Assessment 2
Statistics and probability theory

SS 2007
(With solutions)

Prof. Dr. M.H. Faber

ETH Zurich

Thursday 14th of June 2007
08:15 – 09:45

Surname: ....

Name: ....

Stud. Nr.: ....

Course of studies: ....

Date and duration:

Thursday, 14th of June 2007
Start: 8:15
End: 9:45
Duration: 90 minutes

Aids:

- No communication medium (e.g. cell phones, calculators with Bluetooth etc.)

Hints:

- Please control first, if you have received all the material (listed under: Contents).
- Please place your Legi on your desk.
- Please write your **name on every sheet of paper,** at the bottom left side**.**
- Use only the provided sheets of paper.
- When you have finished, place **all material** back in the envelope and raise your hand to call an assistant to collect it. You are allowed to leave until 9:15. If you finish later than this, please wait quietly until the end of the assessment (9:45).
- **Do not open** the paper fastener.
- Use tables attached at the end of this exercise sheets if necessary.
- The assessment consists of two parts and the points are given as follows:

Part 1: "Multiple choice"	: 18 questions $\times$ 5 points/question =		90 points.
Part 2: "Exercise":	: 5 questions $\times$ 6 points/question =		30 points.
		Total:	120 points.

Contents

- General information, exercises and tables (13 pages. **Page is not collect, since the solution is implemented*).
- 1 sheet of paper (checkered)

Part 1: „Multiple Choice“ (90 points)

In answering the following multiple choice questions it should be noted that for some of the questions several answers may be correct. Tick **ALL** the correct alternative(s) in every question as: ☑

In each multiple-choice question, 5 points are given if and only if all alternatives are ticked correctly. Otherwise, 0 point is given.

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1. The χ^2-goodness of fit test can be used for:

estimating unknown parameters of an assumed family of distribution. ☐

checking the appropriateness of an assumed distribution. ☑

ranking the goodness of fit of several distributions. ☐

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2. According to the specification of a measurement device, the measurement error of the device is supposed to follow the Normal distribution. The true standard deviation of the measurement error is not known. In order to judge, by a hypothesis test, the null hypothesis that the true mean of the measurement error is equal to μ, an engineer plans to take 10 samples and obtain the sample mean $\overline{X}$ and the sample (unbiased) standard deviation $S_{unbiased}$. Which of the following sample statistic is appropriate in the hypothesis test?

$T = \dfrac{\overline{X} - \mu}{S_{unbiased} / \sqrt{10}}$ ☑

$T = \dfrac{\overline{X} - \mu}{S_{unbiased} / 10}$ ☐

$T = \dfrac{\overline{X} / 10 - \mu}{S_{unbiased} / \sqrt{10}}$ ☐

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3. A company checks the safety of a bridge. The type of error that they would make in case they *erroneously* reject the null hypothesis that the bridge is safe is a:

type I error. ☑

type II error. ☐

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4. A new medicine is to be tested against adverse effects before it is allowed to be sold in pharmacies. The null hypothesis is given as:

   H_0: The new medicine has adverse effects.

   Which of the following statement(s) is (are) correct?

   The type I error has to be kept very low in order to ensure small risks associated with the event that medicine will be allowed to be sold, although it has adverse effects. ☑

   The type II error will lead to the judgment that the medicine is not allowed to be sold, although it has no adverse effects. ☑

   None of the above. ☐

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5. In order to check the quality of concrete delivered at a construction site, it is planned to collect n concrete cylinder samples and measure their compressive strength. From past experience, the true value of the variance of the compressive strength is known to be σ^2. Which of the following statement(s) is(are) correct?

   The expected value of the sample variance before the strength measurements are carried out is equal to $\frac{n-1}{n}\sigma^2$. ☑

   The expected value of the sample variance after the strength measurements have been completed always becomes equal to the true variance σ^2. ☐

   The expected value of the sample variance before as well as after the strength measurements are carried out remains the same as $\frac{n-1}{n}\sigma^2$ ☐

   Once the strength measurements have been completed, the sample variance is no longer a random variable and has a fixed value calculated from the measurement results. ☑

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6. Which one(s) of the following statement(s) is(are) correct?

