Exercise 5.3 (Group exercise) - Solution

a. $Z = T + S$. Z is defined within the interval [0;*z*]. The probability density function is expressed as:

$$f_Z(z) = \int_0^z f_T(t) f_S(z-t)dt \qquad (1)$$

Where:

$f_T(t) = \frac{1}{\mu_T} \cdot e^{\frac{-t}{\mu_T}}$,$(t \geq 0)$; $f_S(s) = \frac{1}{\mu_S} \cdot e^{\frac{-s}{\mu_S}} = \frac{1}{\mu_S} \cdot e^{\frac{-(z-t)}{\mu_S}}$,$(s \geq 0)$, (see also Table D.1 in the script for the functions of the exponential distribution).

Hence from Equation (1) it is:

$$f_Z(z) = \int_0^z f_T(t) f_S(z-t)dt = \int_0^z \left(\frac{1}{\mu_T} e^{\frac{-t}{\mu_T}}\right)\left(\frac{1}{\mu_S} e^{\frac{-(z-t)}{\mu_S}}\right)dt$$

$$= \int_0^z \frac{1}{\mu_T \mu_S} e^{\left(1-\left(\frac{1}{\mu_T}-\frac{1}{\mu_S}\right)t-\frac{z}{\mu_S}\right)} dt = \left[\frac{1}{\mu_T \mu_S} \cdot \frac{\mu_T \mu_S}{-\mu_S + \mu_T} e^{\left(\frac{-\mu_S+\mu_T}{\mu_T \mu_S}t - \frac{z}{\mu_S}\right)}\right]_0^z = \frac{1}{\mu_T - \mu_S}\left(e^{-\frac{z}{\mu_T}} - e^{-\frac{z}{\mu_S}}\right)$$

By replacing in the above equation the provided information: $\mu_T = 10$, $\mu_S = \frac{1}{12}$ the probability density function becomes:

$$f_Z(z) = \frac{1}{\mu_T - \mu_S}\left(e^{-\frac{z}{\mu_T}} - e^{-\frac{z}{\mu_S}}\right) = \frac{1}{10 - \frac{1}{12}}\left(e^{-\frac{z}{10}} - e^{-12z}\right) = 0.1008\left(e^{-\frac{z}{10}} - e^{-12z}\right)$$

b.

$P[Z \leq 5] = F_Z(5)$, hence the cumulative distribution function needs first to be found:

$$F_Z(z) = \int_0^z 0.1008\left(e^{-\frac{y}{10}} - e^{-12y}\right)dy = 0.1008\left[-10e^{-\frac{y}{10}} + \frac{1}{12}e^{-12y}\right]_0^z =$$

$$= 0.1008\left[-10e^{-\frac{z}{10}} + 10 + \frac{1}{12}e^{-12z} - \frac{1}{12}\right] = 0.1008\left(9.9167 - 10e^{-\frac{z}{10}} + \frac{1}{12}e^{-12z}\right)$$

For *z*=5 it is:

$$F_Z(5) = 0.1008\left(9.9167 - 10e^{-\frac{5}{10}} + \frac{1}{12}e^{-12 \cdot 5}\right) = 0.389$$

c. The required probability can be expressed as:

$$P[T_1 > 5, T_2 > 5] = P[T_1 > 5]P[T_2 > 5]$$
$$= \left(P[T > 5]\right)^2 = \left(1 - P[T \le 5]\right)^2 \qquad (2)$$

T is exponentially distributed with cumulative distribution (see script Table D.1):

$$F_T(t) = 1 - e^{-\frac{1}{\mu_T}t} = 1 - e^{-0.1t}$$

Hence from Equation (2) it is:

$$P[T_1 > 5, T_2 > 5] = \left(1 - P[T \le 5]\right)^2 = (1 - (1 - e^{-0.1 \cdot 5}))^2 = 0.368$$

Note that: Using Table D.1 of the script you can use directly the cumulative distribution function of the exponential function. It should be clear though that this results by integration of the probability density function and this part of the exercise could also be solved as follows:

$$f_T(t) = \frac{1}{\mu_T} e^{\frac{-t}{\mu_T}}; \mu_T = 10$$

$$F_T(t) = \frac{1}{\mu_T} \int_0^t e^{\frac{-y}{\mu_T}} dy$$
$$= \frac{1}{\mu_T} \left[-\mu_t e^{\frac{-y}{\mu_T}} \right]_0^t$$
$$= \frac{1}{\mu_T} \left[-\mu_t e^{\frac{-t}{\mu_T}} - (-\mu_t e^0) \right]$$
$$= 0.1(-10 e^{\frac{-t}{10}} + 10)$$
$$= 1 - e^{\frac{-t}{10}} = 1 - e^{-0.1t}$$

And hence, as before:

$$P[T_1 > 5, T_2 > 5] = \left(1 - P[T \le 5]\right)^2 = (1 - (1 - e^{-0.1 \cdot 5}))^2 = 0.368$$