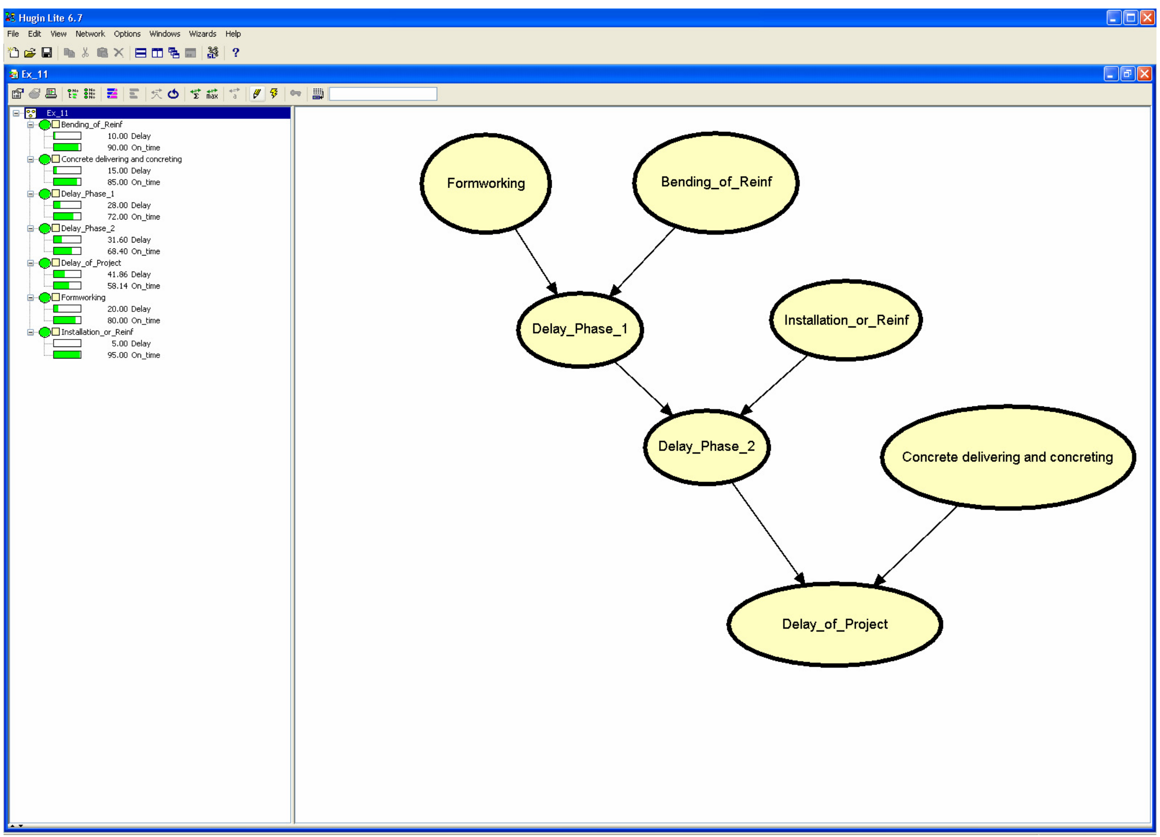
Exercise 11-Solution:

Exercise 11.1:

