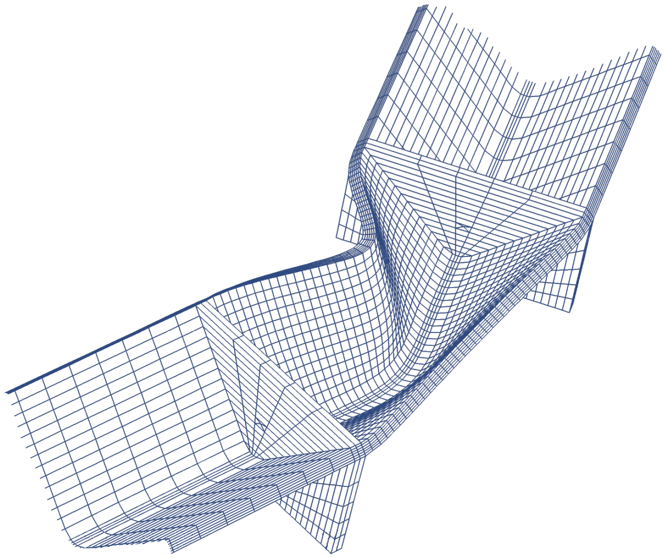
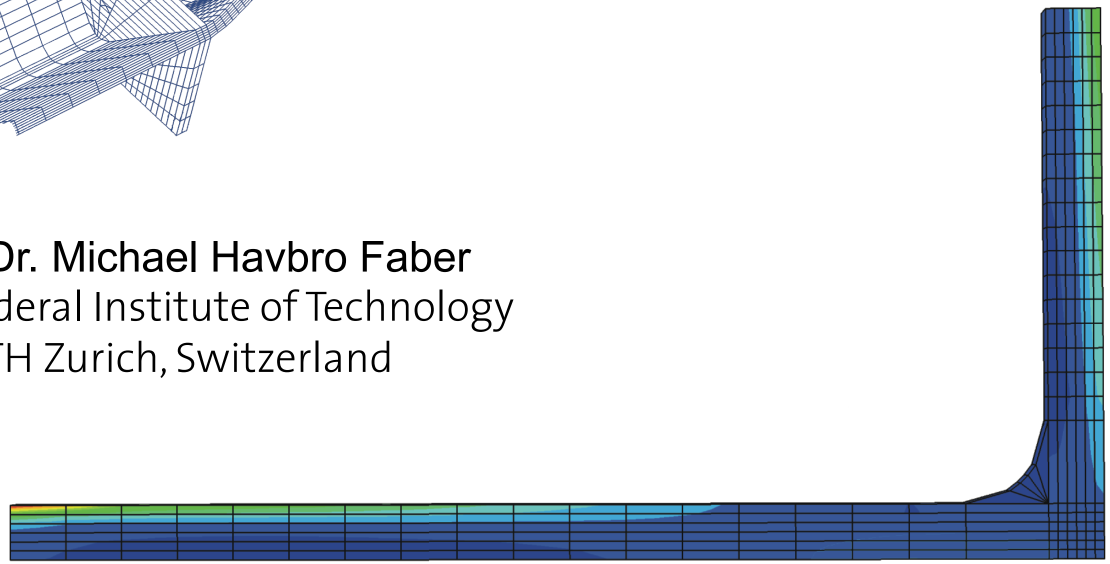
The Finite Element Method for the Analysis of Linear Systems



Prof. Dr. Michael Havbro Faber
Swiss Federal Institute of Technology
ETH Zurich, Switzerland



Contents of Today's Lecture

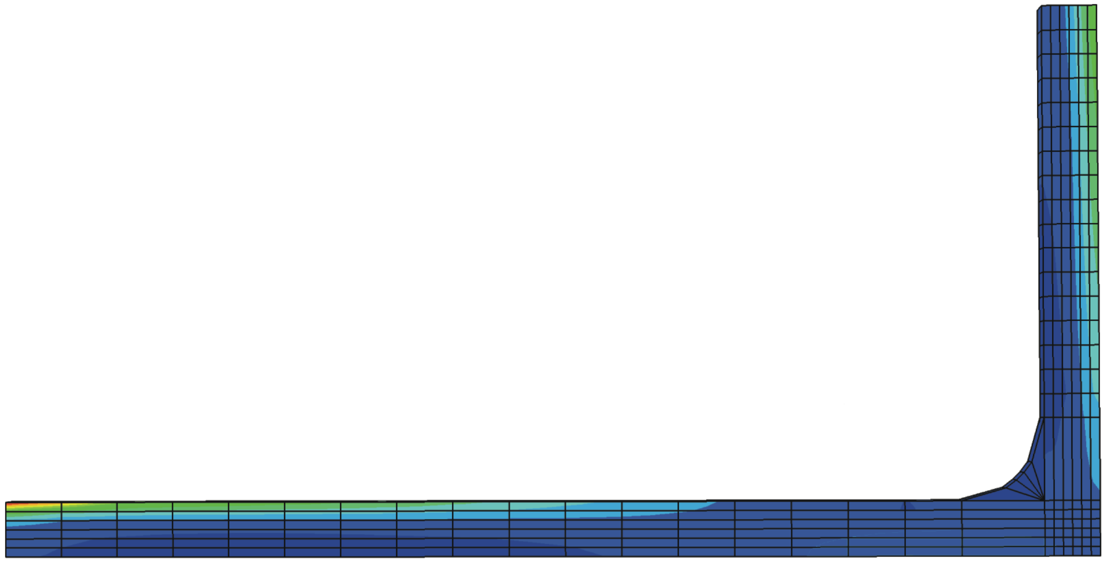
- **Plate Elements**

  The Reissner-Midlin plate theory

  - Pure displacement based formulation

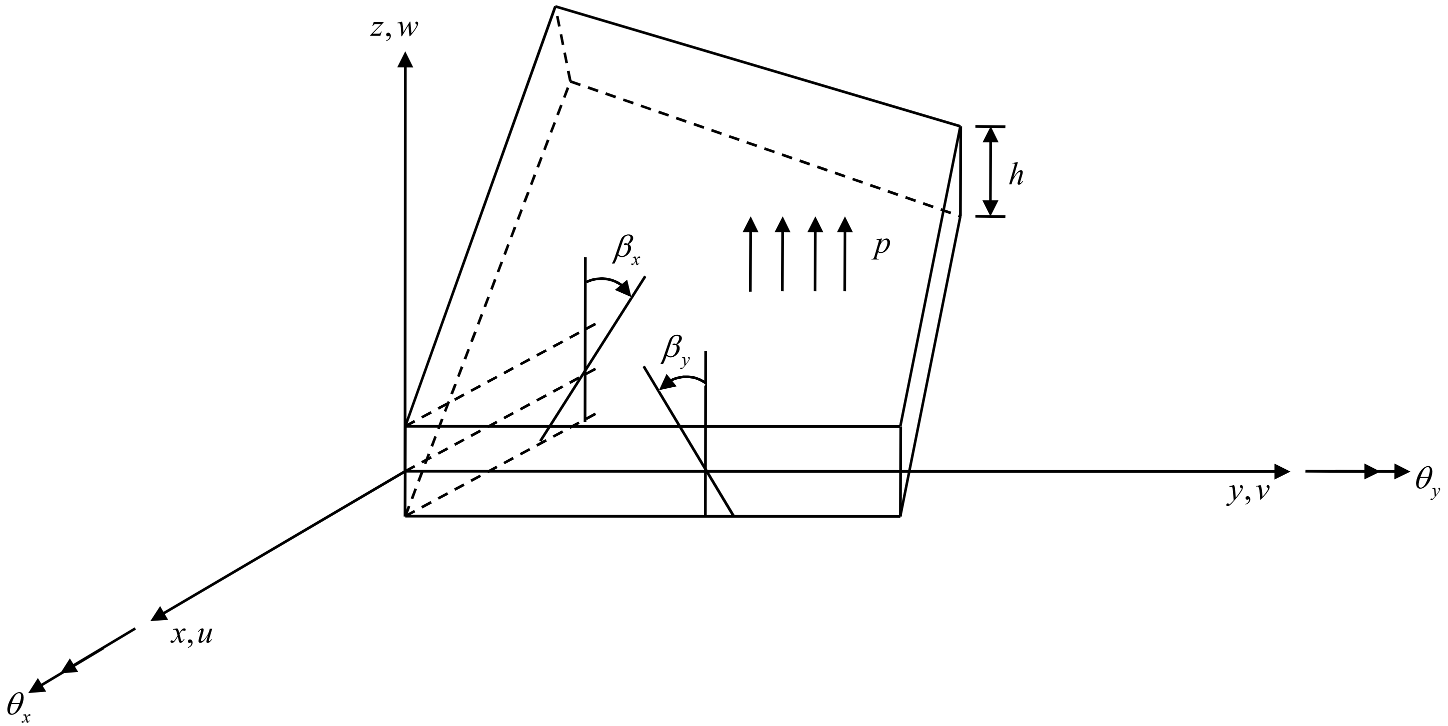
  - Mixed interpolation elements (MITC*n*)