   The probability paper can be used to check whether an assumed family of probability distribution can properly model the data. ☑

   The probability paper can be used to check whether a pair of data sets is linearly related. ☐

   None of the above. ☐

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7. The traveling time of ETH students from home to the ETH is to be modeled with a Normal distributed random variable with the mean μ_X and the standard deviation σ_X. When n samples x_i, ($i = 1,2,3,\ldots,n$) are obtained to estimate the parameters μ_X and σ_X, which of the following likelihood function is (are) correct?

$$L(\mu_X, \sigma_X) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\cdot\sigma_X} \exp\left(-\frac{(x_i - \mu_X)^2}{2\cdot\sigma_X^{\ 2}}\right).$$ ☑

$$L(\mu_X) = \prod_{i=1}^{n} \lambda \exp(-\lambda x_i), \text{where } \lambda = \frac{1}{\mu_X}.$$ ☐

$$L(\mu_X, \sigma_X) = \sum_{i=1}^{n} \frac{1}{\sqrt{2\pi}\cdot\sigma_X} \exp\left(-\frac{(x_i - \mu_X)^2}{2\cdot\sigma_X^{\ 2}}\right).$$ ☐

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8. The traveling time by bus from Rapperswil to Zurich is represented by the random variable X, and is known to be Normal distributed. The standard deviation σ_X is assumed to be known as 10 minutes. In the next 30 days a traveler plans to measure the traveling time every day and will obtain a sample mean $\overline{X}$. Which of the following equation(s) is (are) to be used to establish a confidence interval of the mean value μ_X, if the confidence interval is to be estimated with a significance level of $\alpha = 5\%$?

$$P\left[-k_{\alpha/2}\frac{10}{\sqrt{30}} < \overline{X} - \mu_X < \frac{10}{\sqrt{30}}k_{\alpha/2}\right] = 0.95$$ ☑

$$P\left[-k_{\alpha/2}\frac{10}{\sqrt{30}} < \overline{X} - \mu_X < \frac{10}{\sqrt{30}}k_{\alpha/2}\right] = 0.05$$ ☐

$$P\left[-k_{\alpha/2} < \frac{\overline{X} - \mu_X}{10} < k_{\alpha/2}\right] = 0.95$$ ☐

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9. Weights of trucks are measured before the trucks cross a highway bridge. Assume that the standard deviation of truck weights is known to be 3 tons. How many sample observations of truck weights are approximately required in order that the sample mean lies within ± 1 ton of the true mean value with 99% confidence?

45. ☐

54. ☐

60. ☑

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10. The exterior of a building consists of one hundred 3m x 5m glass panels. Past records indicate that on average, one flaw is found in every 50 m^2 of this type of glass panels. A panel containing two or more flaws will cause breakage problems and has to be replaced. If the occurrence of flaws can be assumed to be a Poisson process, the probability that a given panel will be replaced is equal to:

    0.037. ☑

    0.143. ☐

    0.259. ☐

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11. The speed of vehicles on a highway is to be investigated. The standard deviation of the speed of vehicles is known to be 6 km/h. Consider that two persons – Martin and Anna – are assigned the tasks to collect data on the speed of vehicles on this highway. Martin and Anna will collect data for n vehicles respectively. The sample means can be assumed to be Normal distributed with mean μ (unknown true mean speed) and standard deviation $6/\sqrt{n}$. When Martin and Anna separately (statistically independently) collect data for 10 vehicles, the probability that Martin's sample mean will exceed Anna's sample mean by at least 2 km/h is equal to

    0.009. ☐

    0.015. ☐

    0.165. ☐

    0.228. ☑

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12. The occurrence of traffic accidents at a junction of a highway system may be modeled as a Poisson process. Based on past records, the average rate of accidents is 1/3 per year. It is assumed that in every accident at this junction, the probability that there is at least one fatality is 5%. The probability that there is at least one fatality at this junction over a period of 3 years is equal to:

    0.017. ☐

    0.049. ☑

    0.189. ☐

    0.263. ☐

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13. The following values were obtained from the compressive strength tests carried out on 5 concrete test specimens: 29, 23, 24, 28 and 26 N/mm^2. Assume that the standard deviation of the compressive strength is known to be 3 N/mm^2. If the sample mean is to lie within ± 2 N/mm^2 of the true mean value, the corresponding confidence level based on the 5 measurements given above is equal to:

- 81.1%. ☐
- 86.4%. ☑
- 93.4%. ☐

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14. Z_1 and Z_2 are two independent Chi-Square distributed random variables with 3 and 4 degrees of freedom respectively. The variable Z is then defined as $Z = Z_1 + Z_2$. Which of the following values exceed the 92.5% quantile of the variable Z?

- 12.1. ☐
- 13.9. ☑
- 14.5. ☑
- 16.7. ☑

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15. The lower bound fracture strength of a welded joint is 28 N/mm^2. If the actual strength of the joint is modeled with the Type III extreme minimum value distribution with parameters u = 105 N/ mm^2 and k = 1.75, the probability that the strength of the joint will be at least 115 N/mm^2 is equal to:

- 0.13. ☐
- 0.29. ☑
- 0.45. ☐

Hint: The cumulative distribution function of the Type III extreme minimum value distribution is written as:

$$F_X(x) = 1 - \exp\left(-\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^k\right),$$

where ε is a lower bound.

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16. Which of the following statement(s) about the t-distribution is(are) correct?

The shape of the probability density function curve for a t-distribution is symmetric and bell-shaped like the normal distribution and has its peak at 0. ☑

The standard deviation of a t-distributed random variable becomes larger, as the degrees of freedom become larger. ☐

The t-distribution converges to the standard normal distribution as the degrees of freedom increase. ☑

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17. An engineer wants to check if the results of an experiment support his theory. For this purpose, he made 10 measurements on the characteristic parameter relevant to his theory. Figure 1 shows the comparison between the cumulative distribution function of the parameter derived from his theory and the observed cumulative distribution function.

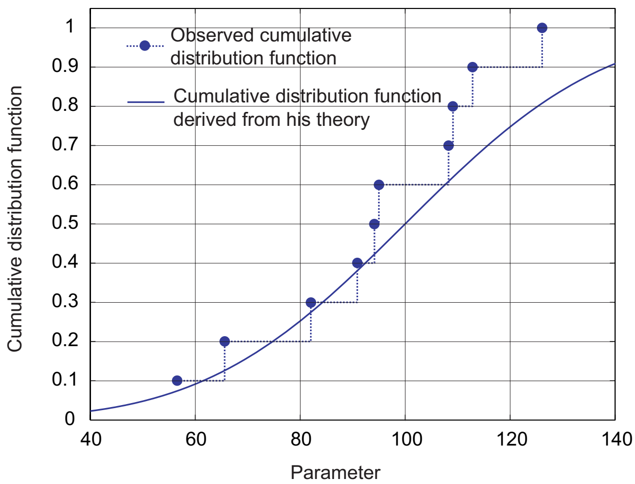


Figure 1: Comparison between the cumulative distribution function of the parameter derived from his theory and the observed cumulative distribution function.

Answer the following questions **17.1** and **17.2** using the above information.

17.1. The sample statistic for the Kolmogorov-Smirnov test is obtained from Figure 1 as:

0.11. ☐

0.23. ☑

25. ☐

47. ☐

17.2. Using the Kolmogorov-Smirnov test the hypothesis that his theory is correct is:

rejected at the 1% significance level. ☐

rejected at the 5% significance level. ☐

accepted (cannot be rejected) at the 5% significance level. ☑

Part 1: Full Solution

1. Check in script Section E.7.

2. See "Testing of the mean with unknown variance", page E-14 in the script for the sample statistic to use in this case.

3. A picture to keep in mind concerning the type I and type II errors is the one from the exercise tutorial 9, slide 4 (see webpage). As in this case, erroneously rejecting the null hypothesis means that H_0 would be the real state, and by rejecting it and so accepting H_1, we make a type I error. Check also the script, section E.4, page E.13 the definitions of the type I and II errors.