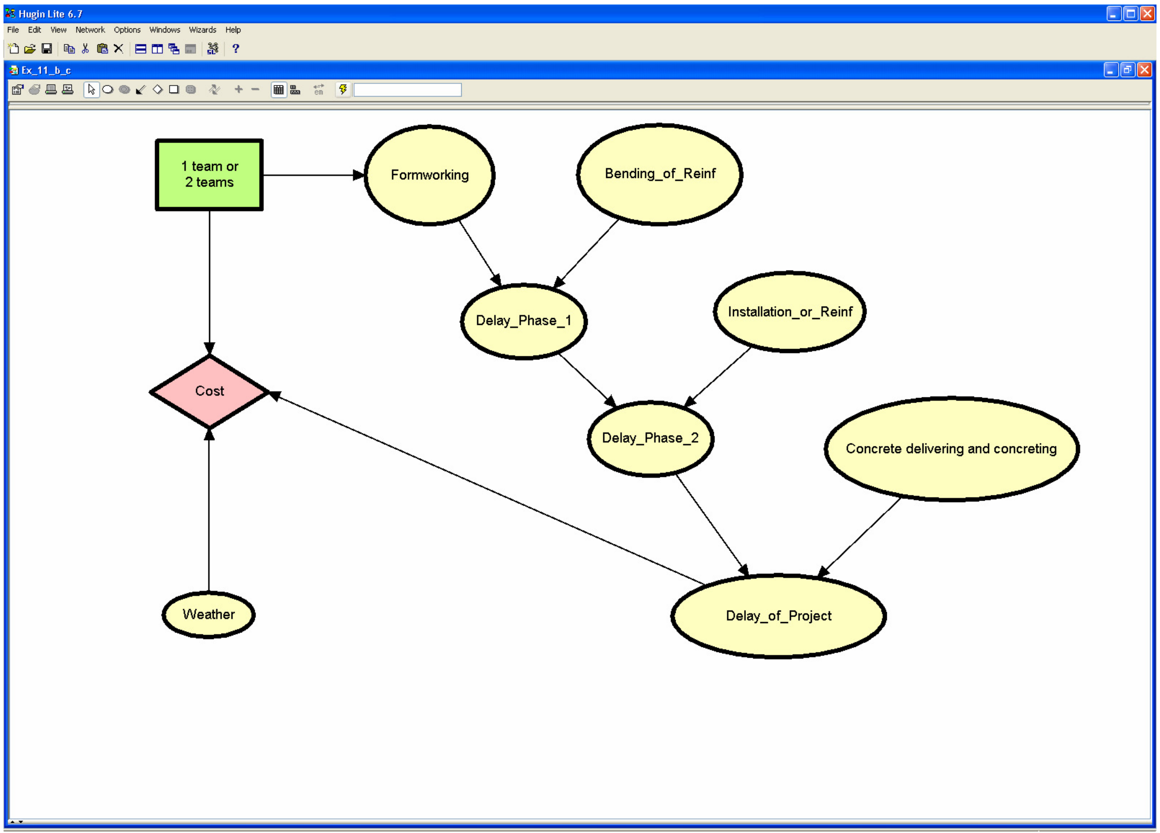
Download Hugin Lite and install it in your pc.

a. The problem described in Exercise 4 can be represented by a BPN as the one shown in the following Figure. The probabilities associated with the various states of each node can be found in the Hugin file provided together with the solution of this exercise.

After inserting all the necessary values save the file and click the compile button (the one that is like a thunder). You should then be able to view the results as th list of results on the left hand column of the following figure. It can be see that the probability o f delay of the construction project is equal to 41.86% (the same as the one estimated in Exercise 4).

