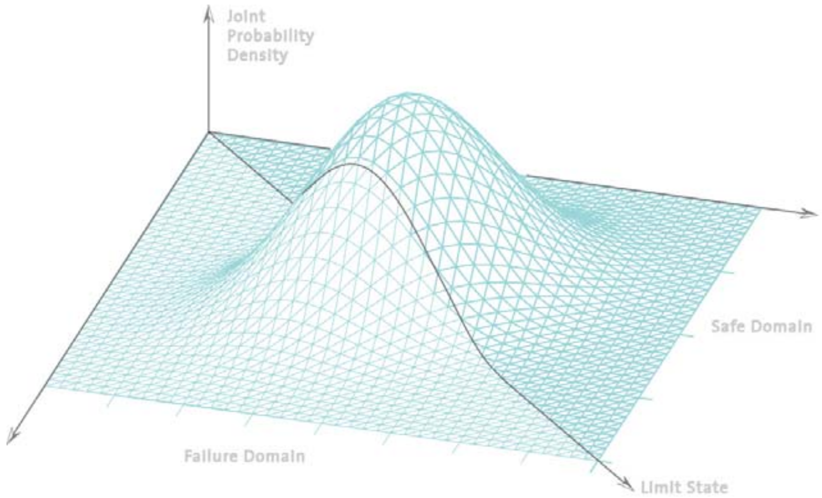

Exercises Tutorial 4

Statistics and Probability Theory

Prof. Dr. Michael Havbro Faber
Swiss Federal Institute of Technology Zurich
ETHZ

Exercise 3.3

- Make a Q-Q plot (Quantile-Quantile plot) to compare the two data sets of Table 3.1.1.
- What do you observe in regard to the traffic flows in directions 1 and 2?
- Provide an approximate value of the difference in the daily traffic flow between the two directions using a Tukey mean-difference plot.

Date	Direction 1	Direction 2
01.04.2001	32618	24609
02.04.2001	33380	29965
03.04.2001	34007	30629
04.04.2001	33888	30263
05.04.2001	35237	31405
06.04.2001	35843	31994
07.04.2001	33197	26846
08.04.2001	30035	22762
09.04.2001	32158	30366
10.04.2001	33406	29994
11.04.2001	34576	30958
12.04.2001	34013	30680
13.04.2001	24846	19735
14.04.2001	28252	21145
15.04.2001	25365	17805
16.04.2001	24862	18123
17.04.2001	32472	28117
18.04.2001	33245	28858
19.04.2001	33788	29080
20.04.2001	34076	30313
21.04.2001	29976	23141
22.04.2001	29224	20903
23.04.2001	32962	27746
24.04.2001	33937	29586
25.04.2001	33198	30788
26.04.2001	34455	31074
27.04.2001	35852	32384
28.04.2001	33091	26525
29.04.2001	30613	22828
30.04.2001	34425	28877

Exercise 3.3

- Make a Q-Q plot (Quantile-Quantile plot) to compare the two data sets of Table 3.1.1.
- What do you observe in regard to the traffic flows in directions 1 and 2?

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01.04.2001	32618	24609
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Steps

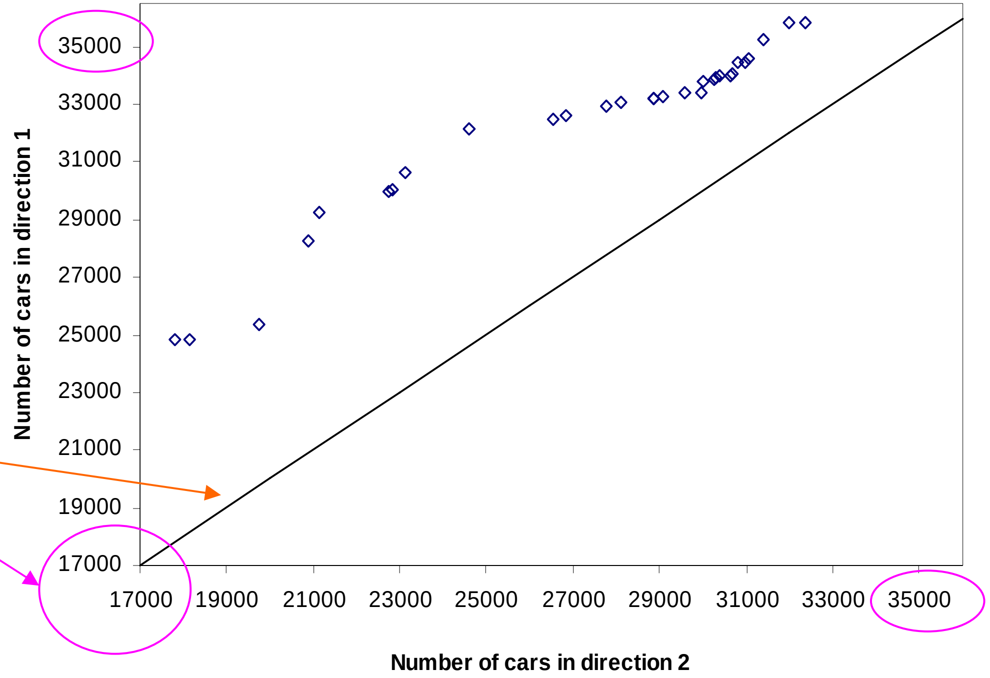
1. sort the data (if not sorted)
2. If $n_x = n_y$ plot the data in an x-y system using the same scale and origin for x and y
3. Draw the line x=y
4. Compare the two data sets

Exercise 3.3

- Make a Q-Q plot (Quantile-Quantile plot) to compare the two data sets of Table 3.1.1.

Steps

1. sort the data (if not sorted)
2. If $n_x = n_y$ plot the data in an x-y system using the same scale and origin for x and y
3. Draw the line x=y (symmetry line)
4. Compare the two data sets



Exercise 3.3

- Make a Q-Q plot (Quantile-Quantile plot) to compare the two data sets of Table 3.1.1.

