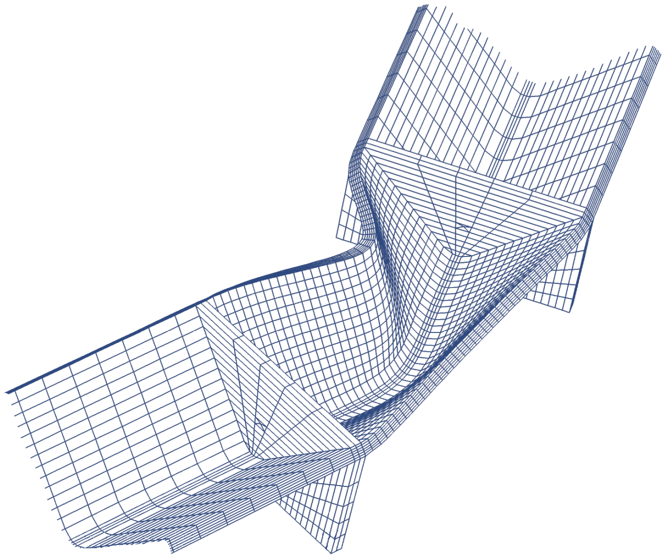
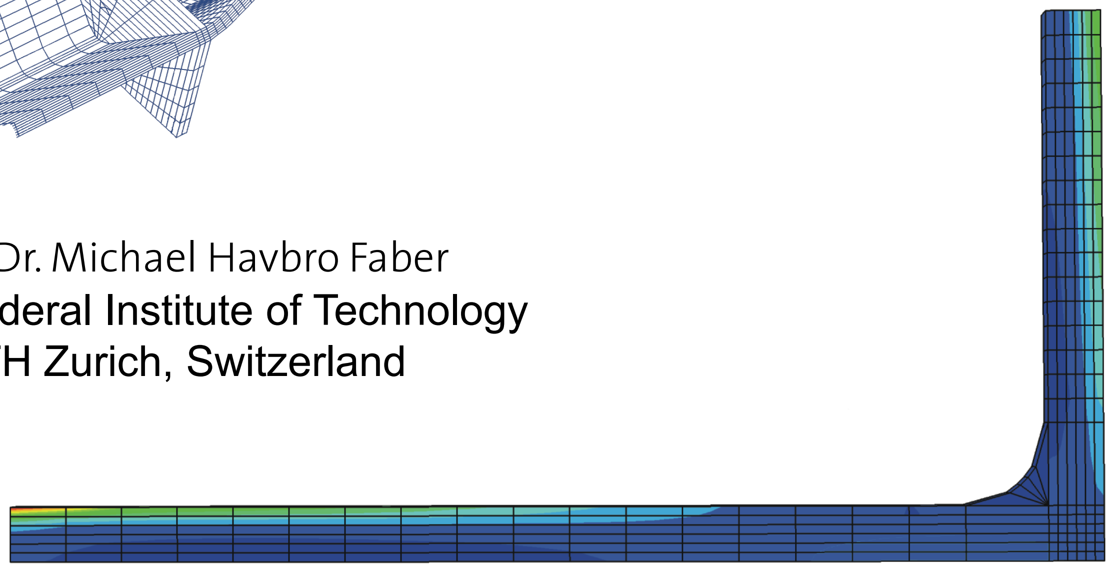
The Finite Element Method for the Analysis of Linear Systems