4. Check in script, section E.4, page E.13 the definitions of the type I and II errors.

5 See the Equation (E.19) for the expectation of the variance on page E-9. As can be seen from the Equation (E.19), the sample size has an influence on the expected value of the variance.

Remember the difference between a random variable as a parameter carrying the entire uncertain input to a considered model, as a realization of a random variable is a real value (D-4).

6 For the probability paper, see section E.5.

7. The likelihood function is a product. The method is aiming to fit the parameters μ_X and σ_X of the Normal distribution (in this case) so as to maximise the probability (likelihood) of the observed random sample. Check also in script Equation (E.47)

8. See "Confidence Intervals on Estimators", page E-10, Equation (E.22).

9. The double sided and symmetrical $1-\alpha$ confidence interval for the mean value is given by:

$$P\left[-k_{\alpha/2} < \frac{\bar{X} - \mu_X}{\sigma_X \frac{1}{\sqrt{n}}} < k_{\alpha/2}\right] = 1-\alpha \qquad \text{Script Equation (E.22)}$$

From the given infor*mation*,

$1-\alpha = 99\% \Rightarrow \alpha = 0.01$ and $k_{\alpha/2} = 2.5758$

$\bar{X} - \mu_X = \pm 1$

$\sigma_X = 3$ tons

$n = ??$

$$P\left[-2.5758 < \frac{\pm 1}{3\frac{1}{\sqrt{n}}} < 2.5758\right] = 1-0.01$$

Solving the above equation, n is obtained as approximately equal to 60

10. The mean occurrence rate of flaws for one 3m x 5m glass panel is given by: $\int_0^{15} \frac{1}{50} dx = \frac{15}{50} = 0.3$

Let X represent the number of flaws in a glass panel. If a panel has two or more flaws, it has to be replaced.
Therefore the probability that a given panel will be replaced is:

$$P(X \geq 2) = 1 - P(X \leq 2) = 1 - \left(P(X=0) + P(X=1)\right)$$
$$= 1 - \left(\frac{(0.3)^0}{0!} e^{-0.3} + \frac{(0.3)^1}{1!} e^{-0.3}\right) = 0.037$$

11. Let M and A be variables representing the sample mean speeds measured by Martin and Anna respectively.

Let X be a variable representing the difference between the sample mean speeds measured by Martin and Anna. Since M and A are Normal distributed, X is also Normal distributed.

$X = M - A$

$\mu_X = \mu_M - \mu_A = \mu - \mu = 0$

${\sigma_X}^2 = {\sigma_M}^2 + {\sigma_A}^2 - 2C_{MA}$ where C_{MA} is the covariance between M and A.

Since M and A are independent, C_{MA} is zero.

$${\sigma_X}^2 = {\sigma_M}^2 + {\sigma_A}^2 = \frac{36}{n} + \frac{36}{n}$$

For n=10, $\sigma_X = 2.6833$

The probability that Martin's sample mean will exceed Anna's sample mean by at least 2 km/h needs to be determined. This can be expressed as $P(X \geq 2)$.

$$P(X \geq 2) = 1 - P(X \leq 2) = 1 - P\left(\frac{X - 0}{2.6833} \leq \frac{2 - 0}{2.6833}\right)$$
$$= 1 - \Phi(0.7454) = 0.228$$

12. A rough estimate of the probability can be obtained as follows:
The average rate of accidents at the traffic junction is given as 1/3 per year. Over a 3-year period, this means there is 1 accident on an average. Further it is given that in every accident at this junction, the probability that there is at least one fatality is 5%. Hence the probability that there is at least one fatality at this junction over a period of 3 years can be estimated as 5% or 0.05 (close to the choice of 0.049).

The exact value of the probability can be calculated as follows:

Probability(at least one fatality in 3 years) = 1 - Probability(no fatality in 3 years)

Let N represent the number of accidents at the junction.

P(no fatality in 3 years) = P(N=0 in 3 years) + $\sum_{\mathrm{i}=1}^{\infty}$P(no fatality|N=i) P(N=i)

The occurrence of accidents follows a Poisson process.