Steps

1. sort the data (if not sorted)
2. If $n_x = n_y$ plot the data in an x-y system using the same scale and origin for x and y
3. Draw the line x=y (symmetry line)
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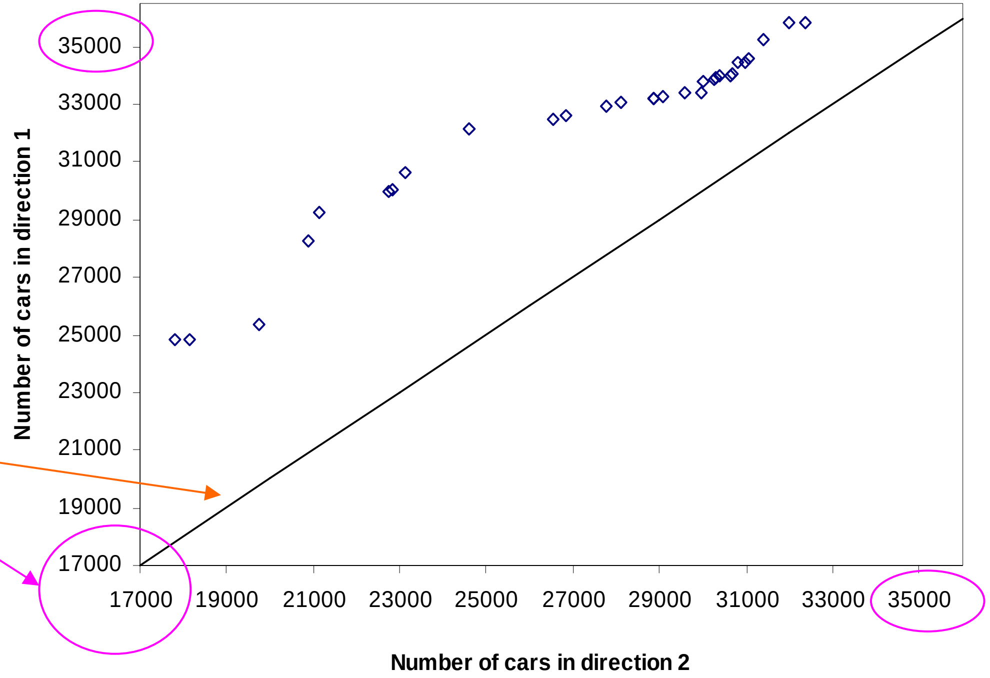

The data lie far from the symmetry line

Concentrated on the side of direction 1- higher traffic flow in direction 1

Exercise 3.3

- Provide an approximate value of the difference in the daily traffic flow between the two directions using a Tukey mean-difference plot.

Date	Direction 1	Direction 2
01.04.2001	32618	24609
02.04.2001	33380	29965
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30.04.2001	34425	28877

Steps

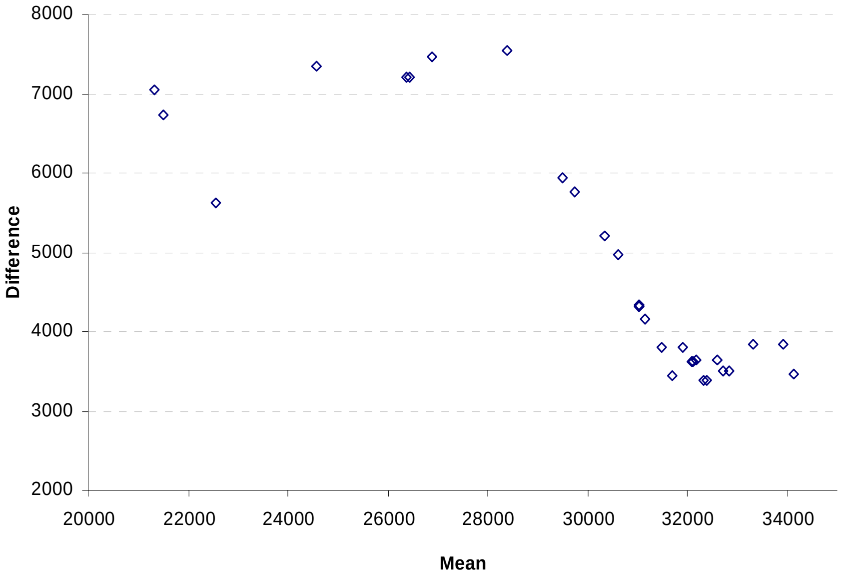
1. sort the data (if not sorted)
2. Calculate $y_i - x_i$ and plot it on the y-axis
3. Calculate $(y_i + x_i)/2$ and plot it on the x-axis
4. Discuss…

Exercise 3.3

- Provide an approximate value of the difference in the daily traffic flow between the two directions using a Tukey mean-difference plot.

Steps

1. sort the data (if not sorted)
2. Calculate $y_i - x_i$ and plot it on the y-axis
3. Calculate $(y_i + x_i)/2$ and plot it on the x-axis



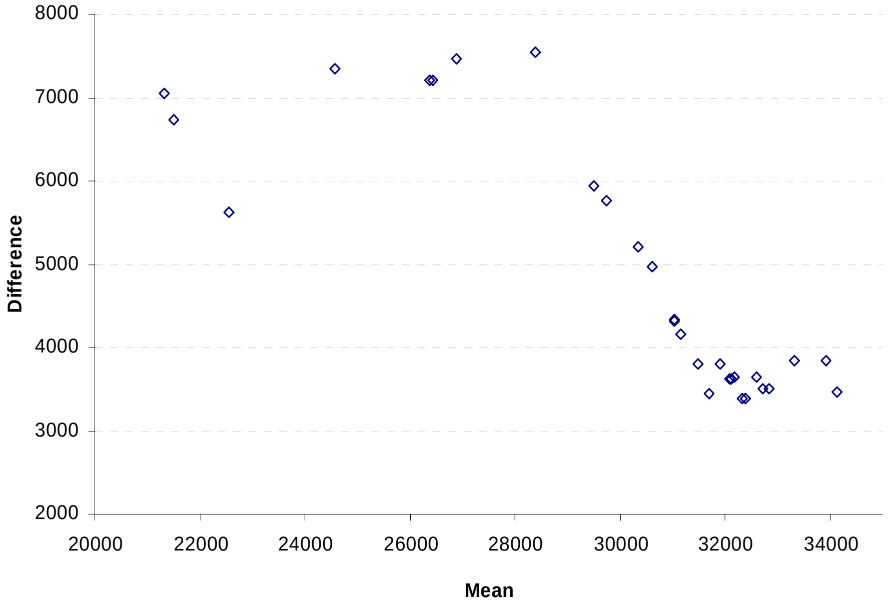
x_i	y_i	$y_i - x_i$	$(y_i + x_i)/2$
Direction 2	**Direction 1**	**y_i-x_i**	**(y_i+x_i)/2**
17805	24846	7041	21325.5
18123	24862	6739	21492.5
19735	25365	5630	22550.0
20903	28252	7349	24577.5
21145	29224	8079	25184.5
22762	29976	7214	26369.0
22828	30035	7207	26431.5
23141	30613	7472	26877.0
24609	32158	7549	28383.5
26525	32472	5947	29498.5
26846	32618	5772	29732.0
27746	32962	5216	30354.0
28117	33091	4974	30604.0
28858	33197	4339	31027.5
28877	33198	4321	31037.5
29080	33245	4165	31162.5
29586	33380	3794	31483.0
29965	33406	3441	31685.5
29994	33788	3794	31891.0
30263	33888	3625	32075.5
30313	33937	3624	32125.0
30366	34007	3641	32186.5
30629	34013	3384	32321.0
30680	34076	3396	32378.0
30788	34425	3637	32606.5
30958	34455	3497	32706.5
31074	34576	3502	32825.0
31405	35237	3832	33321.0
31994	35843	3849	33918.5
32384	35852	3468	34118.0