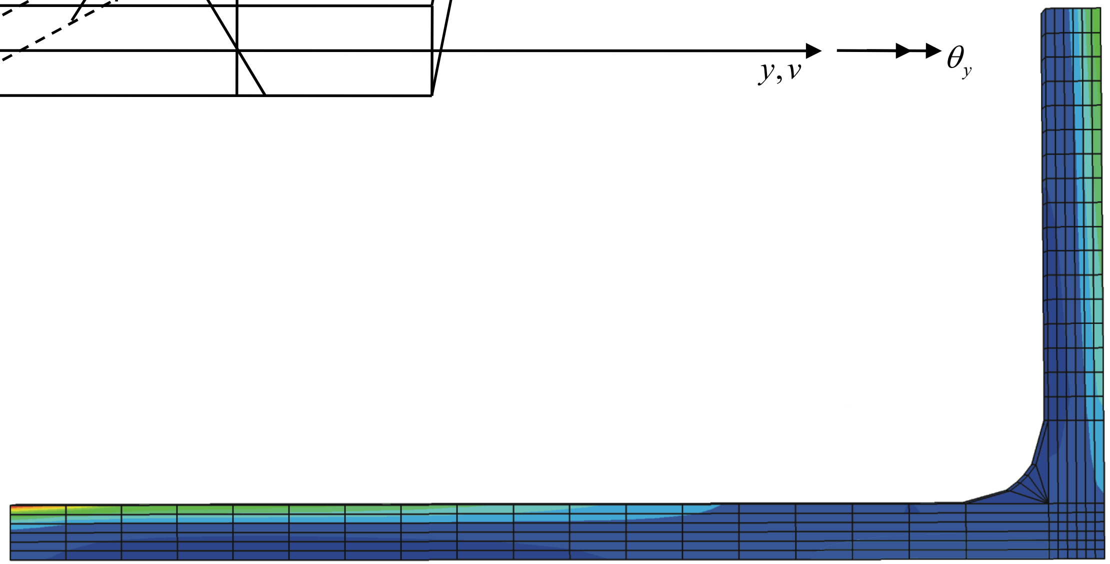
  - Performance considerations



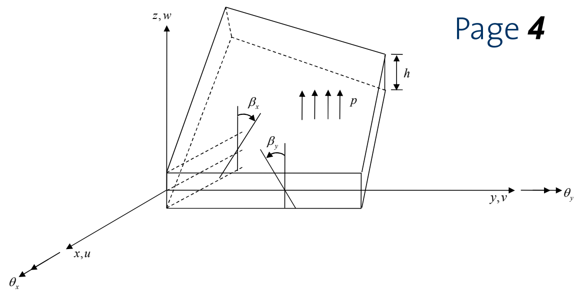
The Reissner-Midlin plate theory

- **We assume the following deformation assumptions**





The Reissner-Midlin plate theory



- **We assume the following deformation assumptions**

$$u = -z\beta_x(x, y)$$

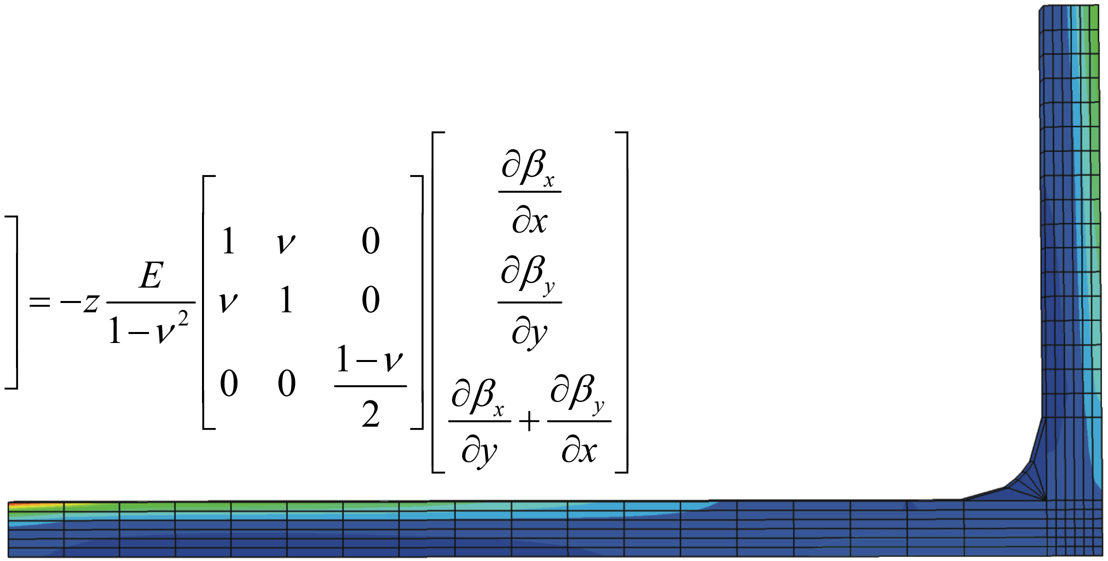
$$v = -z\beta_y(x, y)$$

$$w = w(x, y)$$

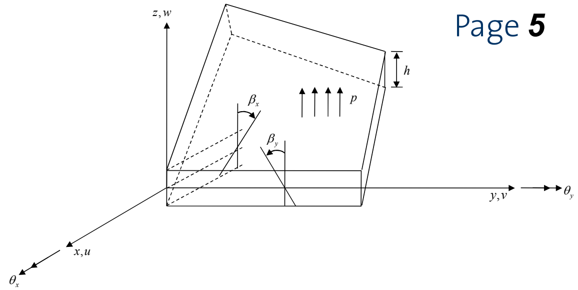
The bending strains and stresses are found from:

$$\tau_{zz} = 0 \qquad \text{Plane stress}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = -z \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{bmatrix} \qquad \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = -z \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{bmatrix}$$



The Reissner-Midlin plate theory



- **We assume the following deformation assumptions**

$$u = -z\beta_x(x,y)$$

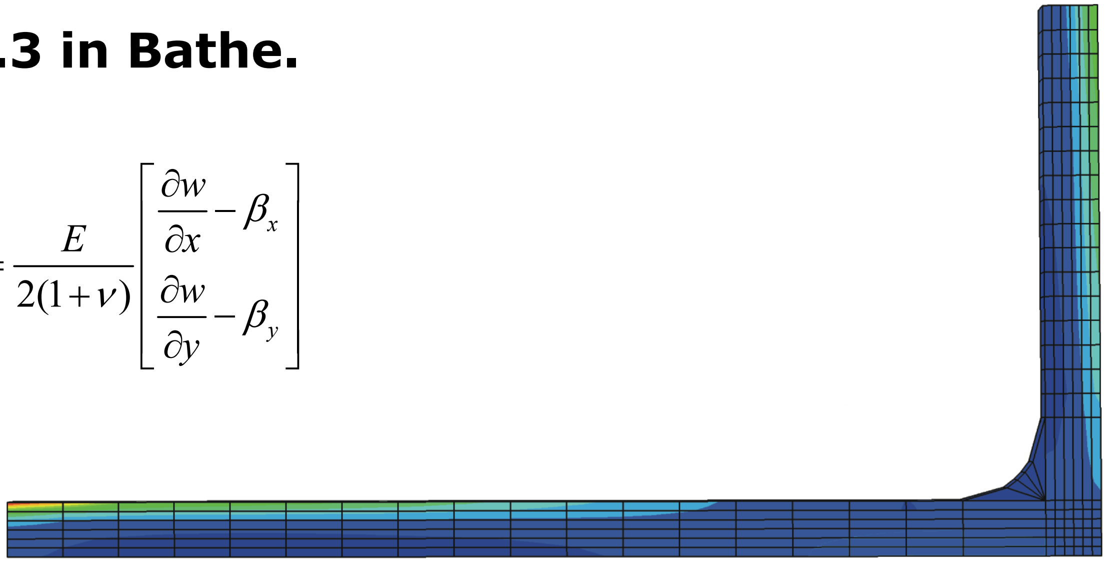
$$v = -z\beta_y(x,y)$$

$$w = w(x,y)$$

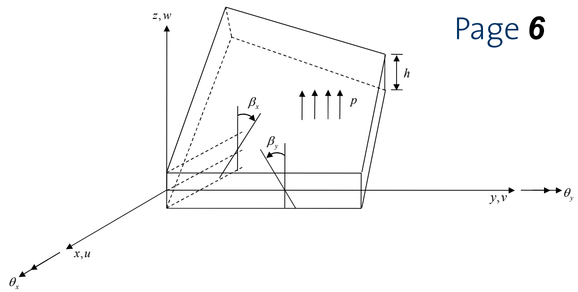
The transverse shear strains and stresses are found from:

see also Tables 4.2, 4.3 in Bathe.

$$\begin{bmatrix} \gamma_{xx} \\ \gamma_{yx} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{bmatrix} \qquad \begin{bmatrix} \tau_{xx} \\ \tau_{yx} \end{bmatrix} = \frac{E}{2(1+\nu)} \begin{bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{bmatrix}$$



The Reissner-Midlin plate theory



- **Now we can apply the principle of virtual work**

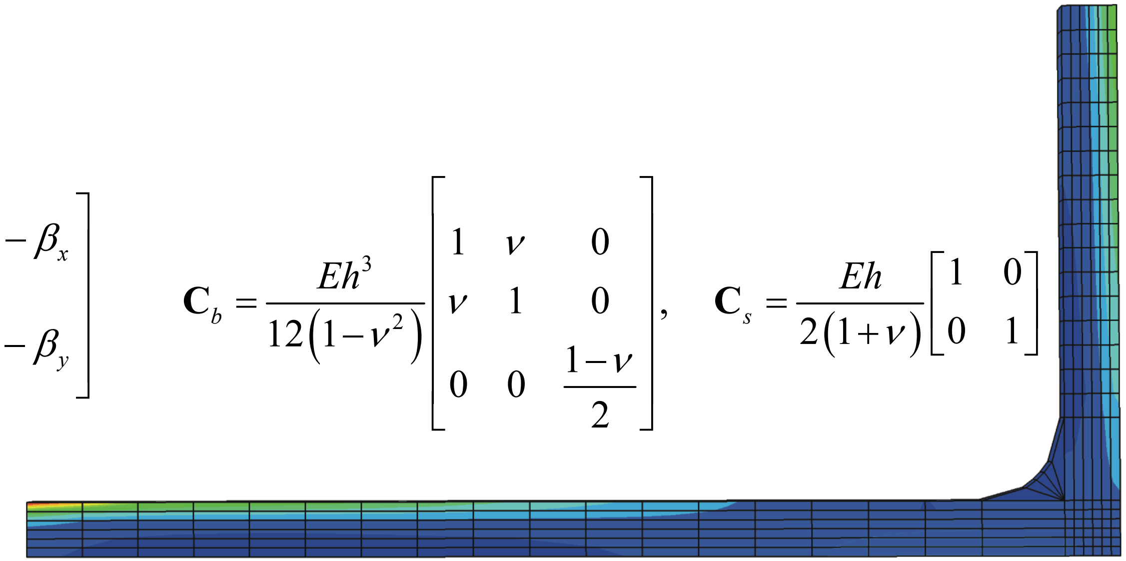
$$\int_A \int_{-h/2}^{h/2} \begin{bmatrix} \bar{\varepsilon}_{xx} & \bar{\varepsilon}_{yy} & \bar{\gamma}_{xy} \end{bmatrix} \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} dz dA + k \int_A \int_{-h/2}^{h/2} \begin{bmatrix} \bar{\gamma}_{xz} & \bar{\gamma}_{yz} \end{bmatrix} \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} dz dA = \int_A \bar{w} p dA$$

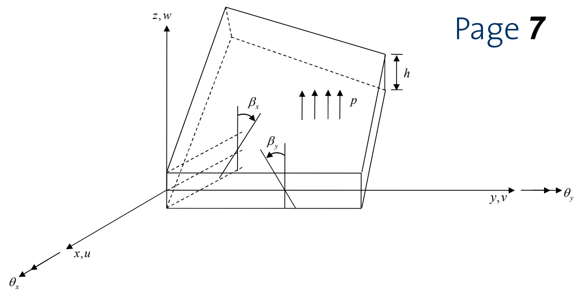
Substituting the strain-stress relationships we get:

$$\int_A \bar{\boldsymbol{\kappa}}^T \mathbf{C}_b \boldsymbol{\kappa} dA + k \int_A \bar{\boldsymbol{\gamma}}^T \mathbf{C}_s \boldsymbol{\gamma} dA = \int_A \bar{w} p dA$$

where:

$$\boldsymbol{\kappa} = \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{bmatrix} \qquad \boldsymbol{\gamma} = \begin{bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{bmatrix} \qquad \mathbf{C}_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}, \qquad \mathbf{C}_s = \frac{Eh}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$





The Reissner-Midlin plate theory

- **For the developed element equations it is emphasized that the independent variables are the displacements and the rotations i.e.:**

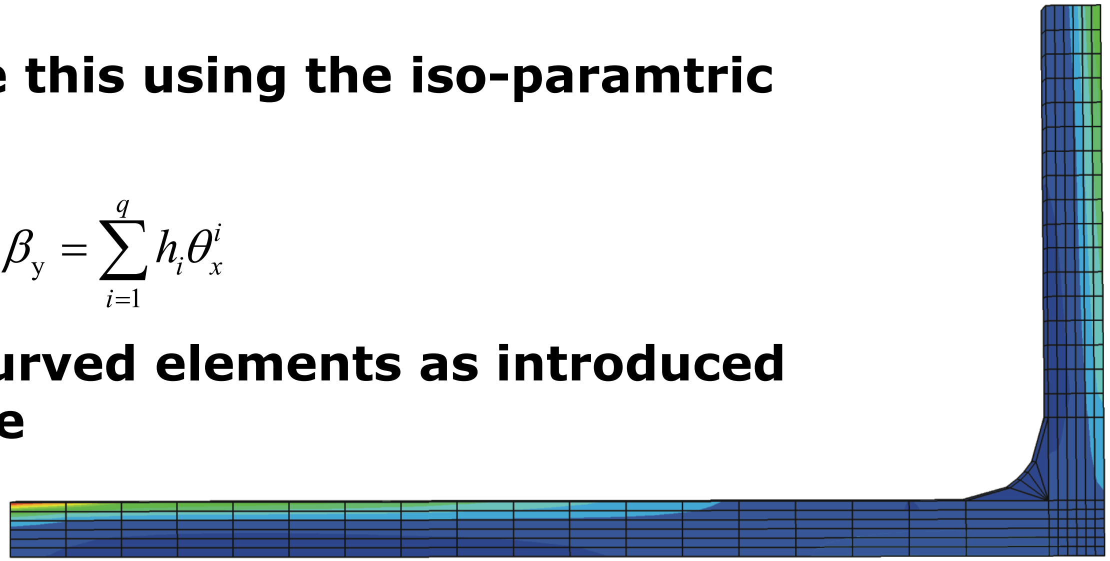
$$w, \beta_x, \beta_y$$

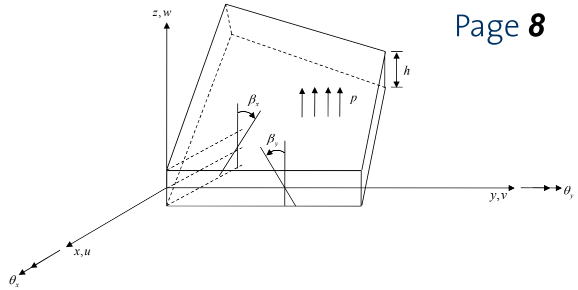
continuity to these variables are required over the boundaries of elements – but not for their derivatives!