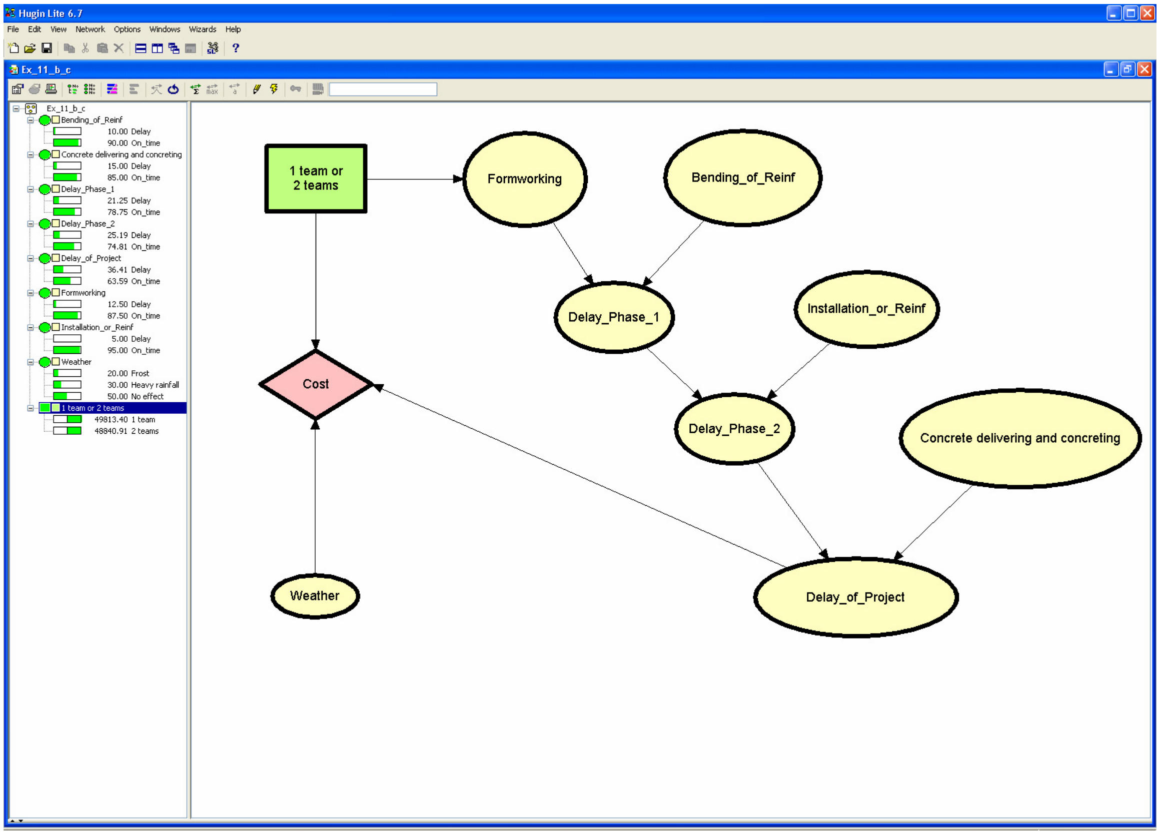


b and c. For parts b. and c. of Exercise 4 the following Bayesian Probabilistic net can be used:



It is seen that the use of 1 or 2 teams, the effect of the weather and the delay of the project are all linked to the node of "Cost" as they have an effect on that.

By completing the associated tables correctly the cost associated with the use of 1 or 2 teams is equal to 49813.40 CHF and 48840.91 CHF respectively. As it can be seen the expected cost of the delay reduces if the extra team for the form working is used.


Hugin Lite 6.7
File Edit View Network Options Windows Wizards Help
Ex_11_b_c
Ex_11_b_c
Bending_of_Reinf
10.00 Delay
90.00 On_time
Concrete delivering and concreting
15.00 Delay
85.00 On_time
Delay_Phase_1
21.25 Delay
78.75 On_time
Delay_Phase_2
25.19 Delay
74.81 On_time
Delay_of_Project
36.41 Delay
63.59 On_time
Formworking
12.50 Delay
87.50 On_time
Installation_or_Reinf
5.00 Delay
95.00 On_time
Weather
20.00 Frost
30.00 Heavy rainfall
50.00 No effect
1 team or 2 teams
49813.40 1 team
48840.91 2 teams
1 team or 2 teams
Formworking
Bending_of_Reinf
Delay_Phase_1
Installation_or_Reinf
Cost
Delay_Phase_2
Concrete delivering and concreting
Weather
Delay_of_Project

Assignment 13-Solution:

Exercise 13.1:

Based on previous knowledge the bending strength of timber is assumed normally distributed with standard deviation equal to $\sigma = 11\,MPa$ and uncertain mean μ assumed also normally distributed with $\mu' = 43.8\,\text{MPa}$ and standard deviation $\sigma' = 3.5\,\text{MPa}$.

Assume now that 10 four point tests are carried out with the results shown in Table 11.1.

Nr.	f_c [MPa]
1	31.5
2	38.3
3	36.6
4	38.2
5	35.1
6	44.9
7	50.8
8	42.5
9	39.6
10	36.9

Table 11.1 Density strength of the timber samples.

Based on the new test results, update the prior probabilistic model for the mean value of the bending strength. Plot the prior and posterior probability density functions.