The mean occurrence rate of accidents for a 3 year period is: v = $\int_0^3 \frac{1}{3}dt = 1$

The probability of fatality at an accident is given as 5% or 0.05.

Hence the probability that there is no fatality at an accident is 0.95.

$$\begin{aligned}\text{P(no fatality in 3 years)} &= \frac{1^0}{0!}e^{-1} + \sum_{\mathrm{i}=1}^{\infty}(0.95)^{\mathrm{i}}\frac{1^i}{i!}e^{-1} = e^{-1}\left(1+\sum_{\mathrm{i}=1}^{\infty}\frac{(0.95)^{\mathrm{i}}}{i!}\right)\\ &= e^{-1}\left(\sum_{\mathrm{i}=0}^{\infty}\frac{(0.95)^{\mathrm{i}}}{i!}\right) = e^{-1}e^{0.95} \text{ (Taylor series expansion of } e^x \text{ is } \sum_{i=0}^{\infty}\frac{x^i}{i!})\\ &= 0.9512\end{aligned}$$

$$\begin{aligned}\text{Probability(at least one fatality in 3 years)} &= 1 - \text{Probability(no fatality in 3 years)}\\ &= 1-0.951 = 0.049\end{aligned}$$

13. The double sided and symmetrical 1-α confidence interval for the mean value is given by:

$$P\left[-k_{\alpha/2} < \frac{\bar{X}\text{ - }\mu_X}{\sigma_X\,\frac{1}{\sqrt{n}}} < k_{\alpha/2}\right] = 1-\alpha \qquad \textit{Script Equation E.22}$$

From the given infor*mation*,

$\bar{X}$ - $\mu_X = \pm 2$

$\sigma_X = 3$ tons

$n = 5$

$\alpha = ??$

$$P\left[-k_{\alpha/2} < \frac{\pm 2}{3\frac{1}{\sqrt{5}}} < k_{\alpha/2}\right] = 1-\alpha$$

From the above equation, $k_{\alpha/2}$ is obtained as 1.491

The corresponding α is 0.136 or 13.6%. Hence the confidence level is 86.4%

14. The Chi-Square distribution is regenerative and hence the sum of two Chi-Square distributed variables i.e. $Y_{n_1} + Y_{n_2}$ is also Chi-Square distributed with $n_1 + n_2$ degrees of freedom. (Refer to Script Section E.2 – The Chi-Square (χ^2) - Distribution)

Hence in this case, Z is a Chi-Square distributed variable with 3+4 = 7 degrees of freedom.

From Script Table T.3, the 90% and 95% quantile values of the Chi-Square distribution for 7 degrees of freedom are 12.0170 and 14.0671.

The 92.5% quantile value of the Chi-Square distribution for 7 degrees of freedom is hence obtained as 13.0421.

From the given choices, the values 13.9, 14.5 and 16.7 exceed the 92.5% quantile of the variable Z .

15. Let the strength of the joint be represented by the variable X.

Probability (strength of the joint will be at least 115 N/mm^2) = $P(X \geq 115)$

$$P(X \geq 115) = 1\text{-}P(X \leq 115) = 1 - F_X(x = 115)$$

$$= 1 - \left(1 - \exp\left(-\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^k\right)\right) = \exp\left(-\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^k\right)$$

From the given information,

$x = 115,\ \varepsilon = 28,\ u = 105,\ k = 1.75$

Substituting the above values,

$P(X \geq 115) = 0.29$

16. The probability density function of t-distribution is given Equation (E.8) in the script:

$$f_S(s) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\,\Gamma(n/2)}\left(1 + \frac{s^2}{n}\right)^{-(n+1)/2}$$