We can easily achieve this using the iso-paramtric formulations

$$w = \sum_{i=1}^{q} h_i w_i, \quad \beta_{\mathrm{x}} = \sum_{i=1}^{q} h_i \theta_y^i, \quad \beta_{\mathrm{y}} = \sum_{i=1}^{q} h_i \theta_x^i$$

– and also consider curved elements as introduced in the previous lecture



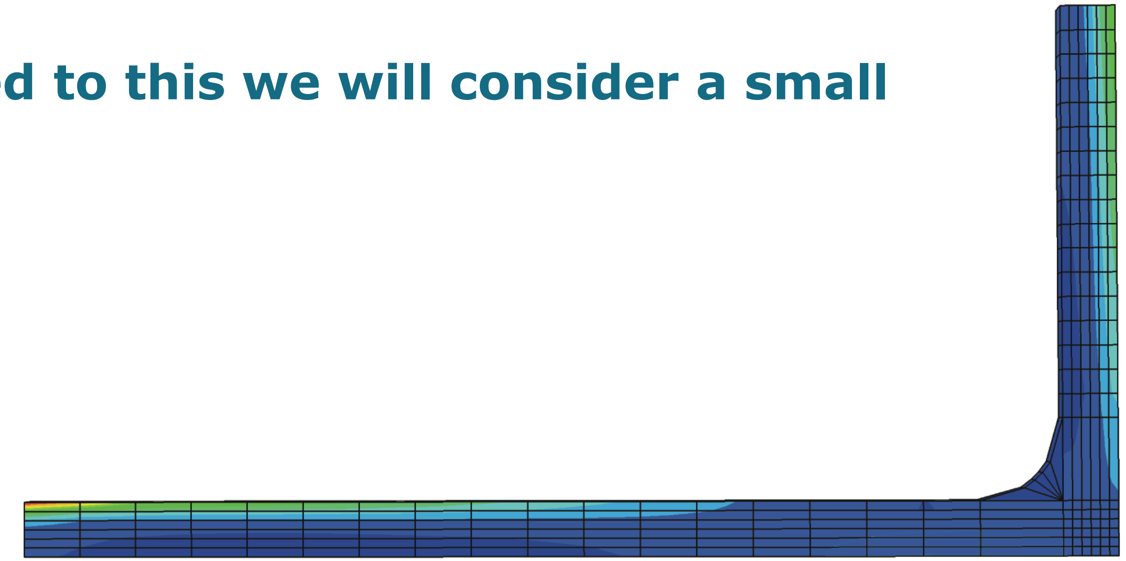


The Reissner-Midlin plate theory

- As for the beam elements considered previously – when we use a pure displacement based formulation we will have problems related to shear locking for lower order elements

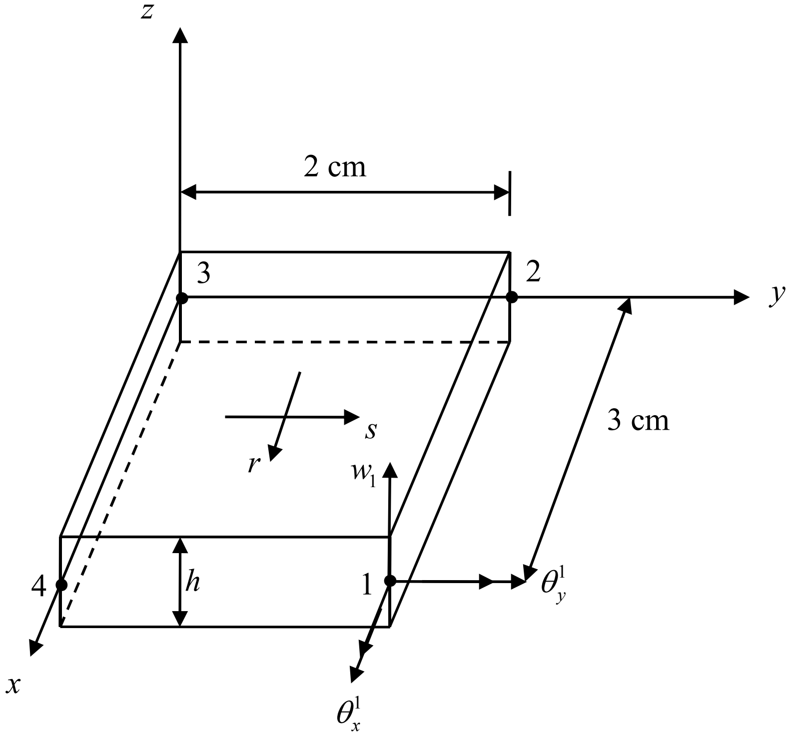
  – however as for the beams these can be solved by mixed interpolation

Before we will proceed to this we will consider a small example ☺



The Reissner-Midlin plate theory

- **Small example – four node plate element**



$$x = \frac{3}{2}(1+r)$$

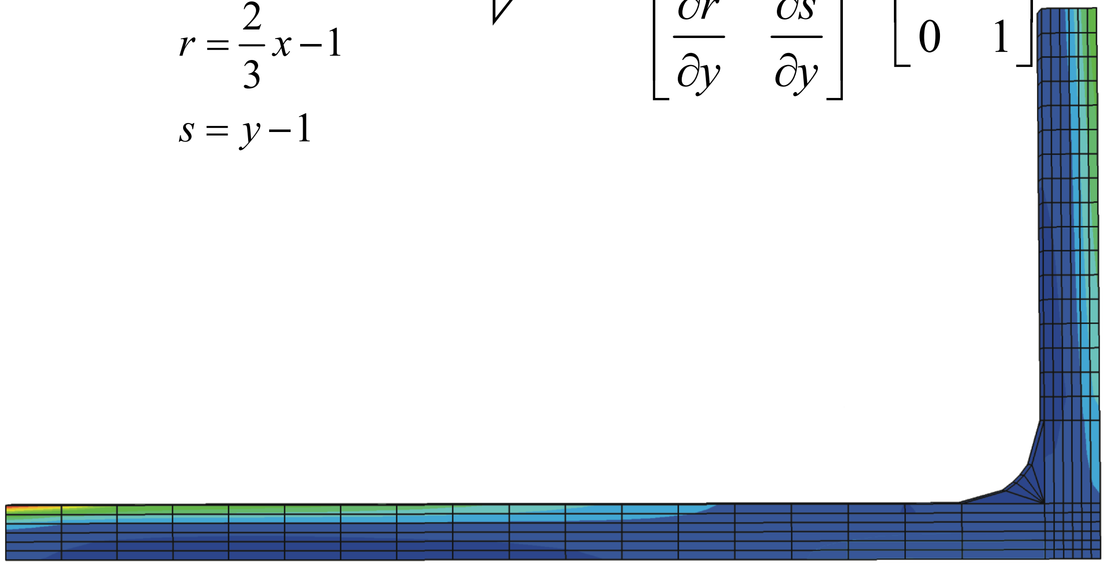
$$y = 1+s$$

$$\Downarrow$$

$$r = \frac{2}{3}x - 1$$

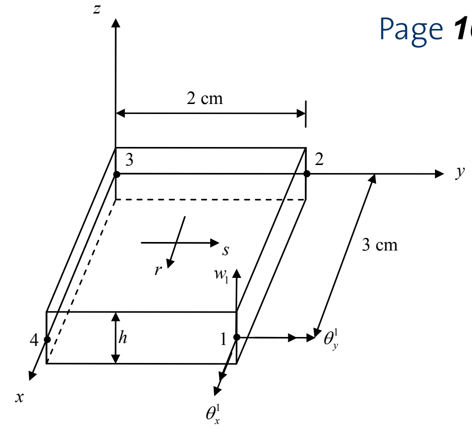
$$s = y - 1$$

$$\Rightarrow \quad \mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}$$



The Reissner-Midlin plate theory

- **Small example – four node plate element**



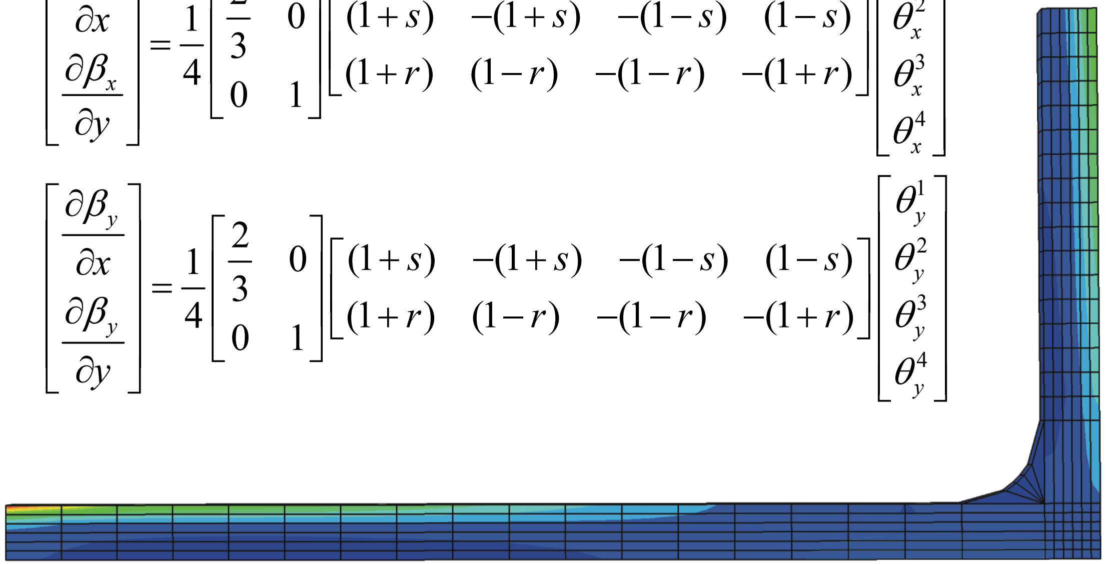
$$\boldsymbol{\kappa} = \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{bmatrix}$$

$$\boldsymbol{\gamma} = \begin{bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \dfrac{2}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

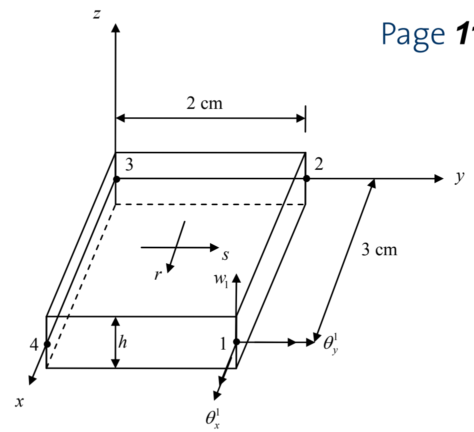
$$\begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_x}{\partial y} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \dfrac{2}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} \theta_x^1 \\ \theta_x^2 \\ \theta_x^3 \\ \theta_x^4 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \beta_y}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \dfrac{2}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} \theta_y^1 \\ \theta_y^2 \\ \theta_y^3 \\ \theta_y^4 \end{bmatrix}$$



The Reissner-Midlin plate theory

- **Small example – four node plate element**



$$\boldsymbol{\kappa}(r,s) = \mathbf{B}_\kappa \hat{\mathbf{u}}$$

$$\boldsymbol{\gamma}(r,s) = \mathbf{B}_\gamma \hat{\mathbf{u}},$$

$$w(r,s) = \mathbf{H}_w \hat{\mathbf{u}}$$

$$\hat{\mathbf{u}}^T = \begin{bmatrix} w_1 & \theta_x^1 & \theta_y^1; & \cdots & ; & w_4 & \theta_x^4 & \theta_y^4 \end{bmatrix}$$