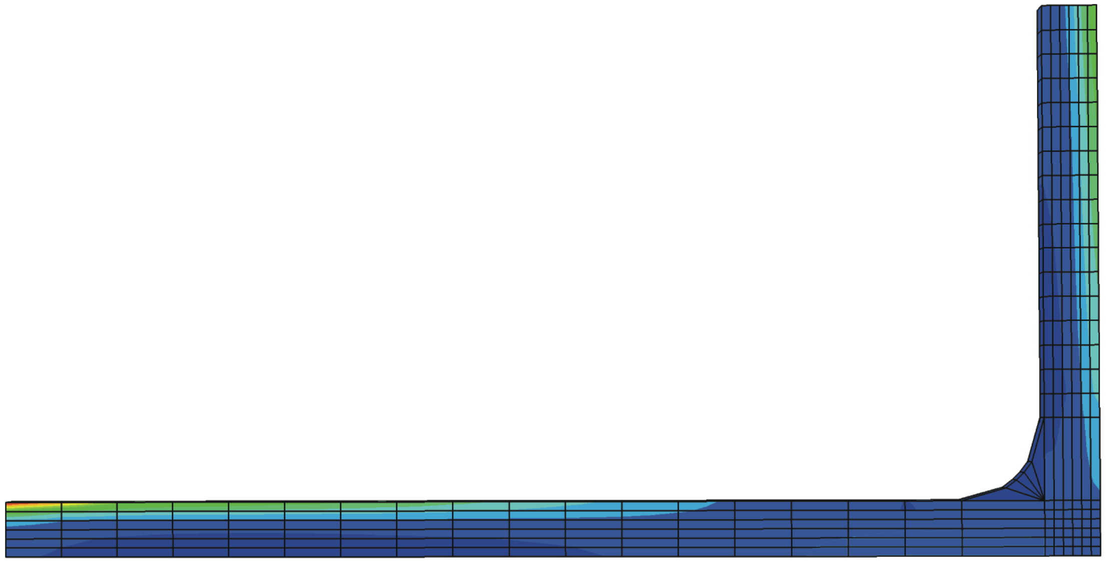
Prof. Dr. Michael Havbro Faber
Swiss Federal Institute of Technology
ETH Zurich, Switzerland



Contents of Today's Lecture

- **Formulation of structural elements**

  Beam and axis-symmetric shell elements
  - Straight beam elements
  - General curved beam elements
  - Transition elements
  - Axis-symmetric shell elements



Formulation of Structural Elements

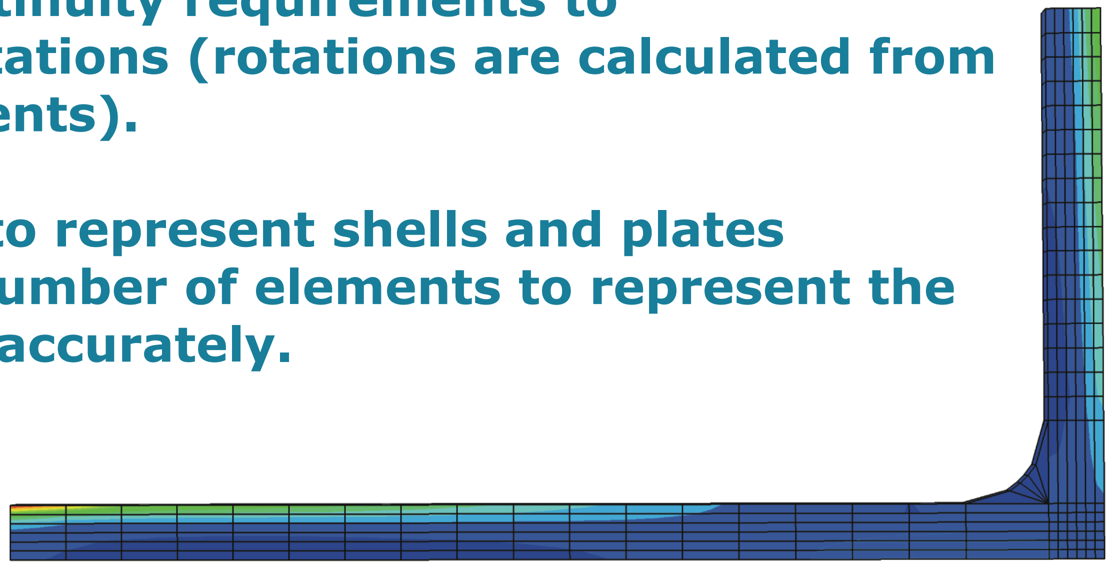
Introduction:

Previously we considered beam elements in which we neglected the shear deformations (Bernoulli beams).