Solution 13.1

Prior probabilistic model of the mean value of bending strength:

Bending strength is normally distributed with

Standard deviation, known: $\sigma = 11\,MPa$
Mean, uncertain, normally distributed: $\mu' = 43.8\,\text{MPa}$ and $\sigma' = 3.5\,\text{MPa}$

Based on the prior probabilistic model and the observations from the tests carried out we can update the probabilistic model for the mean and standard deviation of the mean value of the bending strength.

For normally distributed variables with known standard deviation and unknown mean it is:

$$\varphi_{\mu_f}(\mu_f) = \frac{1}{\sqrt{2\pi}\sigma''} exp\left(-\frac{1}{2}\left(\frac{\mu_f - \mu''}{\sigma''}\right)^2\right)$$

where

$$\mu'' = \frac{\frac{\mu'}{n} + \frac{\bar{x}}{n'}}{\frac{1}{n'} + \frac{1}{n}}$$

and

$$\sigma'' = \sqrt{\frac{\frac{\sigma_f^2}{n'} \cdot \frac{\sigma'^2}{n}}{\frac{\sigma'^2}{n'} + \frac{\sigma'^2}{n}}}$$

n is the sample size of the test carried out, that is $n = 10$, while n' is the sample size assumed for the prior distribution and is given as

$$n' = \frac{\sigma'^2{}_f}{\sigma'^2} = \frac{11^2}{3.5^2} = 9.89$$

$\bar{x}$ is the sample mean of the observations and that is

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{10}(31.5 + ... + 36.9) = 39.44 \text{ MPa}$$

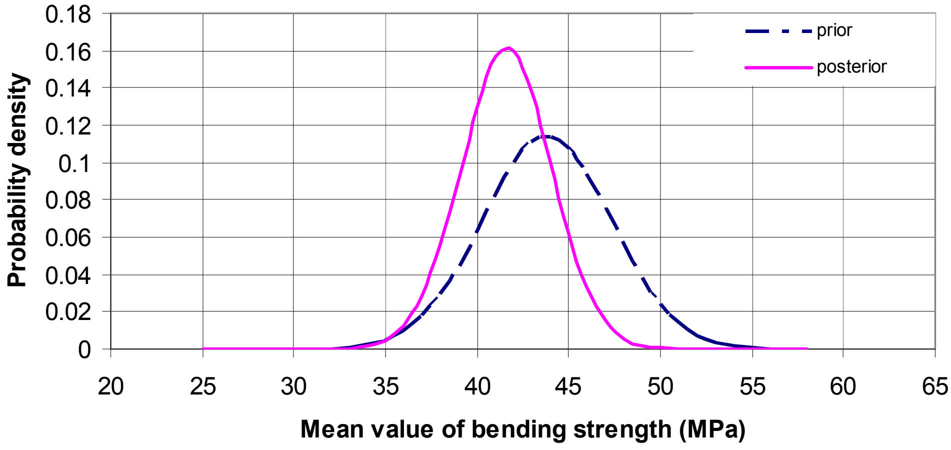
Based on the above the posterior mean and standard deviation are estimated:

$$\mu'' = \frac{\frac{\mu'}{n} + \frac{\bar{x}}{n'}}{\frac{1}{n'} + \frac{1}{n}} = \frac{\frac{43.8}{10} + \frac{39.44}{9.89}}{\frac{1}{9.89} + \frac{1}{10}} = 41.61 \text{ MPa}$$

and

$$\sigma'' = \sqrt{\frac{\frac{\sigma_f^2}{n'} \cdot \frac{\sigma'^2}{n}}{\frac{\sigma'^2}{n'} + \frac{\sigma'^2}{n}}} = \sqrt{\frac{\frac{11^2}{9.89} \cdot \frac{3.5^2}{10}}{\frac{11^2}{9.89} + \frac{3.5^2}{n}10}} = 2.47 \text{ MPa}$$

We can now plot the prior and posterior density functions as shown in Figure 11.1. It is seen that the effect of the test results is quite significant on the updating of the distribution.


0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
Probability density
20
25
30
35
40
45
50
55
60
65
Mean value of bending strength (MPa)
prior
posterior