$\Downarrow$

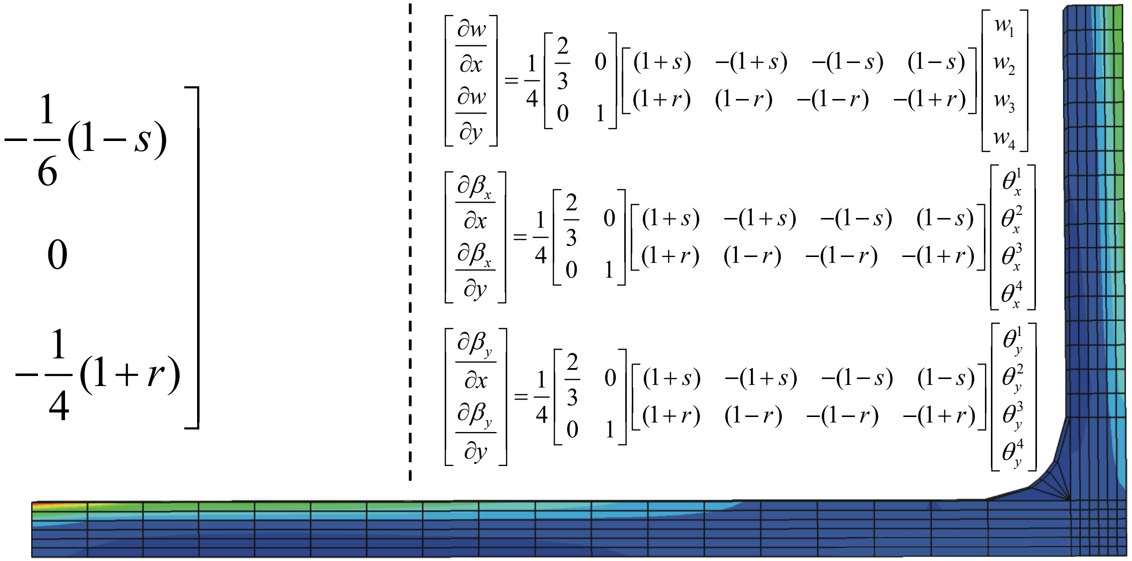
$$\mathbf{B}_\kappa = \begin{bmatrix} 0 & 0 & -\frac{1}{6}(1+s) & \cdots & -\frac{1}{6}(1-s) \\ 0 & \frac{1}{4}(1+r) & 0 & \cdots & 0 \\ 0 & \frac{1}{6}(1+s) & -\frac{1}{4}(1+r) & \cdots & -\frac{1}{4}(1+r) \end{bmatrix}$$

$$\boldsymbol{\kappa} = \begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \end{bmatrix} \qquad \boldsymbol{\gamma} = \begin{bmatrix} \frac{\partial w}{\partial x} - \beta_x \\ \frac{\partial w}{\partial y} - \beta_y \end{bmatrix}$$

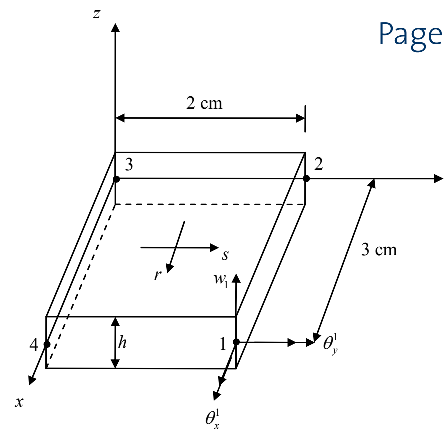
$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_x}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} \theta_x^1 \\ \theta_x^2 \\ \theta_x^3 \\ \theta_x^4 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial \beta_y}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} \theta_y^1 \\ \theta_y^2 \\ \theta_y^3 \\ \theta_y^4 \end{bmatrix}$$



The Reissner-Midlin plate theory



- **Small example – four node plate element**

$$\boldsymbol{\kappa}(r,s) = \mathbf{B}_{\kappa}\hat{\mathbf{u}}$$

$$\boldsymbol{\gamma}(r,s) = \mathbf{B}_{\gamma}\hat{\mathbf{u}}$$

$$w(r,s) = \mathbf{H}_{w}\hat{\mathbf{u}}$$

$$\hat{\mathbf{u}}^T = \begin{bmatrix} w_1 & \theta_x^1 & \theta_y^1; & \cdots & ; & w_4 & \theta_x^4 & \theta_y^4 \end{bmatrix}$$

$$\Downarrow$$

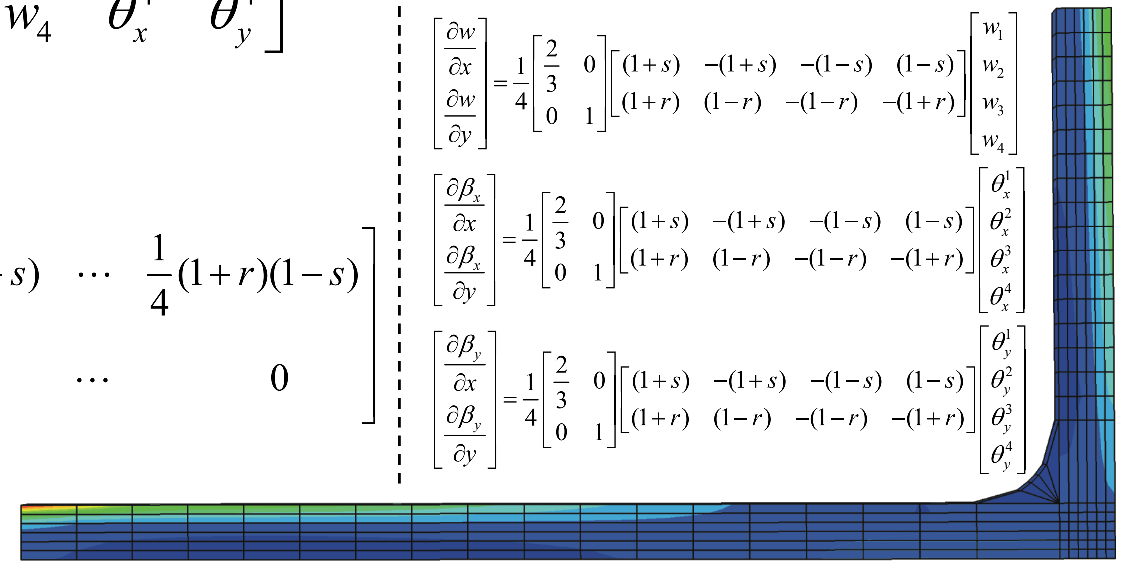
$$\mathbf{B}_{\gamma} = \begin{bmatrix} \frac{1}{6}(1+s) & 0 & \frac{1}{4}(1+r)(1+s) & \cdots & \frac{1}{4}(1+r)(1-s) \\ \frac{1}{4}(1+r) & -\frac{1}{4}(1+r)(1+s) & 0 & \cdots & 0 \end{bmatrix}$$

$$\boldsymbol{\kappa} = \begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \end{bmatrix} \qquad \boldsymbol{\gamma} = \begin{bmatrix} \frac{\partial w}{\partial x} - \beta_x \\ \frac{\partial w}{\partial y} - \beta_y \end{bmatrix}$$

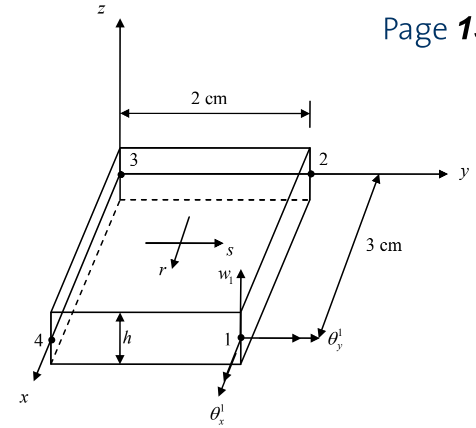
$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_x}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} \theta_x^1 \\ \theta_x^2 \\ \theta_x^3 \\ \theta_x^4 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial \beta_y}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} \theta_y^1 \\ \theta_y^2 \\ \theta_y^3 \\ \theta_y^4 \end{bmatrix}$$



The Reissner-Midlin plate theory



- **Small example – four node plate element**

$$\boldsymbol{\kappa}(r,s) = \mathbf{B}_\kappa \hat{\mathbf{u}}$$

$$\boldsymbol{\gamma}(r,s) = \mathbf{B}_\gamma \hat{\mathbf{u}},$$

$$\boxed{w(r,s) = \mathbf{H}_w \hat{\mathbf{u}}}$$

$$\hat{\mathbf{u}}^T = \begin{bmatrix} w_1 & \theta_x^1 & \theta_y^1; & \cdots & ; & w_4 & \theta_x^4 & \theta_y^4 \end{bmatrix}$$

$$\Downarrow$$

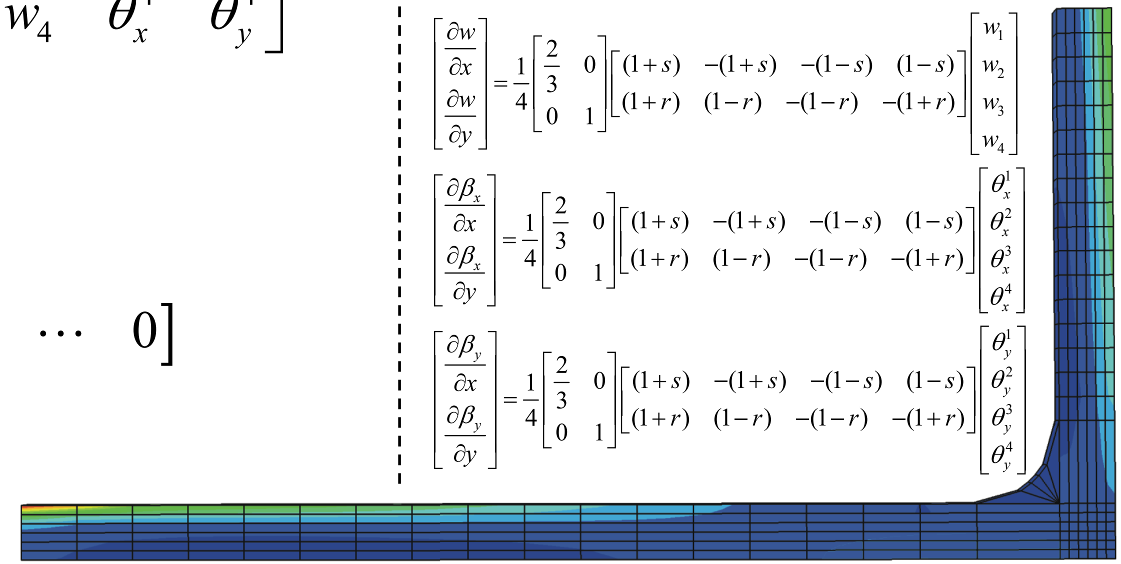
$$\mathbf{H}_w = \frac{1}{4}\begin{bmatrix}(1+r)(1+s) & 0 & 0 & \cdots & 0\end{bmatrix}$$

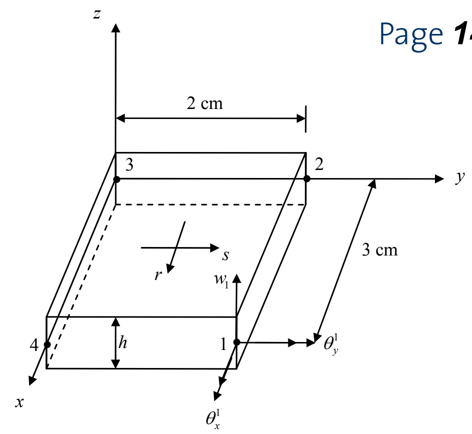
$$\boldsymbol{\kappa} = \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{bmatrix} \qquad \boldsymbol{\gamma} = \begin{bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_x}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} \theta_x^1 \\ \theta_x^2 \\ \theta_x^3 \\ \theta_x^4 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \beta_y}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} \theta_y^1 \\ \theta_y^2 \\ \theta_y^3 \\ \theta_y^4 \end{bmatrix}$$





The Reissner-Midlin plate theory

- **Small example – four node plate element**

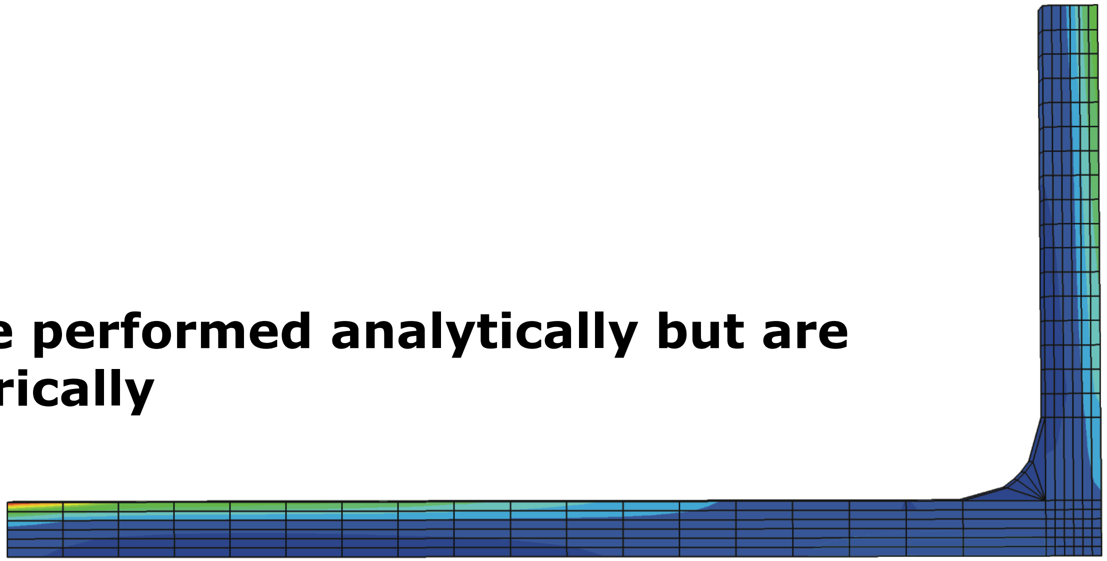
The element stiffness matrix then becomes:

$$\mathbf{K} = \frac{3}{2}\int_{-1}^{1}\int_{-1}^{1}\left(\mathbf{B}_{\kappa}^{T}\mathbf{C}_{b}\mathbf{B}_{\kappa} + \mathbf{B}_{\gamma}^{T}\mathbf{C}_{s}\mathbf{B}_{\gamma}\right)drds$$

