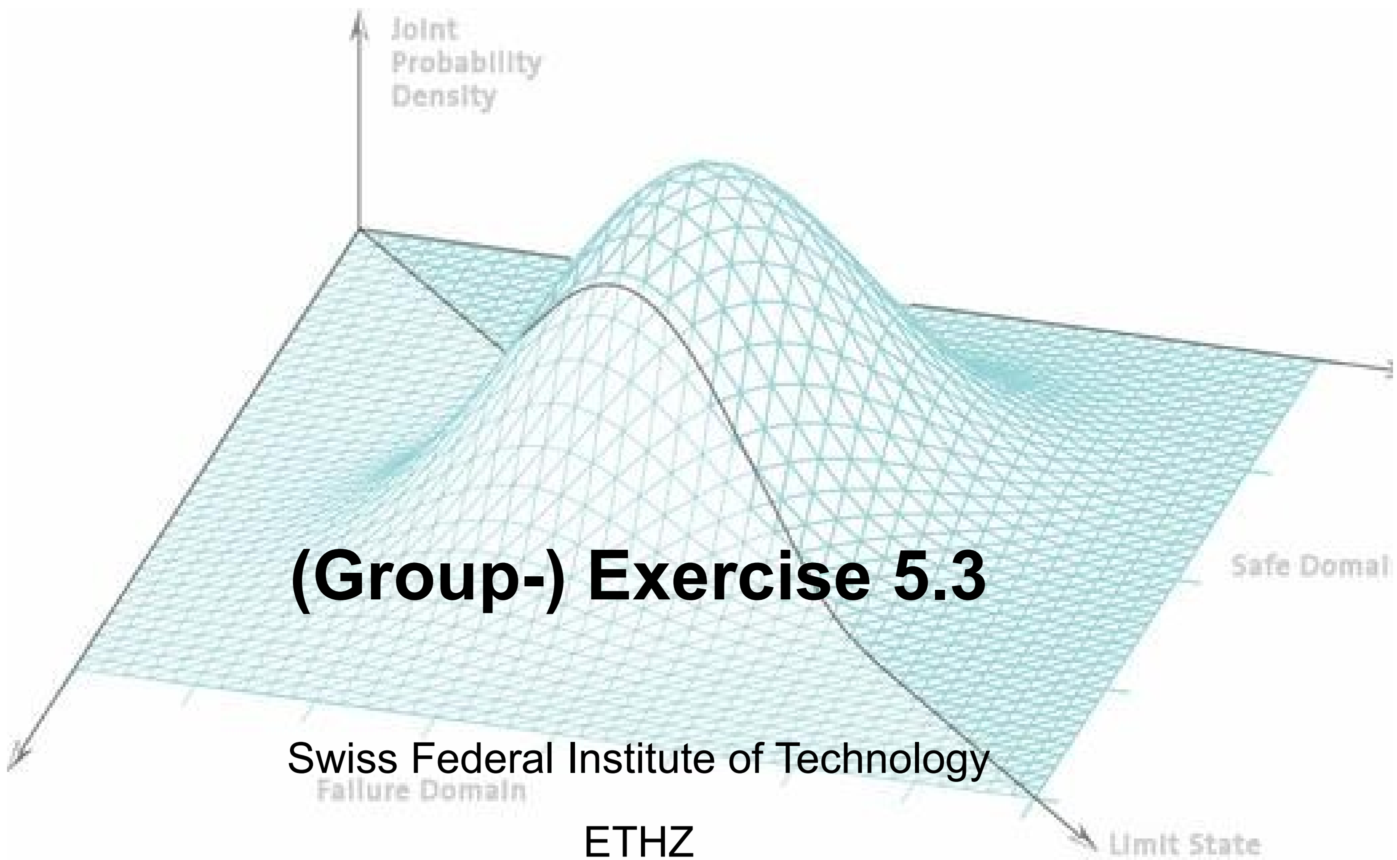
Statistics and Probability Theory



(Group-) Exercise 5.3

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ETHZ

D-BAUG

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Use the given information

The Time T is assumed exponentially distributed with the mean value of 10 years

The Time S is assumed exponentially distributed with the mean value of 1/12 year

S and T are independent

$$\mu_T = 10 \qquad \mu_S = \frac{1}{12}$$

Formula in Table D.1

Probability Density Function: (Page D-17)

