Exercise 8.4 - Solution

a. The probability of the event that a measurement is less than $23\,ng\,/\,ml$ is calculated as:

$$P[X < 23] = P\left[\frac{X - 23}{4.3} < \frac{23 - 23}{4.3}\right] = \Phi(0) = 0.5.$$

where X represents the measurement result of the density of the chemical agent and $\Phi(\cdot)$ is the cumulative distribution function of the standard Normal distribution.

The probability that a measured density lies in the required interval is calculated as:

$$P[19.5 < X \leq 20.5] = P\left[\frac{19.5 - 23.0}{4.3} < \frac{X - 23.0}{4.3} \leq \frac{20.5 - 23.0}{4.3}\right]$$
$$= \Phi(-0.58) - \Phi(-0.81) = 0.073$$

b. The sample mean value of 30 measurement results follows the Normal distribution with mean of $23\,ng\,/\,ml$ and standard deviation of $4.3 / \sqrt{30} = 0.79\,ng\,/\,ml$.

The probability of the daily sample mean being less than $20\,ng\,/\,ml$ is calculated as:

$$P(\overline{X} < 20) = P\left(\frac{\overline{X} - \mu_X}{\sigma_X / \sqrt{n}} < \frac{20 - \mu_X}{\sigma_X / \sqrt{n}}\right) = \Phi\left(\frac{20 - 23}{0.79}\right) = 7.3 \times 10^{-5}.$$

c)
1. Specify what to judge: H_0 (null hypothesis) and H_1 (alternative hypothesis).

$$H_0 : \mu_0 = 23$$
$$H_1 : \mu_0 \neq 23$$

2. Choose the level of significance:

$$\alpha = 5\%$$.

3. Determine the condition of sampling (what kind of and how many data?)

15 measurements are undertaken.

4. Create the operation rule (as a function of sampling statistics).

The operation rule for accepting the null hypothesis is represented with the sample mean $\overline{x}$, which follows the t-distribution:

$$\mu - t_{\alpha/2} \frac{s}{\sqrt{n}} \leq \overline{x} \leq \mu + t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ is the $\alpha/2$ quantile of t-distribution.

5. Execute the sampling and obtain the result.

$$\overline{x} = 19\,ng\,/\,ml$$
$$s = 5\ ng\,/\,ml\,.$$

6. Judge the null hypothesis H_0.

$$\mu - t_{\alpha/2}\frac{s}{\sqrt{n}} = 23 - 2.13 \cdot \frac{5}{\sqrt{15}} = 20.25$$

$$\mu + t_{\alpha/2}\frac{s}{\sqrt{n}} = 23 + 2.13 \cdot \frac{5}{\sqrt{15}} = 25.75$$

The obtained sample statistics do not satisfy the operation rule for accepting the null hypothesis. Therefore, the null hypothesis is rejected at the significant level of 5%.