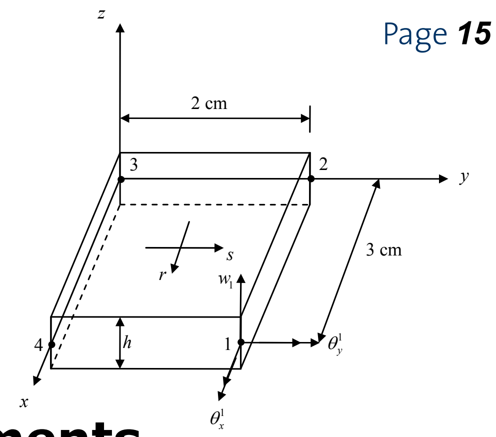
The consistent nodal force vector becomes:

$$\mathbf{R}_{S} = \frac{3}{2}\int_{-1}^{1}\int_{-1}^{1}\mathbf{H}_{w}^{T}p\,drds$$

These integrals can be performed analytically but are generally made numerically



The Reissner-Midlin plate theory



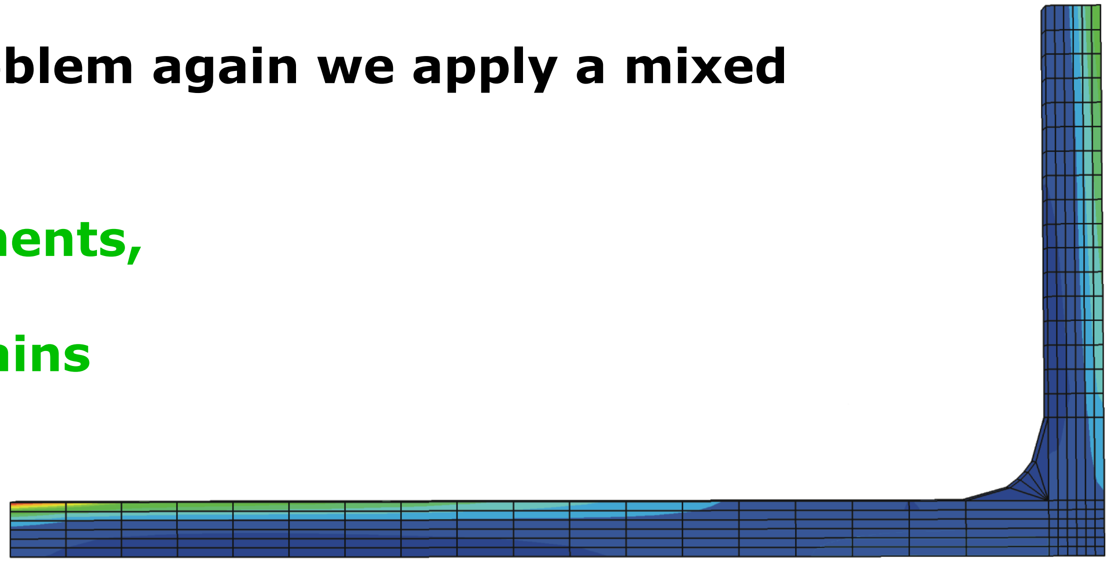
- **Small example – four node plate element**

This element formulated using only displacements is insufficiently precise unless a 16-node quadrilateral or 10-node triangular elements are used – as for the beam element the problem is – shear locking!

Shear locking is especially a problem for low-order elements, thin elements and distorted geometries

To circumvent this problem again we apply a mixed interpolation of

- transverse displacements,
- section rotations,
- transverse shear strains



The MITCn Element

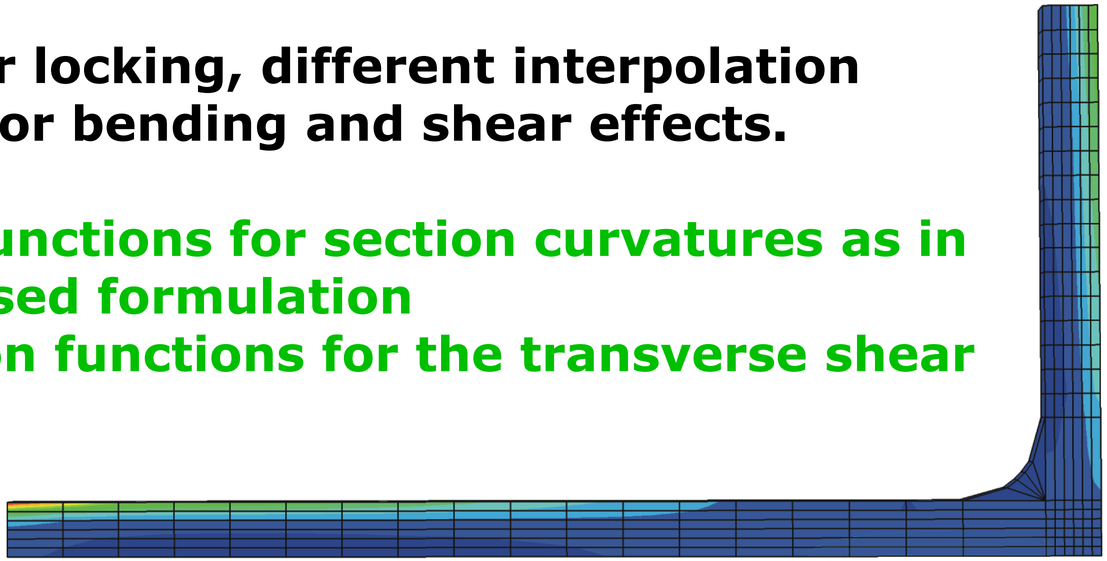
- **The MITC*n* element refers to:**

Mixed Interpolation of Tensorial Components and *n* refers to the number of nodes applied in the element

A main feature of this element is that it applies tensorial components of the shear strain in order to achieve a distortional insensitive representation of the shear strains.

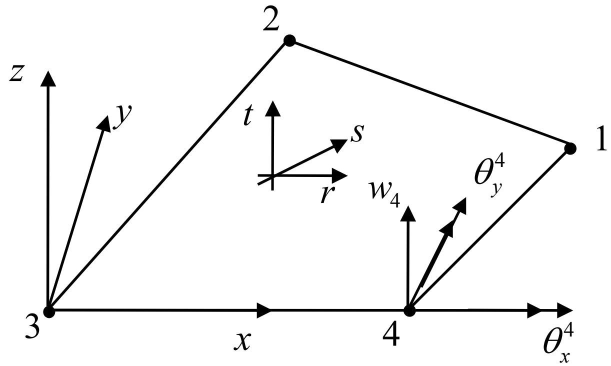
In order to avoid shear locking, different interpolation functions are applied for bending and shear effects.

- Same interpolation functions for section curvatures as in the displacement based formulation
- Different interpolation functions for the transverse shear strains

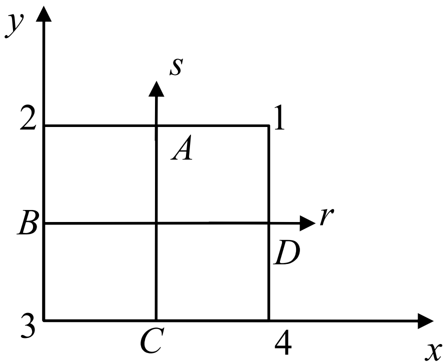


The MITC*n* Element

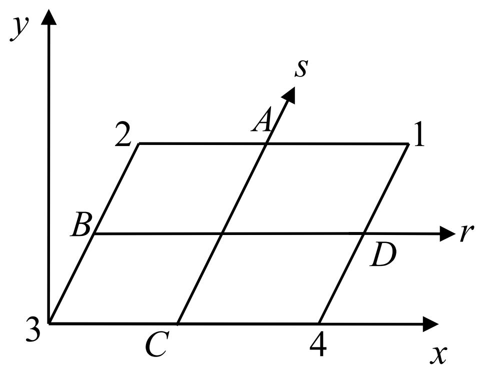
- **The MITC*4* element:**



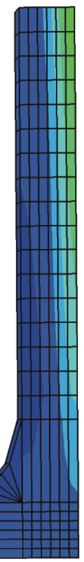
General quadrilateral element



2x2 element in x,y plane

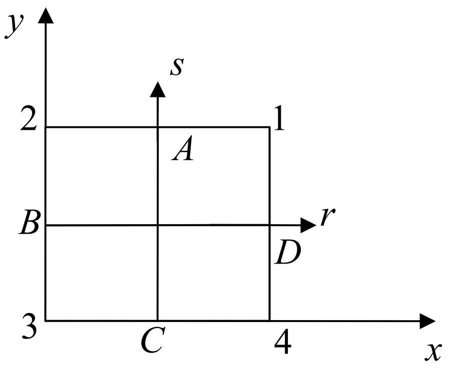


General element in x,y plane



The MITC*n* Element

- **The MITC*4* element:**



Let us consider first the 2x2 element

For this element we use the interpolation:

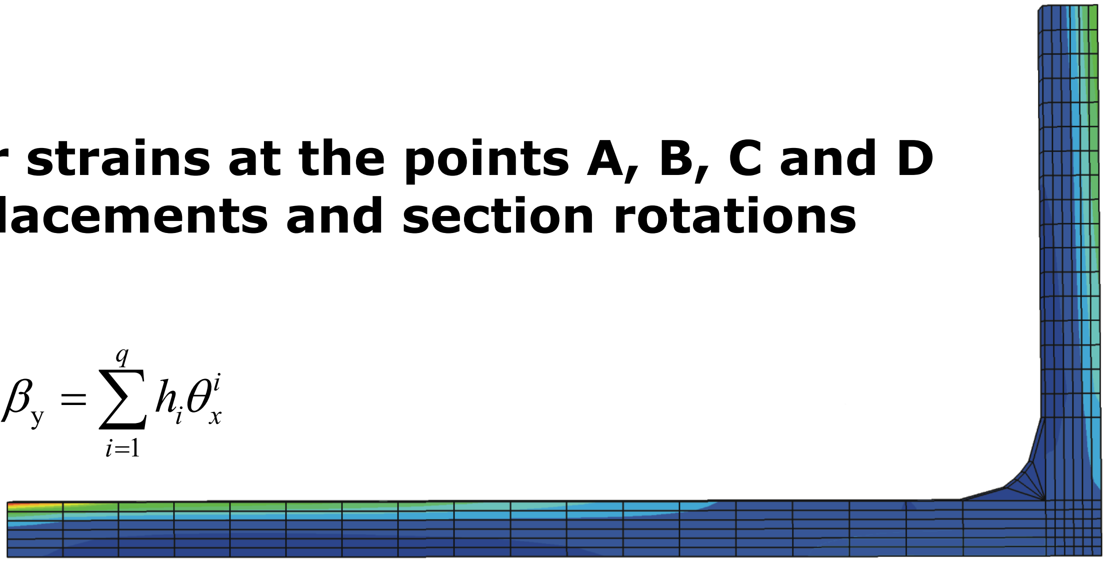
$$\gamma_{rz} = \frac{1}{2}(1+s)\gamma_{rz}^{A} + \frac{1}{2}(1-s)\gamma_{rz}^{C}$$
$$\gamma_{sz} = \frac{1}{2}(1+r)\gamma_{sz}^{D} + \frac{1}{2}(1-r)\gamma_{sz}^{B}$$

