Exercise 4.3 (Group exercise) - Solution:

a. Under given conditions, the probability density function $f_X(x)$ can be written as:

$$f_X(x) = \begin{cases} ax & (0 < x \le 4) \\ k(x-12)^2 & (4 < x \le 12) \\ 0 & (\text{otherwise}) \end{cases}$$

Since the probability density function is continuous at $x = 4$, it is:

$$4a = k(4-12)^2 \tag{4.3.1}$$

The integral of the probability density function over the entire support must be unity:

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^4 axdx + \int_4^{12} k(x-12)^2 dx = 1 \tag{4.3.2}$$

From Equation (4.3.1), it follows:

$$k = \frac{1}{16}a \tag{4.3.3}$$

By substituting this into Equation (4.3.2), it is

$a = \frac{3}{56}$ and $k = \frac{3}{896}$

Therefore,

Point A has coordinates $(4, \frac{3}{14})$ and the probability density function is described as:

$$f_X(x) = \begin{cases} \frac{3}{56}x & (0 < x \le 4) \\ \frac{3}{896}(x-12)^2 & (4 < x \le 12) \\ 0 & (\text{otherwise}) \end{cases}$$

b) The cumulative distribution function $F(x)$ is described as:

$$F_X(x) = \begin{cases} 0 & (x \le 0) \\ \frac{3}{112}x^2 & (0 < x \le 4) \\ 1 - \frac{(12-x)^3}{896} & (4 < x \le 12) \\ 1 & (12 < x) \end{cases}$$

The cumulative distribution function is drawn in Figure 4.3.2.

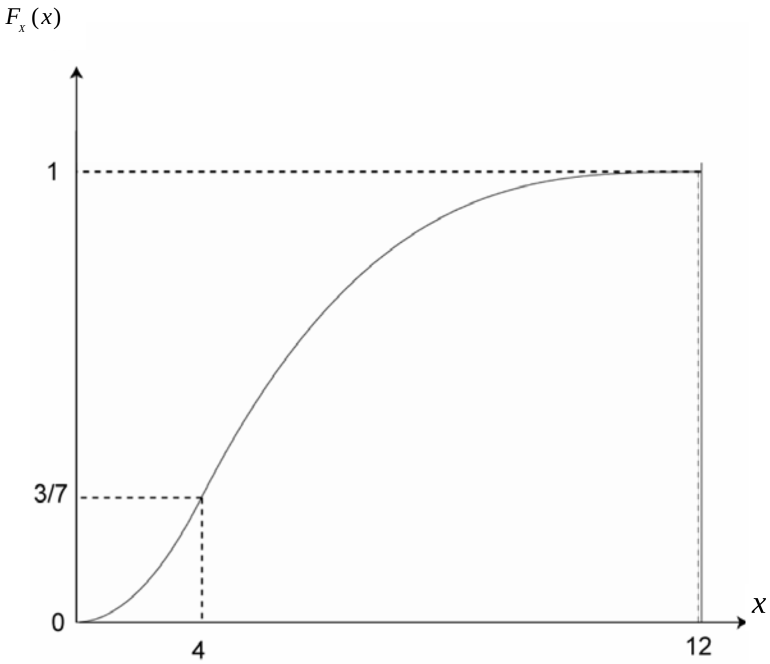

Figure 4.3.2: Cumulative distribution function.

c. The mean value is obtained as:

$$\mu_X = \int_0^{12} x f_X(x)dx = \int_0^4 x \frac{3}{56} x dx + \int_4^{12} x \frac{3}{896}(x-12)^2 dx$$

$$= \frac{32}{7}$$

d. $P[X > 4] = 1 - P[X \le 4] = 1 - \int_0^4 \frac{3}{56} x dx = 1 - \left[\frac{3}{112} x^2\right]_0^4 = \frac{4}{7}$.