Exercise 10.5 (Group Exercise) - Solution

The Weibull distribution with zero lower bound has the following cumulative distribution function:

$$F_X(x) = 1 - \exp\left\{-\left(\frac{x}{u}\right)^k\right\}, x > 0$$

The mean and the standard deviation of the Weibull distribution are

$$\mu = u\Gamma\left(1 + \frac{1}{k}\right)$$

$$\sigma = u\sqrt{\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)}$$

a. The unknown parameters *u* and *k* are estimated in the following way.

The parameter *k* is obtained from

$$\frac{\sigma}{\mu} = \frac{\sqrt{m_2 - m_1^2}}{m_1} = \sqrt{\frac{\Gamma(1 + 2/k)}{\Gamma^2(1 + 1/k)} - 1}$$

k may be approximated as:

$$k \approx \left(\frac{\sigma}{\mu}\right)^{-1.09}$$

Then the parameter *u* is obtained as:

$$u = \frac{m_1}{\Gamma(1 + 1/k)}$$

The estimated parameters are:

$k = 4.21$

$u = 29.05$.

b. The cumulative distribution functions are shown in Figure 10.5.1.

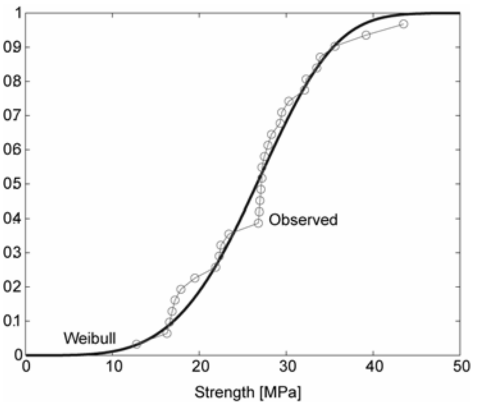

Figure 10.5.1: Comparison of probability distribution function with observed cumulative distribution.

c. The sample statistic for the χ^2 goodness-of-fit test is:

$$\varepsilon_m^2 = \sum_{j=1}^{k} \frac{(N_{o,j} - np_j)^2}{np_j}$$

that follows the χ^2 distribution with 4-1-2=1 degrees of freedom (since both the parameters have been estimated using the data).

Since it is $\alpha = 0.1$, it is $\Delta = 2.7$ (using Table T.3).

The observed sample statistic is $0.30 + 1.02 + 1.04 + 0.27 = 2.63$ (see Table 10.5.1).

The operating rule states that: the null hypothesis should be rejected if $\varepsilon_m^2 \geq \Delta$. It is seen that this does not hold and hence null hypothesis that the wood strength follows the Weibull distribution with $k = 4.21$ and $u = 29.05$ cannot be rejected at the 10% significance level.

Interval	$N_{o,j}$	$p(x_j)$	$N_{p,j} = np(x_j)$	ε_m^2
-20	7	0.19	5.7	0.30
20-25	4	0.22	6.6	1.02
25-30	11	0.27	8.1	1.04
30-	8	0.32	9.6	0.27
Sum	30	1	30	= 2.63

Table 10.5.1: Calculation sheet for the Weibull distribution.

d. The sample statistic for the Kolmogorov-Smirnov test can be calculated by:

$$\varepsilon_{max} = \max_{i=1}^{n} \left[\left| F_o(x_i^o) - F_p(x_i^o) \right| \right]$$

The operation rule states that the null the hypothesis should be rejected if $\varepsilon_{max} \geq \Delta$ where Δ can be determined by:

$$P(\varepsilon_{max} \geq \Delta) = \alpha$$

For $\alpha = 0.10$ and $n = 30$, it is $\Delta = 0.22$ from the table for the Kolmogorov-Smirnov test (Table E.4- also Table T.4 can be used). The observed sample statistic for the assumed distribution is ε_{max} =0.168 see Table 10.2.5.

Therefore, since $\varepsilon_{max} = 0.168 < \Delta = 0.22$, the null hyphothesis that the wood strength follows the Weibull distribution with $k = 4.00$ and $u = 28$ cannot be rejected at the 10% significance level.

Example of calculation:

For $i = 1,\ x_i = 12.8$ *then* $F_o(x_i^o) = \frac{i}{n} = \frac{1}{30} = 0.033$. The predicted cumulative distribution can be calculated from the assumed distribution i.e. Weibull with the assumed parameter values $k = 4$ and $u = 28$. The Weibull cumulative distribution function is expressed as:

$F_X(x) = 1 - \exp\left\{-\left(\frac{x}{u}\right)^k\right\}$ so $F_p(x_i^o) = 1 - exp\left\{-\left(\frac{12.8}{28}\right)^4\right\} = 0.043$. The absolute difference between the observed and the predicted cumulative distributions is shown in the last column of Table 10.5.2.

The sample statistics is eventually the maximum value calculated: $\varepsilon_{max} = \max_{i=1}^{n}\left[\left|F_o(x_i^o) - F_p(x_i^o)\right|\right] = 0.168$

i	x_i	$F_o(x_i^o) = \frac{i}{n}$	$F_p(x_i^o)$	$\left\|F_o(x_i^o) - F_p(x_i^o)\right\|$
1.0	12.8	0.033	0.043	0.009
2.0	16.3	0.067	0.108	0.042
3.0	16.6	0.100	0.116	0.016
4.0	16.9	0.133	0.124	0.009
5.0	17.2	0.167	0.133	0.034
6.0	17.9	0.200	0.154	0.046
7.0	19.5	0.233	0.210	0.024
8.0	21.9	0.267	0.312	0.046
9.0	22.3	0.300	0.331	0.031
10.0	22.5	0.333	0.341	0.008
11.0	23.4	0.367	0.386	0.019
12.0	26.8	0.400	0.568	**0.168**
13.0	26.9	0.433	0.573	0.140
14.0	27.0	0.467	0.579	0.112
15.0	27.1	0.500	0.584	0.084

Table 10.5.2: Calculation table for the Weibull distribution.