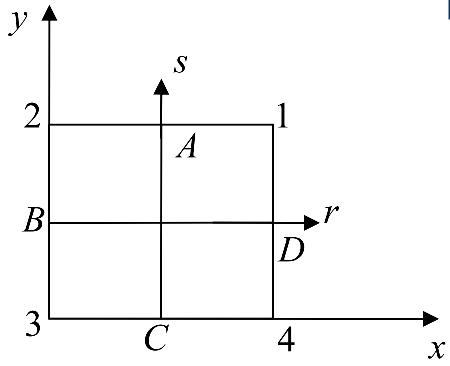
Proposed by Bathe & Dvorkin

where $\gamma_{rz}^{A}, \gamma_{rz}^{C}, \gamma_{sz}^{D}$ and γ_{sz}^{B}

are the physical shear strains at the points A, B, C and D evaluated by the displacements and section rotations established using

$$w = \sum_{i=1}^{q} h_i w_i, \quad \beta_x = \sum_{i=1}^{q} h_i \theta_y^i, \quad \beta_y = \sum_{i=1}^{q} h_i \theta_x^i$$





The MITC*n* Element

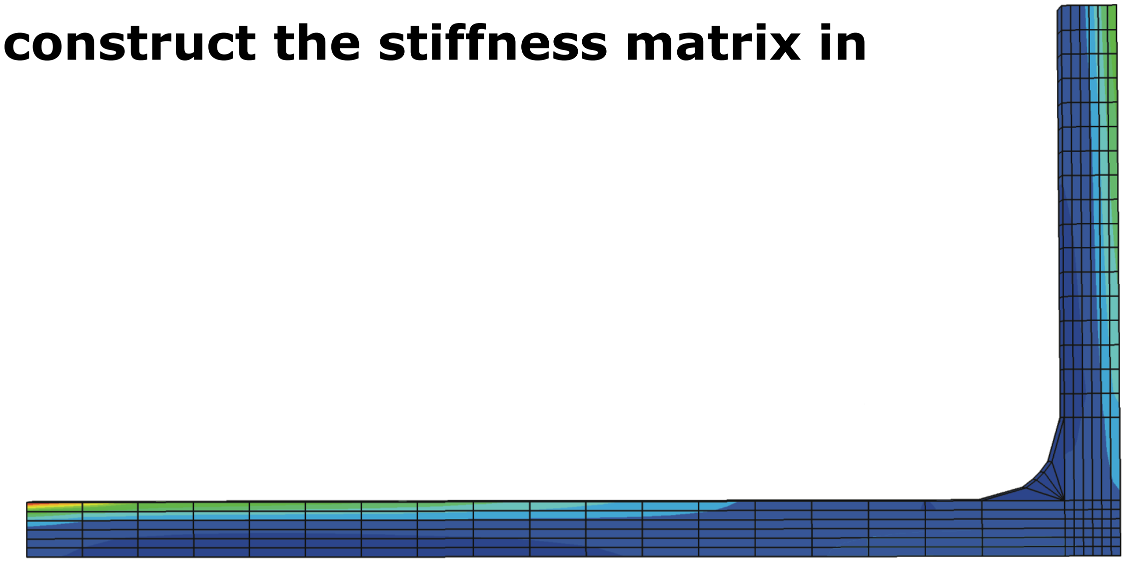
- ## The MITC*4* element:

Thus introducing the interpolation of displacements and section rotations we get:

$$\gamma_{rz} = \frac{1}{2}(1+s)\left(\frac{w_1 - w_2}{2} + \frac{\theta_y^1 - \theta_y^2}{2}\right) + \frac{1}{2}(1-s)\left(\frac{w_4 - w_3}{2} + \frac{\theta_y^4 - \theta_y^3}{2}\right)$$

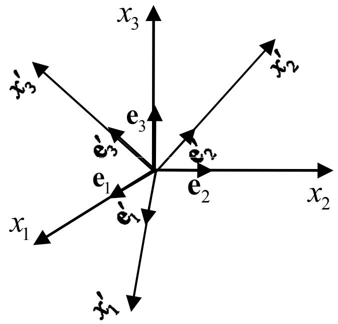
$$\gamma_{sz} = \frac{1}{2}(1+r)\left(\frac{w_1 - w_4}{2} + \frac{\theta_x^1 - \theta_x^4}{2}\right) + \frac{1}{2}(1-r)\left(\frac{w_2 - w_3}{2} + \frac{\theta_x^2 - \theta_x^3}{2}\right)$$

On this basis we can construct the stiffness matrix in the usual manner.



The MITC*n* Element

- **Before proceeding - let us have a short refreshment of tensor calculus:**

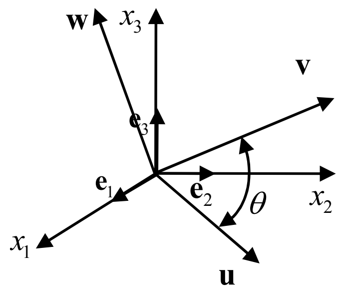


In tensor algebra we define a vector as:

$$\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e}_i = u_i \mathbf{e}_i$$ **Einstein index notation**

The scalar (or "dot") product between two vectors is given as

$$\mathbf{u} \bullet \mathbf{v} = u_i v_i$$

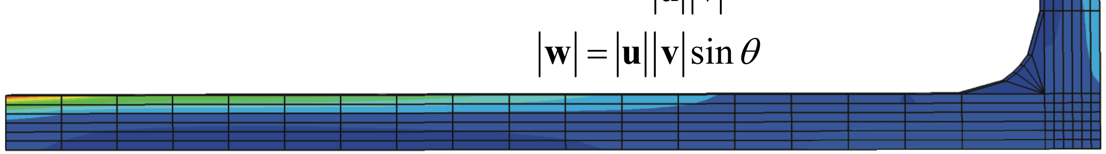


The "cross product" as

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

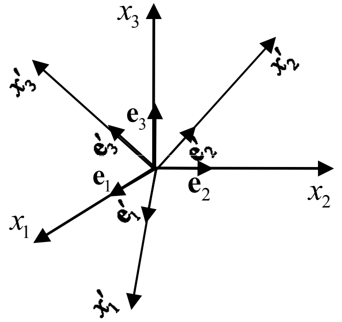
$$\cos\theta = \frac{u_i v_i}{|\mathbf{u}||\mathbf{v}|}$$

$$|\mathbf{w}| = |\mathbf{u}||\mathbf{v}|\sin\theta$$



The MITC*n* Element

- **Let us have a short refreshment of tensor calculus:**



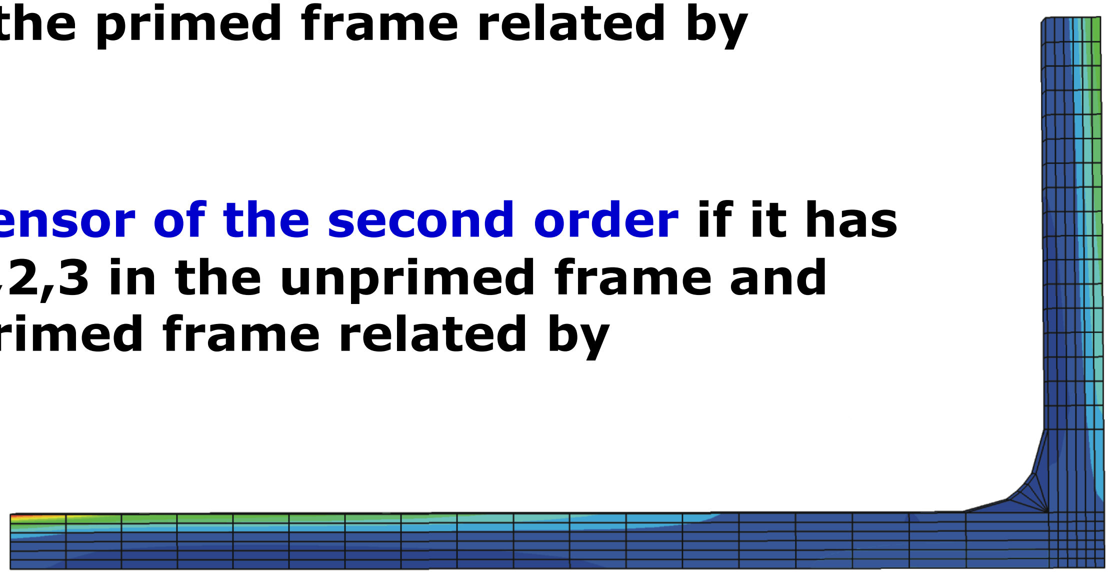
A scalar, or tensor of the 0 order is an entity which has only one component along the x_i-axis which does not change if measured along the x'_i-axis

An entity is called a vector or tensor of the first order if it has three components in the unprimed frame and three components in the primed frame related by

$$\xi'_i = p_{ik}\xi_k$$

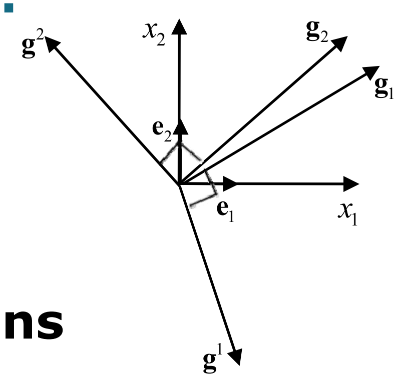
An entity is called a tensor of the second order if it has 9 component t_{ij}, i,j=1,2,3 in the unprimed frame and 9 component in the primed frame related by

$$t'_{ij} = p_{ik}p_{jl}t_{kl}$$



The MITC*n* Element

- **Let us have a short refreshment of tensor calculus:**

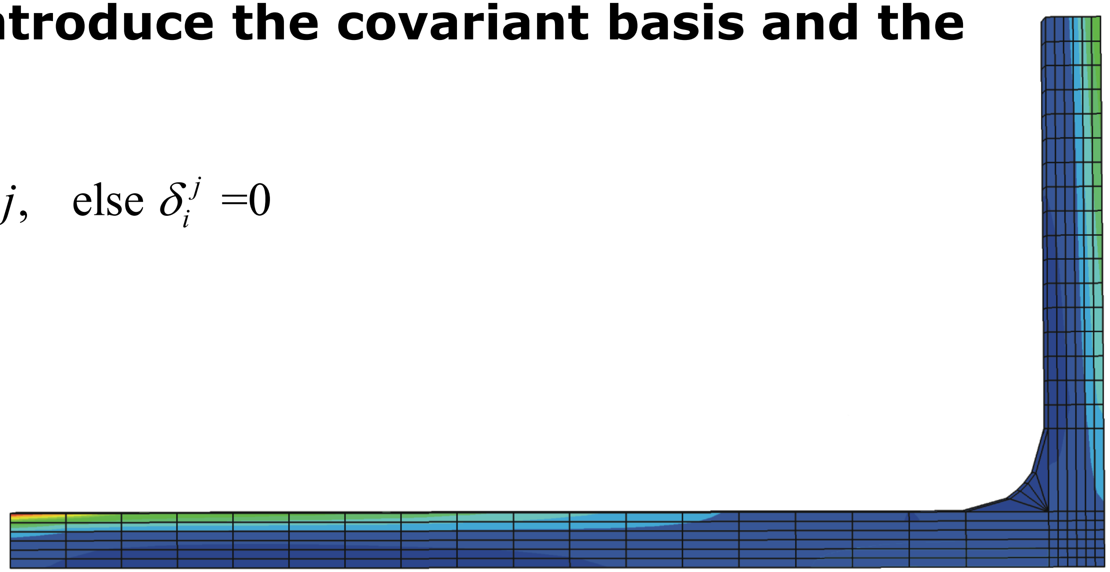


Often we operate with an orthogonal frame

but we can also express tensors in a non-orthogonal frame ! – and this is convenient when we are dealing with element representations in natural coordinates (plates and shells)

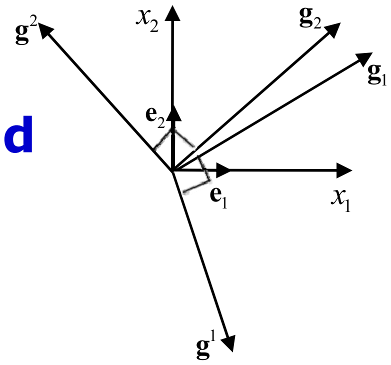
for this purpose we introduce the covariant basis and the contravariant basis

$$\mathbf{g}_i \bullet \mathbf{g}^j = \delta_i^j, \qquad \delta_i^j = 1 \text{ for } i = j, \quad \text{else } \delta_i^j = 0$$



The MITC*n* Element

- **Let us have a short refreshment of tensor calculus:**



To illustrate this consider the calculation of work i.e. the dot product of the force vector R and the displacement vector u