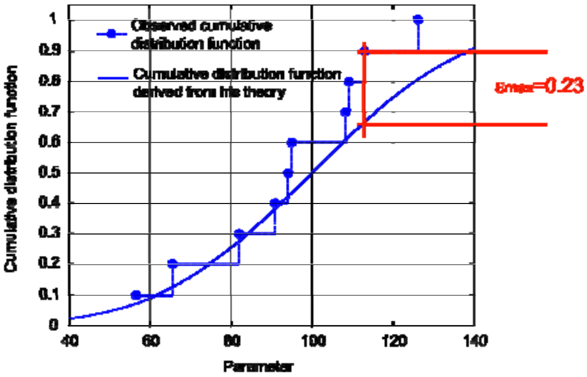
It converges to the standard Normal distribution when the number of the degrees of freedom is large, because as $n \to \infty$,

$$\frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \approx \left(\frac{n}{2}\right)^{1/2}$$

and

$$\left(1 + \frac{s^2}{n}\right)^{-(n+1)/2} = \left(\left(1 + \frac{s^2}{n}\right)^{-n}\right)^{-1/2}\left(1 + \frac{s^2}{n}\right)^{-1/2} \to e^{-s^2/2}.$$

17.1. From the figure, the sample statistic of the Kolmogorov-Smirnov test is obtained as 0.23.



17.2. Since the value of the sample statistic is smaller than 0.409, the null hypothesis cannot be rejected at the 5% level of significance. (Then it is also true that the null hypothesis cannot be rejected at the 1% level of significance). See the following table (a part of Table T.4 in the script).

n	α=0.01	0.02	0.05	0.1	0.2
1	0.995	0.990	0.975	0.950	0.900
2	0.929	0.900	0.842	0.776	0.684
3	0.829	0.785	0.708	0.636	0.565
4	0.734	0.689	0.624	0.565	0.493
5	0.669	0.627	0.563	0.509	0.447
6	0.617	0.577	0.519	0.468	0.410
7	0.576	0.538	0.483	0.436	0.381
8	0.542	0.507	0.454	0.410	0.358
9	0.513	0.480	0.430	0.387	0.339
10	0.489	0.457	0.409	0.369	0.323
11	0.468	0.437	0.391	0.352	0.308
12	0.449	0.419	0.375	0.338	0.296
13	0.432	0.404	0.361	0.325	0.285
14	0.418	0.390	0.349	0.314	0.275
15	0.404	0.377	0.338	0.304	0.266

Table T.4: Critical values of the Kolmogorov-Smirnov test.

Part 2: „Exercise: Extreme value distribution and return period" (30 points)

Assume that the annual maximum earthquake load effect X follows the Gumbel distribution with the mean $\mu_X = 10 N/mm^2$ and the standard deviation $\sigma_X = 3.0 N/mm^2$. The cumulative distribution function of the Gumbel distribution and the relationships between the parameters of the distribution and the mean and the standard deviation are summarized in Table 1. Assume that the annual maxima of the earthquake load effect are independent. Answer the following questions.

Table 1: Characteristics of the Gumbel distribution.

	Gumbel distribution
Cumulative distribution function	$F_X(x) = \exp\left(-\exp\left(-\alpha(x-u)\right)\right)$
Moments	$\mu_X = u + \frac{0.5772}{\alpha}$
	$\sigma_X = \frac{\pi}{\alpha\sqrt{6}}$

1) Calculate the parameters u and α of the cumulative distribution function of the annual maximum earthquake load effect X.

2) Calculate the probability that the earthquake load effect exceeds $20 N/mm^2$ in one year.

3) Calculate the magnitude of the earthquake load effect with 100-year return period.

4) Calculate the cumulative distribution function of the 50-year maximum earthquake load effect.

5) Calculate the probability that the earthquake load effect exceeds $20 N/mm^2$ in 50 years.

Solutions:

1) Solving the following equations:

$$\begin{cases} u+\dfrac{0.5772}{\alpha}=10 \\ \dfrac{\pi}{\alpha\sqrt{6}}=3 \end{cases}$$

we obtain $u=8.65$ and $\alpha=0.43$.

2) $P[X>20]=1-F_X(20)=1-\exp\left(-\exp\left(-0.43(20-8.65)\right)\right)=0.0076$.

3) Solving the following equation:

$F_X(x_{100})=\exp\left(-\exp\left(-0.43(x_{100}-8.65)\right)\right)=1-1/100$

we obtain $x_{100}=19.35$.