Exercise 3.3

- Provide an approximate value of the difference in the daily traffic flow between the two directions using a Tukey mean-difference plot.

Steps

4. Discuss



Exercise 4.1

The monthly expense [CHF] for water consumption including sewage fee for a 2-persons household may be considered as a random variable with the following density function:

$$f_X(x) = \begin{cases} c \cdot x \cdot (60 - x) & \text{for } 0 \le x \le 60 \\ 0 & \text{otherwise} \end{cases}$$

Change to

$$f_X(x) = \begin{cases} c \cdot x \cdot (15 - \frac{x}{4}) & \text{for } 0 \le x \le 60 \\ 0 & \text{otherwise} \end{cases}$$

a. Which value of *c* should be chosen?

b. Describe the cumulative distribution function $F_X(x)$ of the random variable X.

c. Which of the following four values, 30.00 CHF, 40.00 CHF, 50.00 CHF and 60.00 CHF does not exceed the 90%-quantile of the monthly expense?

d. How large is the mean monthly expense for water consumption including sewage fee for a 2-persons household?

Solution 4.1 a. Which value of *c* should be chosen?

Probability density function

$f_X(x) \geq 0$ ← Non-negative

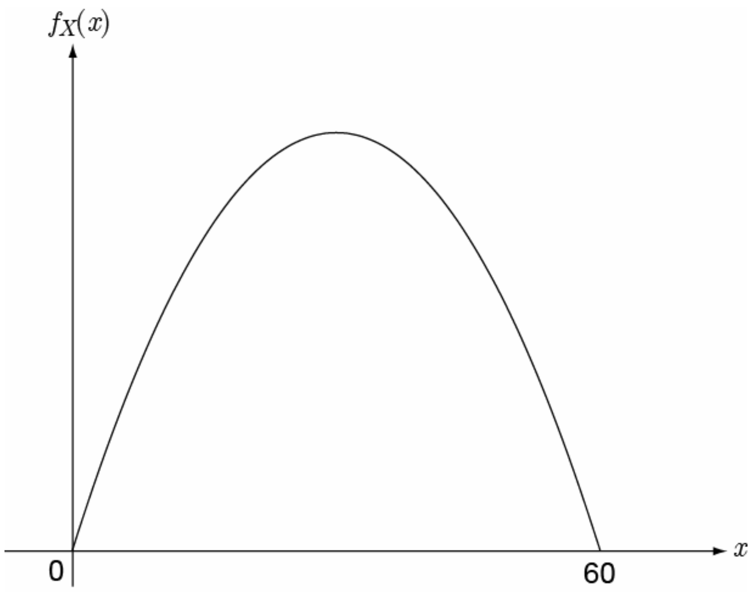
$\int_{\Omega} f_X(x)dx = ??$ ← Area = ??

Solution 4.1 a. Which value of c should be chosen?

Probability density function

$$f_X(x) \geq 0$$ ← Non-negative

$$\int_{\Omega} f_X(x)dx = ?$$ ← Area = ?



$$f_X(x) = \begin{cases} c \cdot x \cdot (60 - x) & \text{for } 0 \leq x \leq 60 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^{60} c \cdot x \cdot (60 - x)\, dx = ? \Rightarrow c = ...$$

Solution 4.1 b. Describe the cumulative distribution function $F_X(x)$ of the random variable X.

Cumulative distribution function

$$F_X(x) = \int_{\Omega} f_X(x) dx$$

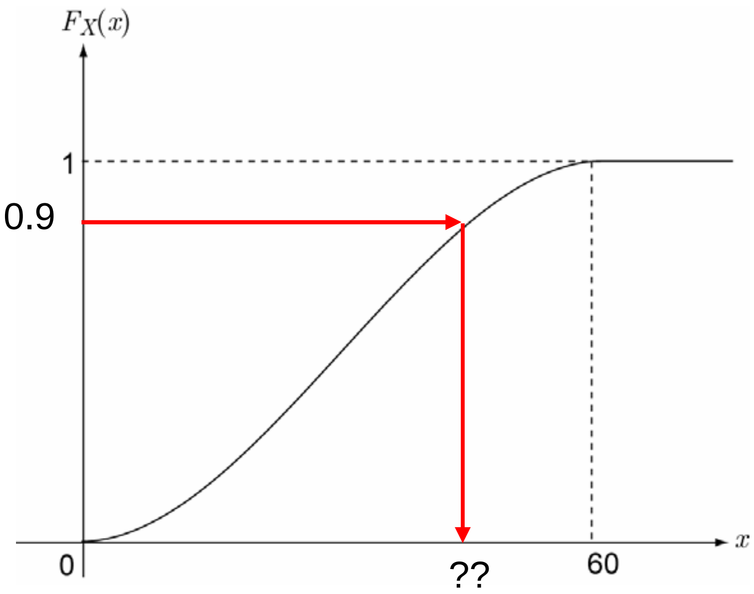
$$f_X(x) = \begin{cases} c \cdot x \cdot (60 - x) & \text{for } 0 \leq x \leq 60 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{36000} \cdot \left(\frac{60}{2} \cdot x^2 - \frac{1}{3} \cdot x^3 \right) & 0 \leq x \leq 60 \\ 1 & 60 < x \end{cases}$$

Solution 4.1

c. Which of the following four values, 30.00 CHF, 40.00 CHF, 50 CHF and 60 CHF does not exceed the 90%-quantile of the monthly expense?