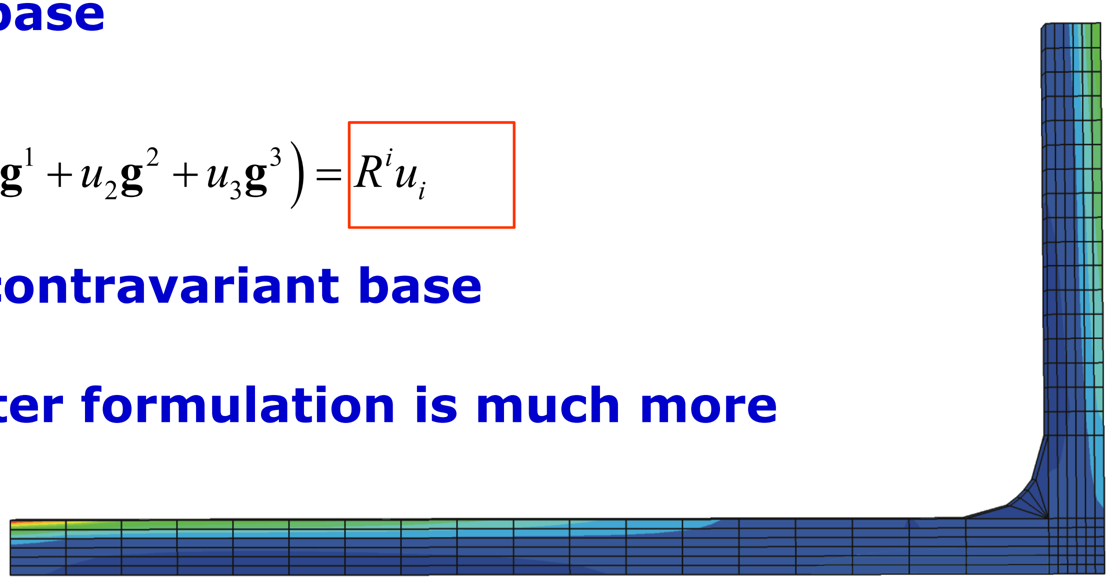
$$\mathbf{R}\bullet\mathbf{u} = \left(R^1\mathbf{g}_1 + R^2\mathbf{g}_2 + R^3\mathbf{g}_3\right)\bullet\left(u^1\mathbf{g}_1 + u^2\mathbf{g}_2 + u^3\mathbf{g}_3\right) = \boxed{R^i u^j g_{ij}}$$

R and u in covariant base

$$\mathbf{R}\bullet\mathbf{u} = \left(R^1\mathbf{g}_1 + R^2\mathbf{g}_2 + R^3\mathbf{g}_3\right)\bullet\left(u_1\mathbf{g}^1 + u_2\mathbf{g}^2 + u_3\mathbf{g}^3\right) = \boxed{R^i u_i}$$

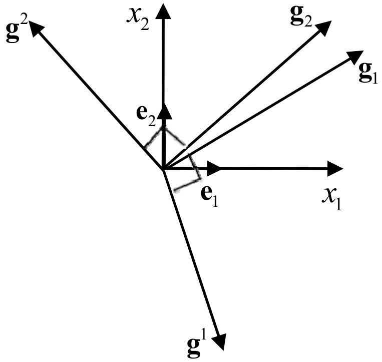
R in covariant – u in contravariant base

It is seen that the latter formulation is much more convenient!



The MITC*n* Element

- **Let us have a short refreshment of tensor calculus:**



Now we can write the stress tensor in a contravariant base as:

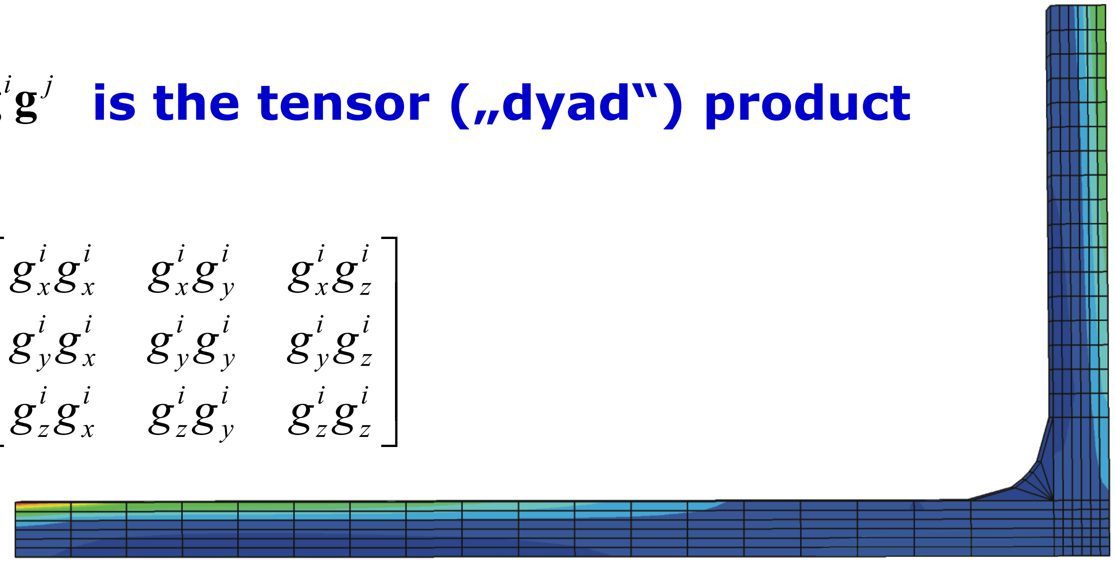
$$\boldsymbol{\tau} = \tilde{\tau}^{mn}\mathbf{g}_m\mathbf{g}_n$$

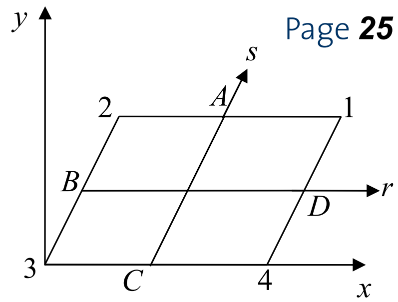
and the strain tensor in a covariant bases as:

$$\boldsymbol{\varepsilon} = \tilde{\varepsilon}_{ij}\mathbf{g}^i\mathbf{g}^j$$

We remember that $\mathbf{g}^i\mathbf{g}^j$ **is the tensor („dyad") product given as:**

$$\mathbf{g}^i\mathbf{g}^j = \begin{bmatrix} g_x^i \\ g_y^i \\ g_z^i \end{bmatrix}\begin{bmatrix} g_x^i & g_y^i & g_z^i \end{bmatrix} = \begin{bmatrix} g_x^i g_x^i & g_x^i g_y^i & g_x^i g_z^i \\ g_y^i g_x^i & g_y^i g_y^i & g_y^i g_z^i \\ g_z^i g_x^i & g_z^i g_y^i & g_z^i g_z^i \end{bmatrix}$$





The MITC*n* Element

- **Example (general quadrilateral 4-node element)**

First we determine the covariant base (generally non-orthogonal) vectors in the natural coordinate system

$$\mathbf{g}_r = \frac{\partial \mathbf{x}}{\partial r}, \quad \mathbf{g}_s = \frac{\partial \mathbf{x}}{\partial s}, \quad \mathbf{g}_z = \frac{h}{2}\mathbf{e}_z$$

x is the vector of coordinates

We recall that in natural coordinates the strain vector can be expressed through the covariant tensor components and the contravariant base vectors as:

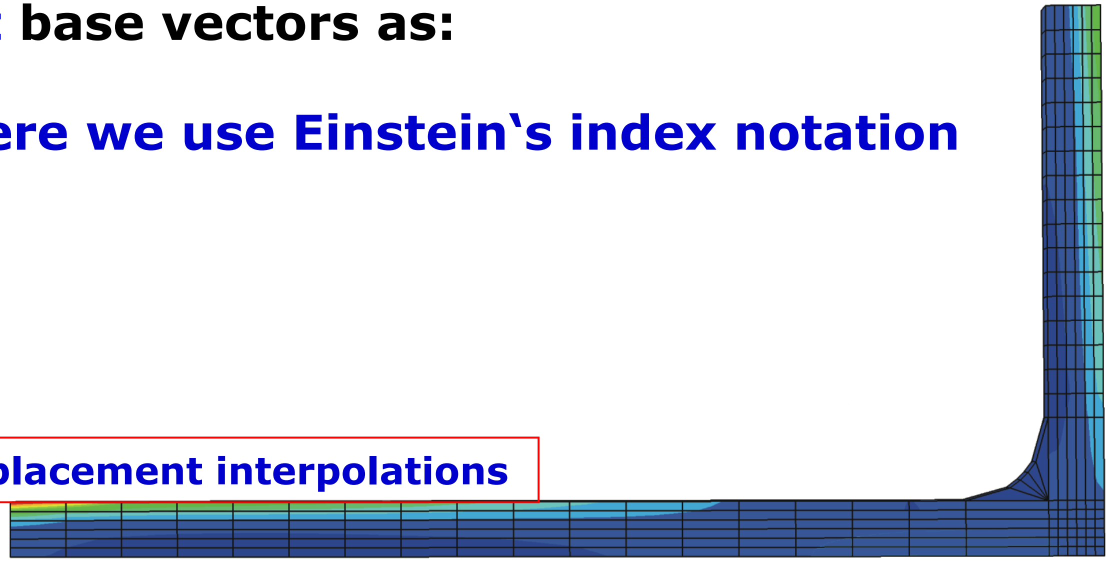
$$\boldsymbol{\varepsilon} = \tilde{\boldsymbol{\varepsilon}}_{ij}\mathbf{g}^i\mathbf{g}^j, \quad i, j = r, s, t$$

Here we use Einstein‘s index notation

$$\tilde{\varepsilon}_{rz} = \frac{1}{2}(1+s)\tilde{\varepsilon}_{rz}^{A} + \frac{1}{2}(1-s)\tilde{\varepsilon}_{rz}^{C}$$

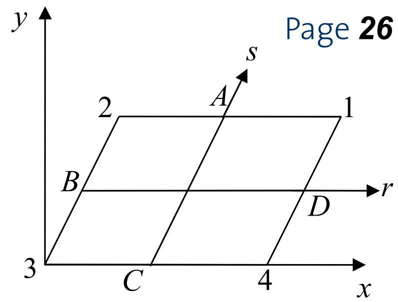
$$\tilde{\varepsilon}_{sz} = \frac{1}{2}(1+r)\tilde{\varepsilon}_{sz}^{D} + \frac{1}{2}(1-r)\tilde{\varepsilon}_{sz}^{B}$$

To be determined from displacement interpolations



The MITC*n* Element

- **Example (general quadrilateral 4-node element)**



$$w = \sum_{i=1}^{q} h_i w_i, \quad \beta_x = \sum_{i=1}^{q} h_i \theta_y^i, \quad \beta_y = \sum_{i=1}^{q} h_i \theta_x^i$$

$$\mathbf{g}_r = \frac{\partial \mathbf{x}}{\partial r}, \quad \mathbf{g}_s = \frac{\partial \mathbf{x}}{\partial s}, \quad \mathbf{g}_z = \frac{h}{2}\mathbf{e}_z$$

We now obtain:

From displacement interpolations

$${}_0^1\tilde{\varepsilon}_{ij} = \frac{1}{2}\left[{}^1\mathbf{g}_i \bullet {}^1\mathbf{g}_j - {}^0\mathbf{g}_i \bullet {}^0\mathbf{g}_j\right] \quad \Rightarrow$$

$$\tilde{\varepsilon}_{rz}^{A} = \frac{1}{4}\left[\frac{h(w_1 - w_2)}{2} + \frac{h(x_1 - x_2)}{4}(\theta_y^1 + \theta_y^2) - \frac{h(y_1 - y_2)}{4}(\theta_x^1 + \theta_x^2)\right]$$