$$f_X(x) = \lambda \exp(-\lambda(x - \varepsilon)) \qquad x > \varepsilon$$

$$\mu = \varepsilon + \frac{1}{\lambda} \qquad \varepsilon = 0$$

$$\Rightarrow \lambda = \frac{1}{\mu}$$

Substitution: X = T, S
replace 'x' by 't' and 's'

$$f_T(t) = \begin{cases} 0 & (t \leq 0) \\ \frac{1}{\mu_T}\exp(-\frac{t}{\mu_T}) & (t > 0) \end{cases}$$

$$f_S(s) = \begin{cases} 0 & (s \leq 0) \\ \frac{1}{\mu_S}\exp(-\frac{s}{\mu_S}) & (s > 0) \end{cases}$$

CONVOLUTION INTEGRAL

(Page D-13)
Special case where both variables are **independent**:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_2}(y - x_1)\, f_{X_1}(x_1)\, dx_1 \qquad \text{(D.36)}$$

$$Y = X_1 + X_2$$

$$X_1 = S$$

$$X_2 = T$$

$$=> \quad Z = S + T$$

$$Y = Z$$

Analyse the convolution integral

$$f_Z(z) = \int_{-\infty}^{\infty} f_T(z-s)\, f_S(s)\, ds$$

$$analyse\ f_S(s)$$

When

- s ≤ 0 $\qquad f_S(\mathrm{s}) = 0$

- s > 0 $\qquad f_S(\mathrm{s}) \neq 0$

Reduce the range

split the range in 2 parts

- $s \leq 0$
- $s > 0$

$$f_Z(z) = \int_{-\infty}^{0} f_T(z-s)\, f_S(s)\, ds + \int_{0}^{\infty} f_T(z-s)\, f_S(s)\, ds$$

= 0

$$\Rightarrow \quad f_Z(z) = \int_{0}^{\infty} f_T(z-s)\, f_S(s)\, ds$$

Analyse the reduced integral

$$f_Z(z) = \int_0^\infty f_T(z-s)\, f_S(s)\, ds$$

$analyse\ f_T(t) = f_T(z-s)$ (z = s + t => t = z – s)

- t ≤ 0 => s ≥ z $f_T(t) = 0$

- t > 0 => s < z $f_T(t) \neq 0$

Reduce the range

split the range in 2 parts

- s ≥ z
- s < z

$$f_Z(z) = \int_0^z f_T(z-s)\, f_S(s)\, ds + \int_z^\infty f_T(z-s)\, f_S(s)\, ds$$

= 0

$$\Rightarrow \quad f_Z(z) = \int_0^z f_T(z-s)\, f_S(s)\, ds$$

Analyse z

$$\Rightarrow \quad f_Z(z) = \int_0^z f_T(z-s)\, f_S(s)\, ds$$

$$\text{if} \quad z < 0 \quad \Rightarrow \quad f_T = 0 \quad \Rightarrow \quad f_Z(z) = 0$$

$$\text{if} \quad z \geq 0 \quad \Rightarrow \quad f_Z(z) = \int_0^z \frac{1}{\mu_T}\, \mathrm{e}^{-\frac{z-s}{\mu_T}} \frac{1}{\mu_S}\, \mathrm{e}^{-\frac{s}{\mu_S}}\, ds$$

Insert the values

$$f_Z(z) = \int_0^z \frac{1}{\mu_T}\, e^{-\frac{z-s}{\mu_T}} \frac{1}{\mu_S}\, e^{-\frac{s}{\mu_S}}\, ds$$

$$= \int_0^z \frac{1}{10} e^{\frac{s-z}{10}}\ 12 e^{-12s}\ ds$$

$$= \frac{6}{5} \int_0^z e^{\frac{-z-119s}{10}}\ ds$$

$$= -\frac{12}{119} [e^{-12z} - e^{-\frac{z}{10}}]$$

Probability density function of the random variable Z

$$f_Z(z) = \begin{cases} 0 & (z < 0) \\ \frac{12}{119}[e^{-\frac{z}{10}} - e^{-12z}] & (z \geq 0) \end{cases}$$