First we need to find the value corresponding to the 90% quantile

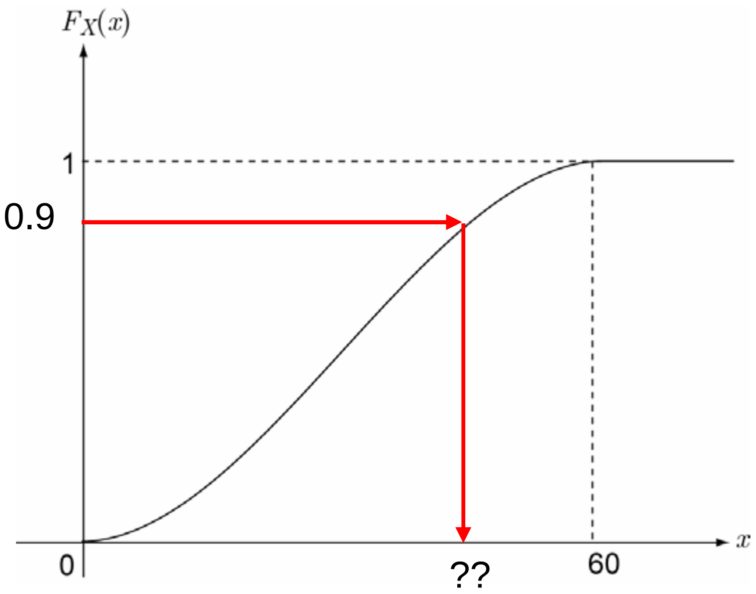


Solution 4.1 c. Which of the following four values, 30.00 CHF, 40.00 CHF, 50 CHF and 60 CHF does not exceed the 90%-quantile of the monthly expense?

First we need to find the value corresponding to the 90% quantile

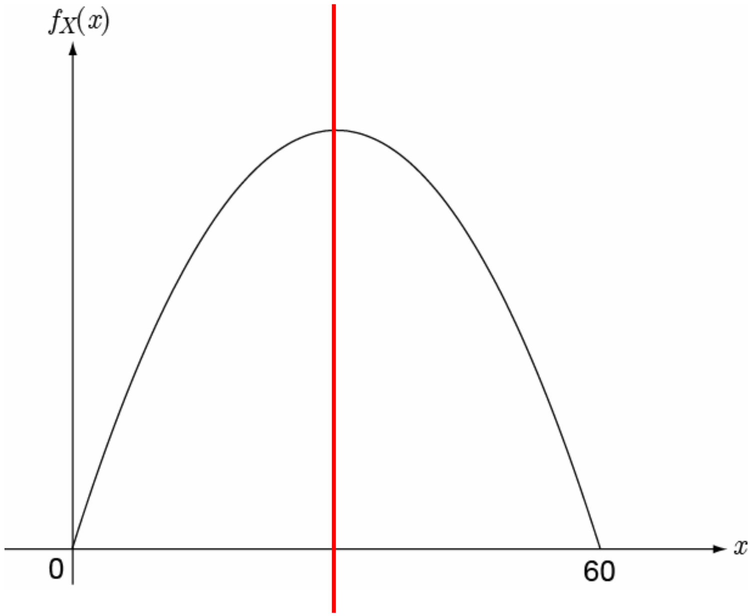
$$P(X \leq \alpha) = F_X(x) = 0.9$$

$$P(X \leq a) = \frac{1}{36000} \cdot \int_0^{\alpha} x(60\text{-x})\, dx \Rightarrow \alpha = ...$$



Solution 4.1

d. How large is the mean monthly expense for water consumption including sewage fee for a 2-persons household?



Mean = 30

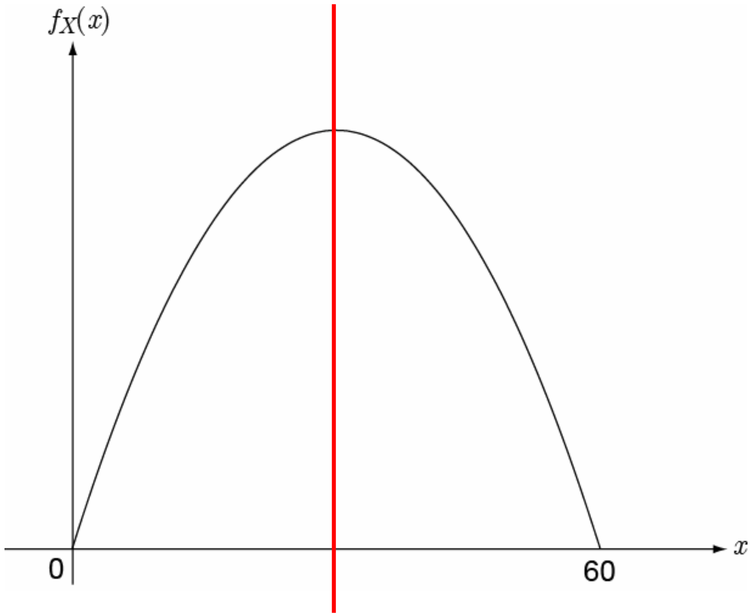
Mean = 30

We can say this directly by looking at the Probability density function. WHY???

Solution 4.1 d. How large is the mean monthly expense for water consumption including sewage fee for a 2-persons household?

Mean---First moment

$$\mu = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$



Mean = 30

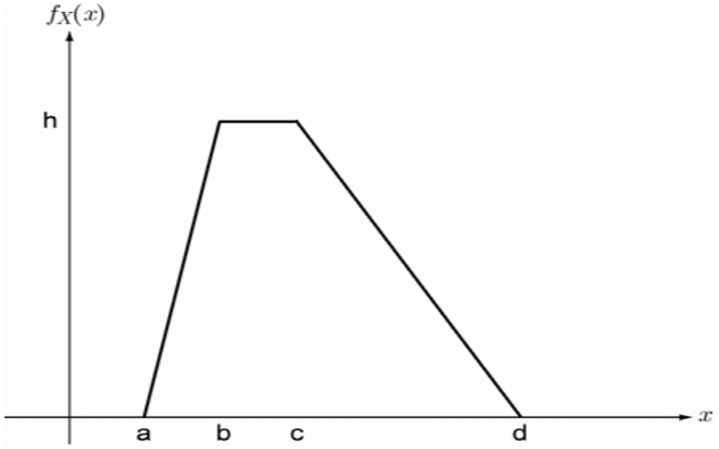
$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) dx = \frac{1}{36000} \cdot \int_{0}^{60} x^2 \cdot (60 - x)\, dx$$