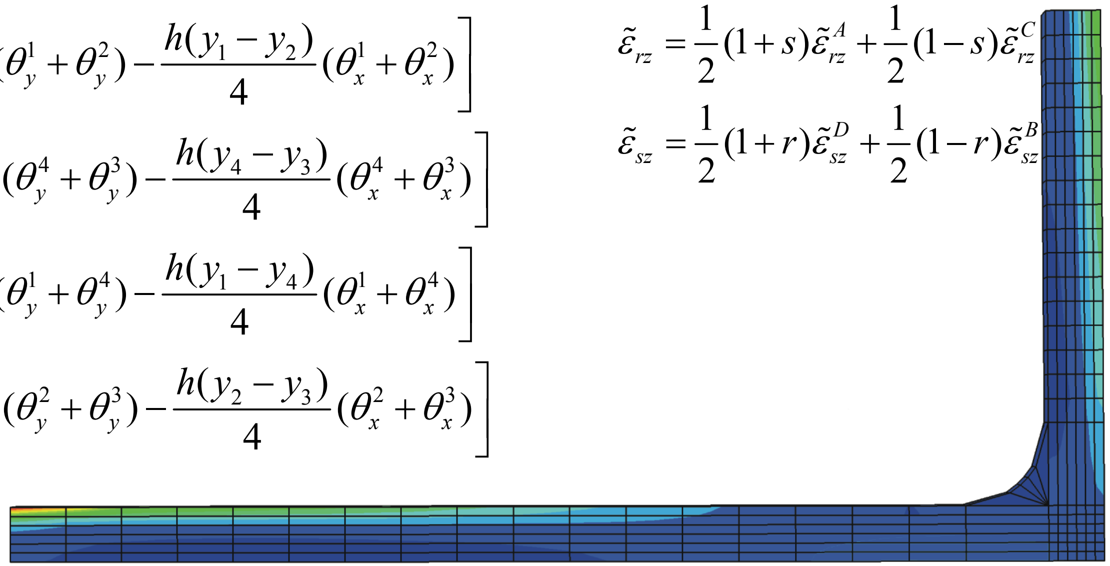
$$\tilde{\varepsilon}_{rz}^{C} = \frac{1}{4}\left[\frac{h(w_4 - w_3)}{2} + \frac{h(x_4 - x_3)}{4}(\theta_y^4 + \theta_y^3) - \frac{h(y_4 - y_3)}{4}(\theta_x^4 + \theta_x^3)\right]$$

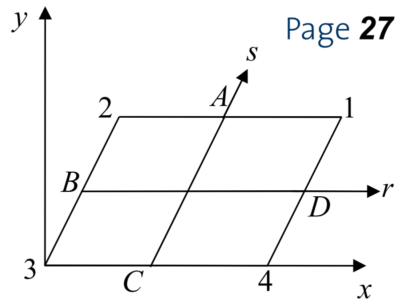
$$\tilde{\varepsilon}_{rz} = \frac{1}{8}(1+s)\left[\frac{h(w_1 - w_2)}{2} + \frac{h(x_1 - x_2)}{4}(\theta_y^1 + \theta_y^2) - \frac{h(y_1 - y_2)}{4}(\theta_x^1 + \theta_x^2)\right]$$
$$+ \frac{1}{8}(1-s)\left[\frac{h(w_4 - w_3)}{2} + \frac{h(x_4 - x_3)}{4}(\theta_y^4 + \theta_y^3) - \frac{h(y_4 - y_3)}{4}(\theta_x^4 + \theta_x^3)\right]$$

$$\tilde{\varepsilon}_{sz} = \frac{1}{8}(1+r)\left[\frac{h(w_1 - w_4)}{2} + \frac{h(x_1 - x_4)}{4}(\theta_y^1 + \theta_y^4) - \frac{h(y_1 - y_4)}{4}(\theta_x^1 + \theta_x^4)\right]$$
$$+ \frac{1}{8}(1-r)\left[\frac{h(w_2 - w_3)}{2} + \frac{h(x_2 - x_3)}{4}(\theta_y^2 + \theta_y^3) - \frac{h(y_2 - y_3)}{4}(\theta_x^2 + \theta_x^3)\right]$$

$$\tilde{\varepsilon}_{rz} = \frac{1}{2}(1+s)\tilde{\varepsilon}_{rz}^{A} + \frac{1}{2}(1-s)\tilde{\varepsilon}_{rz}^{C}$$

$$\tilde{\varepsilon}_{sz} = \frac{1}{2}(1+r)\tilde{\varepsilon}_{sz}^{D} + \frac{1}{2}(1-r)\tilde{\varepsilon}_{sz}^{B}$$





The MITC*n* Element

- **We look at an example:**

We now use that:

$$\tilde{\varepsilon}_{ij}\mathbf{g}^i\mathbf{g}^j = \varepsilon_{kl}\mathbf{e}_k\mathbf{e}_l$$

whereby

$$\gamma_{xz} = 2\tilde{\varepsilon}_{rz}(\mathbf{g}^r \bullet \mathbf{e}_x)(\mathbf{g}^z \bullet \mathbf{e}_z) + 2\tilde{\varepsilon}_{sz}(\mathbf{g}^s \bullet \mathbf{e}_x)(\mathbf{g}^z \bullet \mathbf{e}_z)$$
$$\gamma_{yz} = 2\tilde{\varepsilon}_{rz}(\mathbf{g}^r \bullet \mathbf{e}_y)(\mathbf{g}^z \bullet \mathbf{e}_z) + 2\tilde{\varepsilon}_{sz}(\mathbf{g}^s \bullet \mathbf{e}_y)(\mathbf{g}^z \bullet \mathbf{e}_z)$$

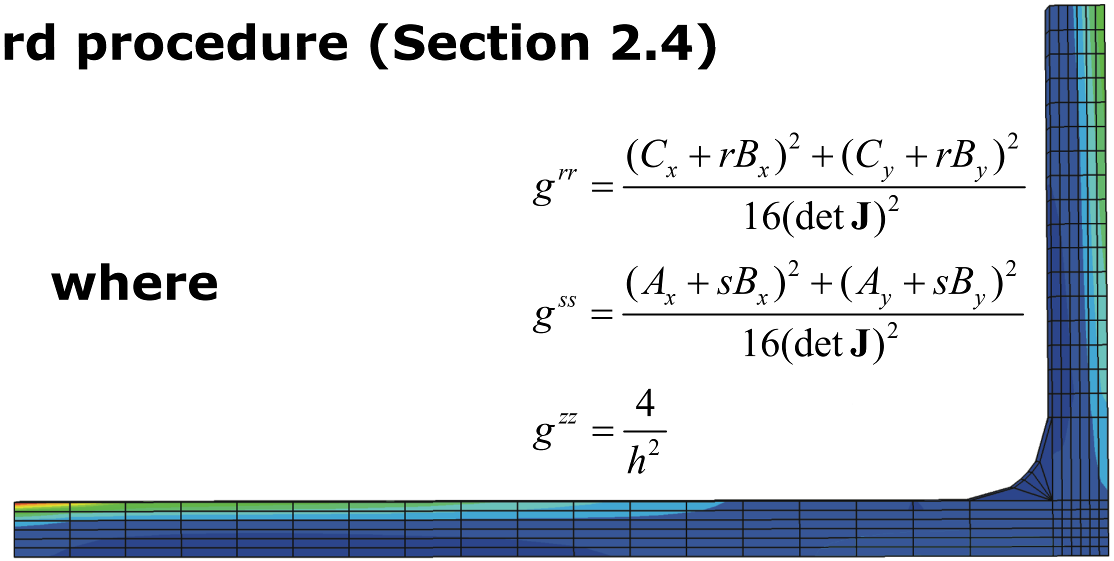
$$A_x = x_1 - x_2 - x_3 + x_4$$
$$A_y = y_1 - y_2 - y_3 + y_4$$
$$B_x = x_1 - x_2 + x_3 - x_4$$
$$B_y = y_1 - y_2 + y_3 - y_4$$
$$C_x = x_1 + x_2 - x_3 - x_4$$
$$C_y = y_1 + y_2 - y_3 - y_4$$

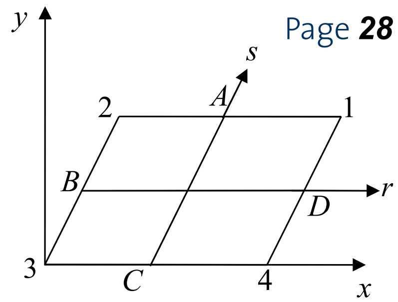
now using the standard procedure (Section 2.4)

$$\mathbf{g}^r = \sqrt{g^{rr}}(\sin\beta\mathbf{e}_x - \cos\beta\mathbf{e}_y)$$
$$\mathbf{g}^s = \sqrt{g^{ss}}(-\sin\alpha\mathbf{e}_x + \cos\alpha\mathbf{e}_y)$$
$$\mathbf{g}^z = \sqrt{g^{zz}}\mathbf{e}_z$$

where

$$g^{rr} = \frac{(C_x + rB_x)^2 + (C_y + rB_y)^2}{16(\det\mathbf{J})^2}$$
$$g^{ss} = \frac{(A_x + sB_x)^2 + (A_y + sB_y)^2}{16(\det\mathbf{J})^2}$$
$$g^{zz} = \frac{4}{h^2}$$





The MITC*n* Element

- **We look at an example:**

Now we can insert into

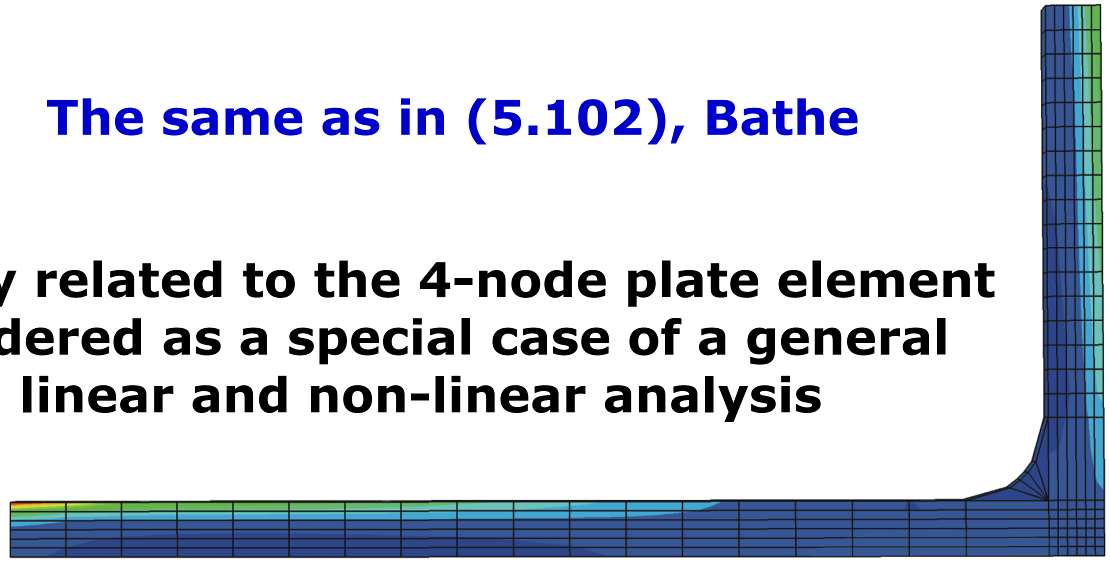
$$\gamma_{xz} = 2\tilde{\varepsilon}_{rz}(\mathbf{g}^r \bullet \mathbf{e}_x)(\mathbf{g}^z \bullet \mathbf{e}_z) + 2\tilde{\varepsilon}_{sz}(\mathbf{g}^s \bullet \mathbf{e}_x)(\mathbf{g}^z \bullet \mathbf{e}_z)$$
$$\gamma_{yz} = 2\tilde{\varepsilon}_{rz}(\mathbf{g}^r \bullet \mathbf{e}_y)(\mathbf{g}^z \bullet \mathbf{e}_z) + 2\tilde{\varepsilon}_{sz}(\mathbf{g}^s \bullet \mathbf{e}_y)(\mathbf{g}^z \bullet \mathbf{e}_z)$$

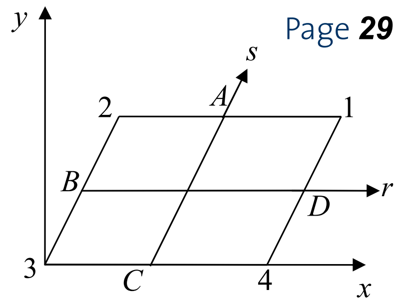
and achieve the relations

$$\gamma_{xz} = \gamma_{rz} \sin\beta - \gamma_{sz} \sin\alpha$$
$$\gamma_{yz} = -\gamma_{rz} \cos\beta + \gamma_{sz} \cos\alpha$$

The same as in (5.102), Bathe

This element is closely related to the 4-node plate element however can be considered as a special case of a general shell element for both linear and non-linear analysis





Performance Considerations

- **General considerations:**

This MITC*4* element is closely related to the 4-node plate element, however can considered a special case of a general shell element for both linear and non-linear analysis

In Bathe pp. 430-432 interpolation functions are provided for MITC 4,9,16,7 and 12 elements

The elements
- are based on full Gauss integration
- do not contain any spurious zero energy modes
- pass the "patch test" (we will look at this again later)