There are drawbacks of this approach:

- Its generalization to shells and plates is difficult, as it is difficult to satisfy continuity requirements to displacements and rotations (rotations are calculated from transverse displacements).

- Using flat elements to represent shells and plates necessitates a large number of elements to represent the geometry sufficiently accurately.



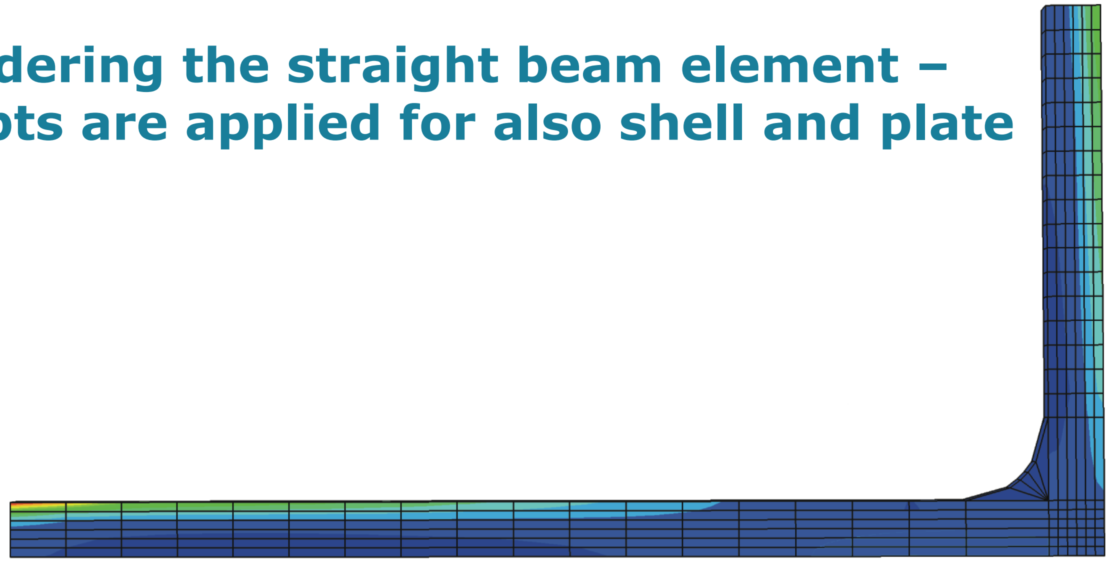
Formulation of Structural Elements

Introduction:

We will now consider beam, shell and plate elements where we take into account the shear deformations.

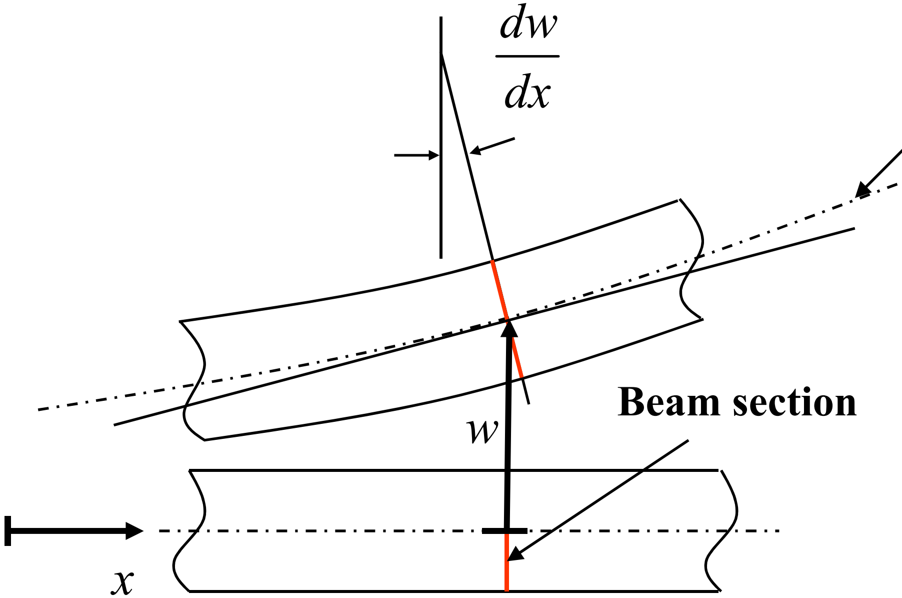
Furthermore we will apply iso-parametric formulations allowing for the accurate representation of general curved shell geometries.