Exercise 4.2

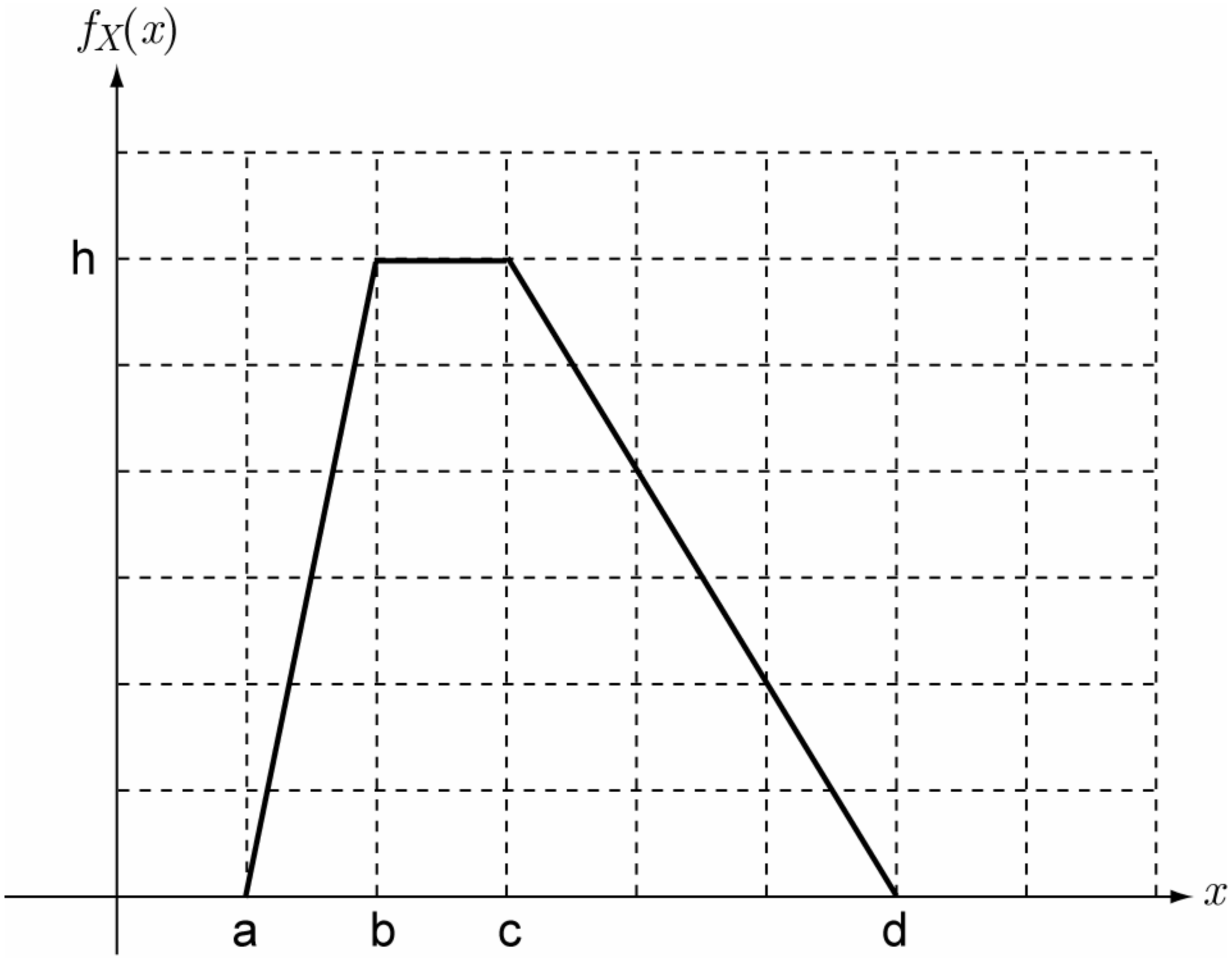
The probability function of a basic variable is shown in the following figure.
a. determine analytically the PDF and the CDF.

Let a=1, b=2, c=3, d=6. (Change location in the exercise)

a. Define the mode and the parameter h.
b. Calculate the mean value.
c. Calculate the value of the median.
d. Obtain graphically the median of the pdf. Discuss how the mean value may be evaluated graphically.

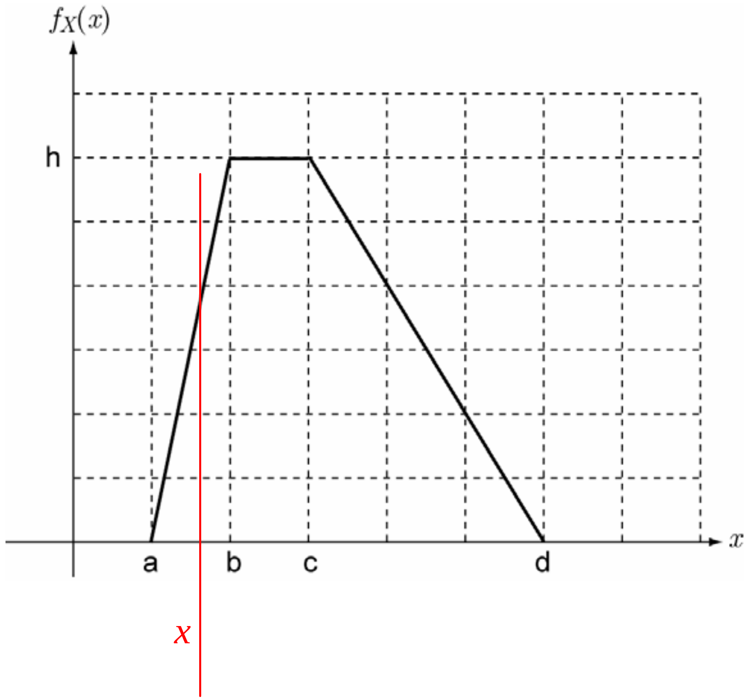


First think along with the definition, then think it again graphically.



Solution 4.2

PDF – Probability Density Function a. determine analytically the PDF and the CDF.