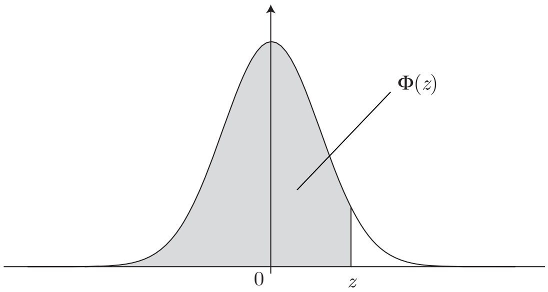
4) The cumulative distribution function of the 50-year maxima Y is:

$$F_Y(y)=\left(F_X(y)\right)^{50}=\exp\left(-50\exp\left(-0.43(y-8.65)\right)\right)$$
$$=\exp\left(-50\exp\left(-0.43y+7.43\right)\right)$$

5) The probability that the earthquake load effect exceeds $20N/mm^2$ in 50 years is:

$P[Y>20]=1-F_Y(20)=1-\exp\left(-50\exp\left(-0.43(20-8.65)\right)\right)=0.32$.

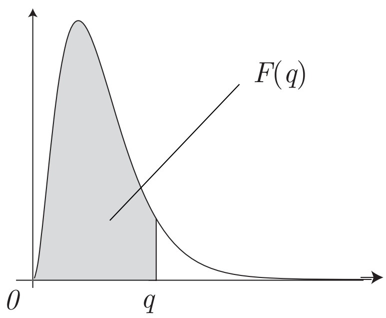
Table A.1: Cumulative distribution function of the standard Normal distribution $\Phi(z)$.



Probability density function of the standard normal random variable.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.9772
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.10	0.9821356
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.20	0.9860966
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.30	0.9892759
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.40	0.9918025
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.50	0.9937903
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.60	0.9953388
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.70	0.9965330
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.80	0.9974449
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.90	0.9981342
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	3.00	0.9986501
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	3.10	0.9990324
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	3.20	0.9993129
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	3.30	0.9995166
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	3.40	0.9996631
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	3.50	0.9997674
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	3.60	0.9998409
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	3.70	0.9998922
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	3.80	0.9999277
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	3.90	0.9999519
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	4.00	0.9999683
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	4.10	0.9999793
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	4.20	0.9999867
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	4.30	0.9999915
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	4.40	0.9999946
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	4.50	0.9999966
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	4.60	0.9999979
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	4.70	0.9999987
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	4.80	0.9999992
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	4.90	0.9999995
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	5.00	0.9999997
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649		
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656		
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664		
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671		
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678		
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686		
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693		
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699		
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706		
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713		
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719		
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726		
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732		
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738		
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744		
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750		
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756		
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761		
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767		

Table A.2: Quantile values of the Chi-square distribution q.



Probability density function of Chi-square distribution.