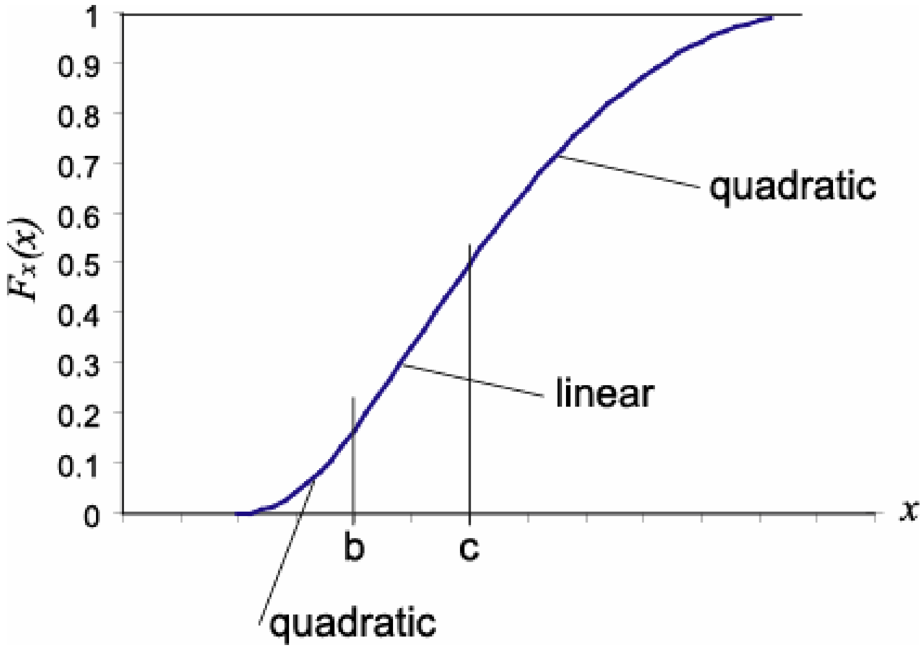
Rule of similar triangles

$$f_X(x) = \begin{cases} 0 & x < a \\ h \cdot \dfrac{(x-a)}{(b-a)} & a \le x < b \\ h & b \le x < c \\ h \cdot \dfrac{(x-d)}{(c-d)} & c \le x < d \\ 0 & d \le x \end{cases}$$

Solution 4.2

CDF – Cumulative Density Function a. determine analytically the PDF and the CDF.

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)dx$$

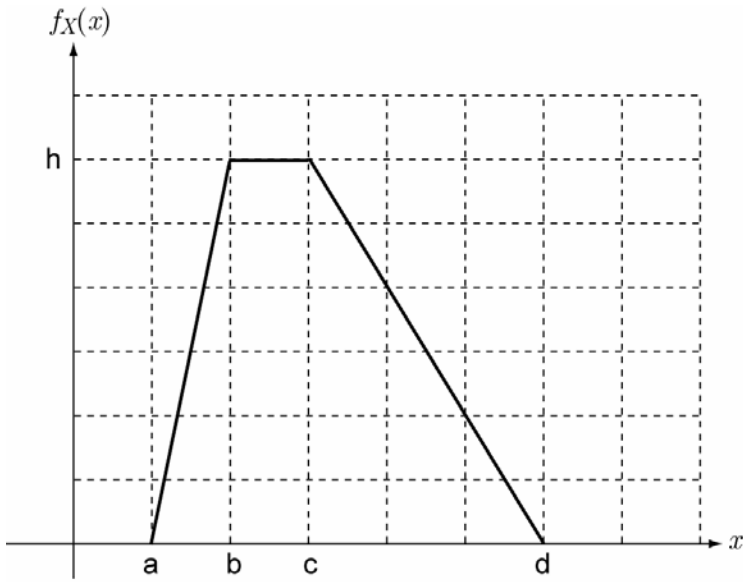


$$F_X(x) = \begin{cases} 0 & x < a \\ h \cdot \dfrac{(x-a)^2}{2 \cdot (b-a)} + C_1 & a \le x < b \\ h \cdot x + C_2 & b \le x < c \\ h \cdot \dfrac{(x-d)^2}{2 \cdot (c-d)} + C_3 & c \le x < d \\ C_4 & d \le x \end{cases}$$

The four constants can be calculated by using the boundary conditions

Solution 4.2

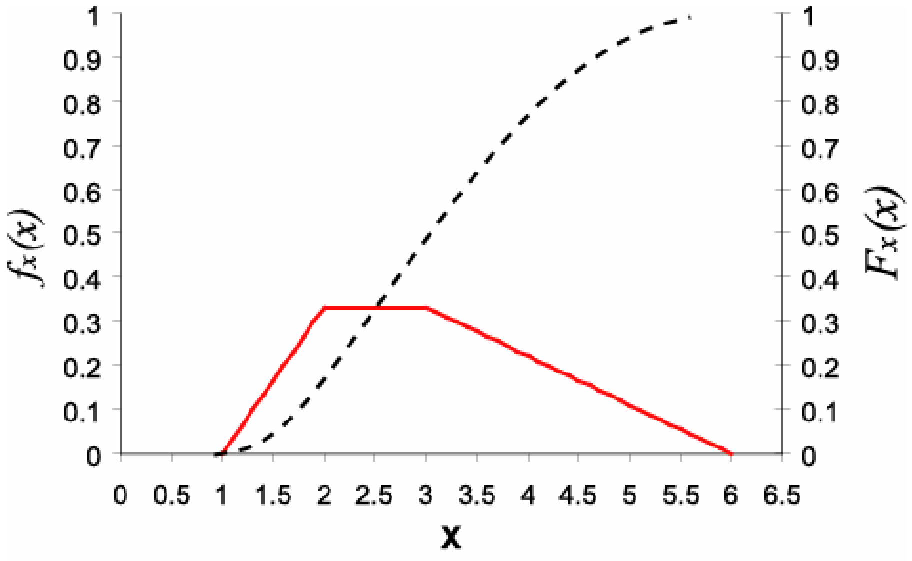
b. define the mode and the parameter *h*. (a=1, b=2, c=3, d=6)



What means the term "mode"????

Solution 4.2

b. define the mode and the parameter *h*. (a=1, b=2, c=3, d=6)



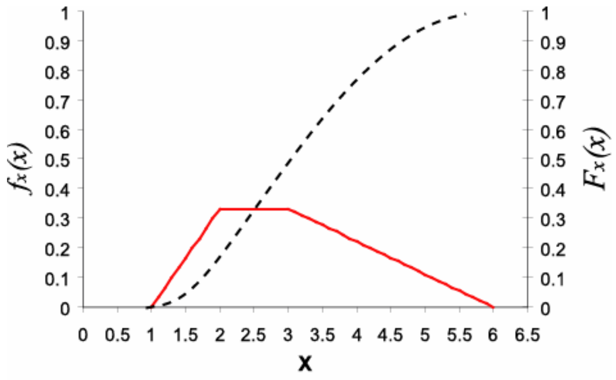
$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

Area under the density function

$$\frac{(d-a)+(c-b)}{2} \cdot h = 1 \Rightarrow ......h = ...$$

Solution 4.2

c. Calculate the value of the mean (a=1, b=2, c=3, d=6)



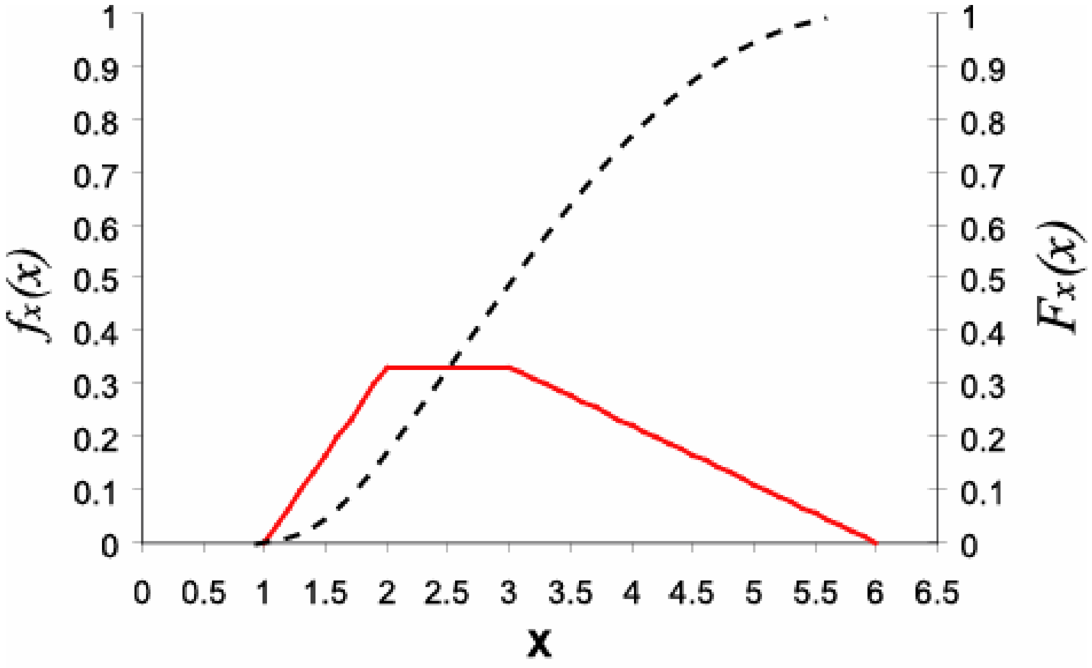
$$f_X(x)=\begin{cases} 0 & x<a \\ h\cdot\dfrac{(x-a)}{(b-a)} & a\le x<b \\ h & b\le x<c \\ h\cdot\dfrac{(x-d)}{(c-d)} & c\le x<d \\ 0 & d\le x \end{cases} \quad\Rightarrow\quad f_X(x)=\begin{cases} 0 & x<1 \\ \dfrac{(x-1)}{3} & 1\le x<2 \\ \dfrac{1}{3} & 2\le x<3 \\ -\dfrac{(x-6)}{9} & 3\le x<5 \\ 0 & 5\le x \end{cases}$$

$$\mu_x = E\left[x\right] = \int_{-\infty}^{\infty} x\cdot f_x(x)\cdot dx = \int_1^2 \frac{x\cdot(x-1)}{3}dx + \int_2^3 \frac{x}{3}\cdot dx + \int_3^6 \frac{-x\cdot(x-6)}{9}dx = ...$$

Solution 4.2

d. Calculate the value of the median.

Graphically from the CDF (???)



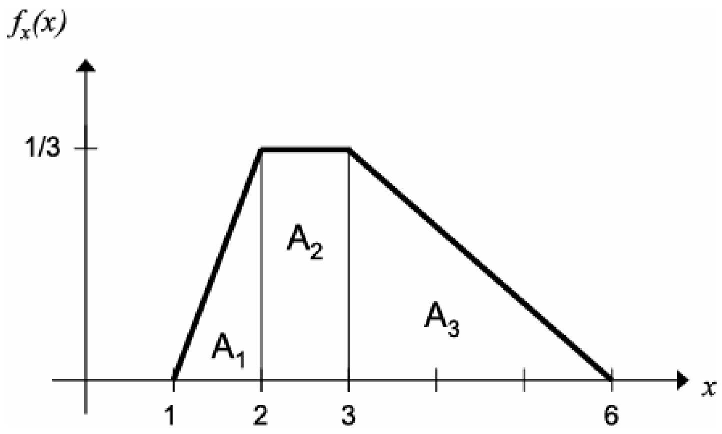
Analytically

(???)

Solution 4.2

e. Obtain graphically the median of the pdf. Discuss how the mean value may be evaluated graphically.

Graphically from the PDF



$$A_1 = (2-1)\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$$A_2 = (3-2)\cdot\frac{1}{3} = \frac{1}{3}$$

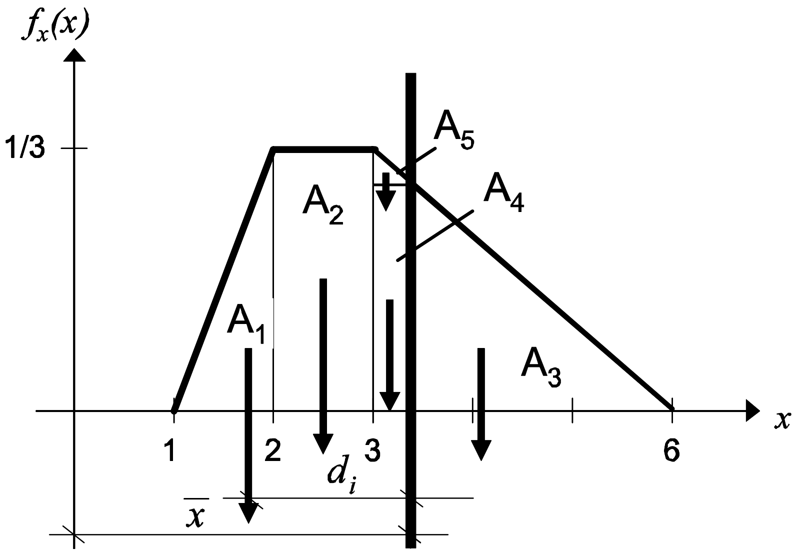
$$A_3 = (6-3)\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{2}$$

Median: ???

Solution 4.2

e. Obtain graphically the median of the pdf. Discuss how the mean value may be evaluated graphically.