ν	F(q)=0.75	0.90	0.95	0.98	0.99	0.995	0.999
1	1.3233	2.7055	3.8415	5.4119	6.6349	7.8794	10.8276
2	2.7726	4.6052	5.9915	7.8240	9.2103	10.5966	13.8155
3	4.1083	6.2514	7.8147	9.8374	11.3449	12.8382	16.2662
4	5.3853	7.7794	9.4877	11.6678	13.2767	14.8603	18.4668
5	6.6257	9.2364	11.0705	13.3882	15.0863	16.7496	20.5150
6	7.8408	10.6446	12.5916	15.0332	16.8119	18.5476	22.4577
7	9.0371	12.0170	14.0671	16.6224	18.4753	20.2777	24.3219
8	10.2189	13.3616	15.5073	18.1682	20.0902	21.9550	26.1245
9	11.3888	14.6837	16.9190	19.6790	21.6660	23.5894	27.8772
10	12.5489	15.9872	18.3070	21.1608	23.2093	25.1882	29.5883
11	13.7007	17.2750	19.6751	22.6179	24.7250	26.7568	31.2641
12	14.8454	18.5493	21.0261	24.0540	26.2170	28.2995	32.9095
13	15.9839	19.8119	22.3620	25.4715	27.6882	29.8195	34.5282
14	17.1169	21.0641	23.6848	26.8728	29.1412	31.3193	36.1233
15	18.2451	22.3071	24.9958	28.2595	30.5779	32.8013	37.6973
16	19.3689	23.5418	26.2962	29.6332	31.9999	34.2672	39.2524
17	20.4887	24.7690	27.5871	30.9950	33.4087	35.7185	40.7902
18	21.6049	25.9894	28.8693	32.3462	34.8053	37.1565	42.3124
19	22.7178	27.2036	30.1435	33.6874	36.1909	38.5823	43.8202
20	23.8277	28.4120	31.4104	35.0196	37.5662	39.9968	45.3147
21	24.9348	29.6151	32.6706	36.3434	38.9322	41.4011	46.7970
22	26.0393	30.8133	33.9244	37.6595	40.2894	42.7957	48.2679
23	27.1413	32.0069	35.1725	38.9683	41.6384	44.1813	49.7282
24	28.2412	33.1962	36.4150	40.2704	42.9798	45.5585	51.1786
25	29.3389	34.3816	37.6525	41.5661	44.3141	46.9279	52.6197
26	30.4346	35.5632	38.8851	42.8558	45.6417	48.2899	54.0520
27	31.5284	36.7412	40.1133	44.1400	46.9629	49.6449	55.4760
28	32.6205	37.9159	41.3371	45.4188	48.2782	50.9934	56.8923
29	33.7109	39.0875	42.5570	46.6927	49.5879	52.3356	58.3012
30	34.7997	40.2560	43.7730	47.9618	50.8922	53.6720	59.7031

ν : Degrees of freedom.

Table A.3: Critical values of the Kolmogorov-Smirnov test.

n	α=0.01	0.02	0.05	0.1	0.2
1	0.995	0.990	0.975	0.950	0.900
2	0.929	0.900	0.842	0.776	0.684
3	0.829	0.785	0.708	0.636	0.565
4	0.734	0.689	0.624	0.565	0.493
5	0.669	0.627	0.563	0.509	0.447
6	0.617	0.577	0.519	0.468	0.410
7	0.576	0.538	0.483	0.436	0.381
8	0.542	0.507	0.454	0.410	0.358
9	0.513	0.480	0.430	0.387	0.339
10	0.489	0.457	0.409	0.369	0.323
11	0.468	0.437	0.391	0.352	0.308
12	0.449	0.419	0.375	0.338	0.296
13	0.432	0.404	0.361	0.325	0.285
14	0.418	0.390	0.349	0.314	0.275
15	0.404	0.377	0.338	0.304	0.266
16	0.392	0.366	0.327	0.295	0.258
17	0.381	0.355	0.318	0.286	0.250
18	0.371	0.346	0.309	0.279	0.244
19	0.361	0.337	0.301	0.271	0.237
20	0.352	0.329	0.294	0.265	0.232
21	0.344	0.321	0.287	0.259	0.226
22	0.337	0.314	0.281	0.253	0.221
23	0.330	0.307	0.275	0.248	0.217
24	0.323	0.301	0.269	0.242	0.212
25	0.317	0.295	0.264	0.238	0.208
26	0.311	0.290	0.259	0.233	0.204
27	0.305	0.284	0.254	0.229	0.200
28	0.300	0.279	0.250	0.225	0.197
29	0.295	0.275	0.246	0.221	0.194
30	0.290	0.270	0.242	0.218	0.190
31	0.285	0.266	0.238	0.214	0.187
32	0.281	0.262	0.234	0.211	0.185
33	0.277	0.258	0.231	0.208	0.182
34	0.273	0.254	0.227	0.205	0.179
35	0.269	0.251	0.224	0.202	0.177
36	0.265	0.247	0.221	0.199	0.174
37	0.262	0.244	0.218	0.196	0.172
38	0.258	0.241	0.215	0.194	0.170
39	0.255	0.238	0.213	0.192	0.168
40	0.252	0.235	0.210	0.189	0.166
$n > 40$	$1.63/\sqrt{n}$	$1.52/\sqrt{n}$	$1.36/\sqrt{n}$	$1.22/\sqrt{n}$	$1.07/\sqrt{n}$

α : Significance level.
n : Sample size.