Graphically from the PDF



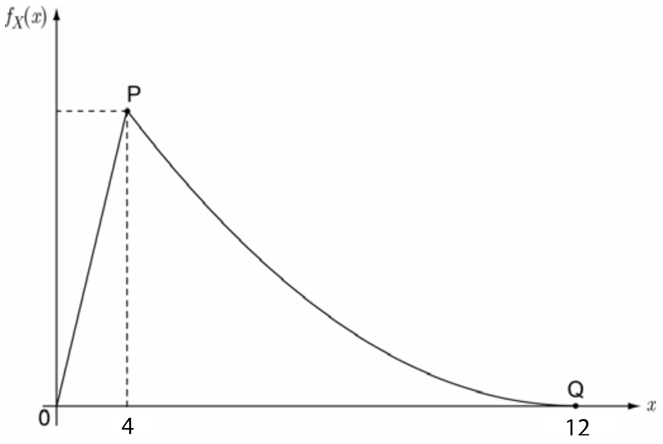
$$\sum_{i=1}^{5} A_i \cdot d_i = 0$$

1. **Estimate moments for each shape**
2. **Take equilibrium around the hypothesized location of the center of gravity**

Mean: center of gravity of the shape of the probability density function.

Exercise 4.3 (Group exercise- to be presented on 19.04.07)

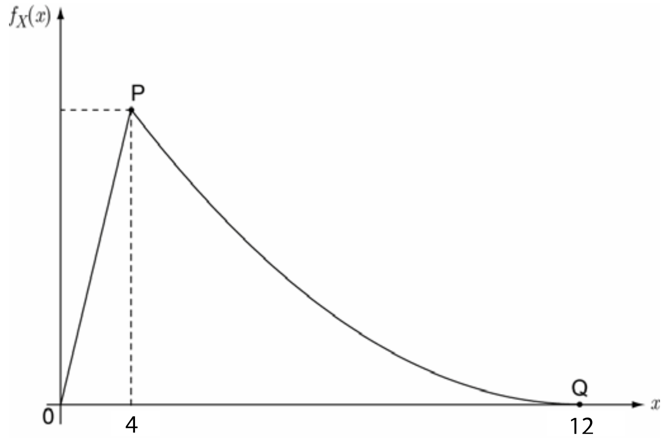
The probability density function of a random variable *X* is shown in Figure 4.3.1. In the interval [0, 4] the function is linear and in the interval [4, 12] the function is parabolic which is tangent to x-axis at point Q.



a. Determine the coordinate of point P(x,y) and then describe the probability density function.
b. Describe and draw the cumulative distribution function of *X* with some characteristic numbers in the figure.
c. Calculate the mean value of *X*.
d. Calculate $P[X>4]$.

Exercise 4.3 (Group exercise- to be presented on 19.04.07)

a. Determine the coordinate of point P(x,y) and describe the probability density function.

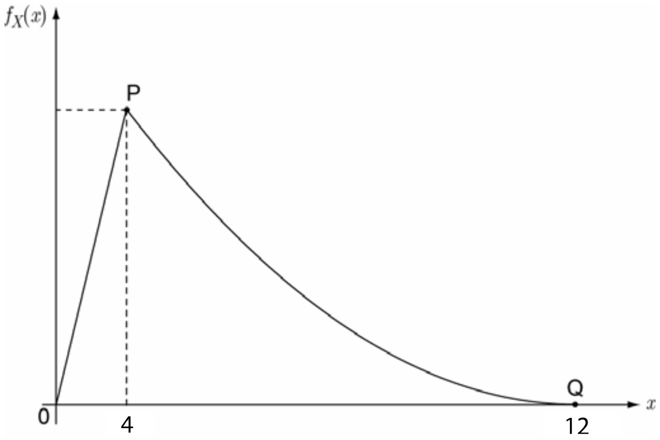


Steps:

- **Define the pdf in the interval [0,12]**
- **Find coordinates of P by remembering that the area under the density function is equal to 1!**

Exercise 4.3 (Group exercise- to be presented on 19.04.07)

b. Describe and draw the cumulative distribution function of X with some characteristic numbers in the figure.



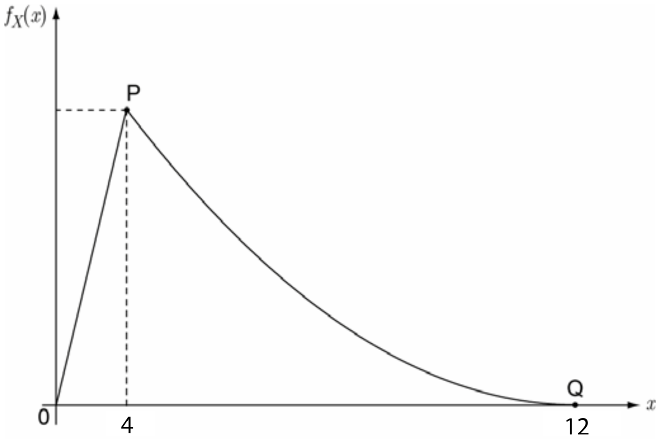
Steps:

1. $\int_{\Omega} f_X(x)dx = 1$

2. Draw…!

Exercise 4.3 (Group exercise- to be presented on 19.04.07)

c. Calculate the mean value

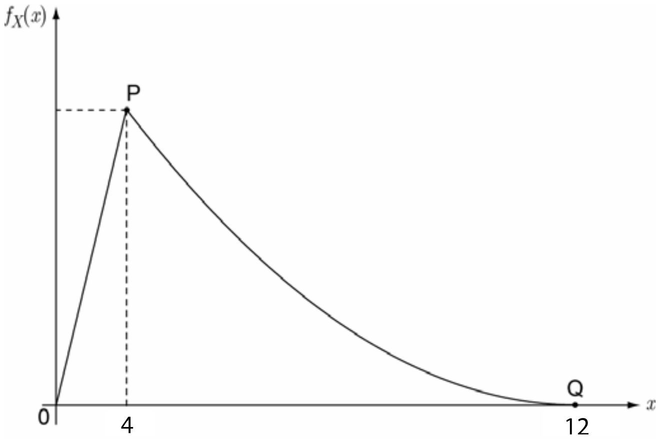


Steps (Remember Exercise 4.2):

1. $\mu_x = E\left[x\right]$

Exercise 4.3 (Group exercise- to be presented on 19.04.07)

d. Calculate $P[X>4]$.



Steps (Remember Exercise 4.2):

Exceedance probability $P[X>\alpha]$ is $1-P[X\leq\alpha]$

1. $P[X>4]=1-P[X\leq 4]$

How can this be expressed???