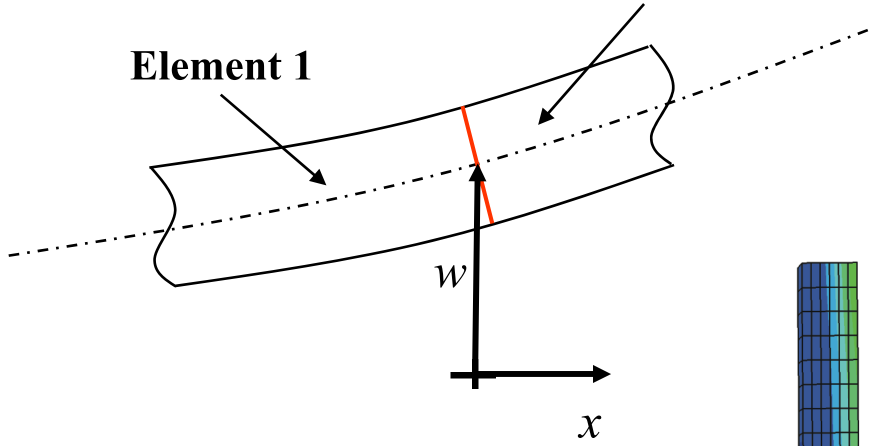
Start is taken in considering the straight beam element – whereafter the concepts are applied for also shell and plate elements.



Formulation of Structural Elements

Straight beam elements: Neglecting shear effects



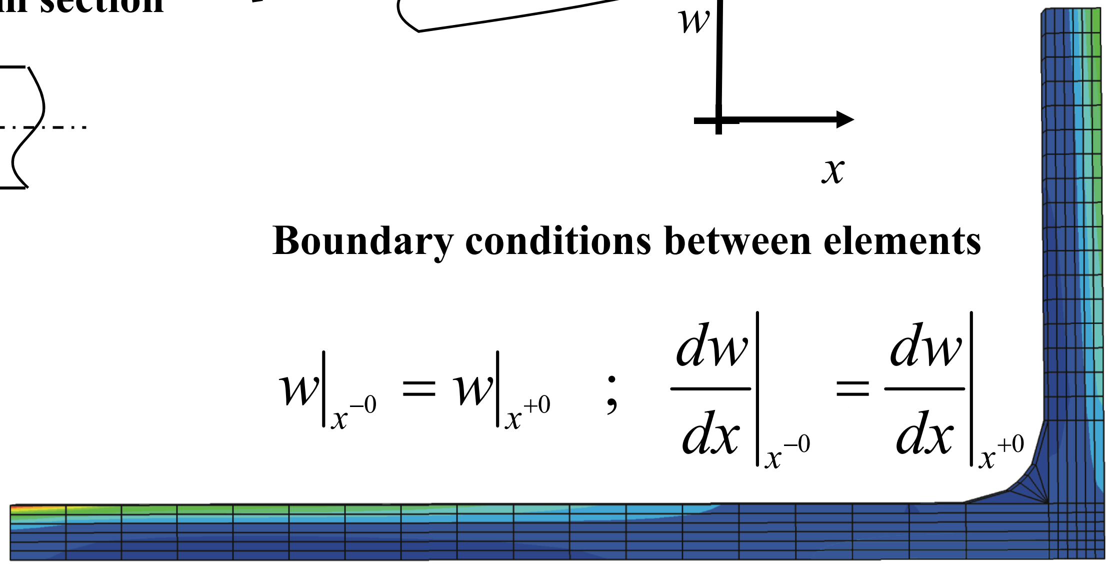


Bernoulli beam theory

- continuity in: $w, \frac{dw}{dx}$

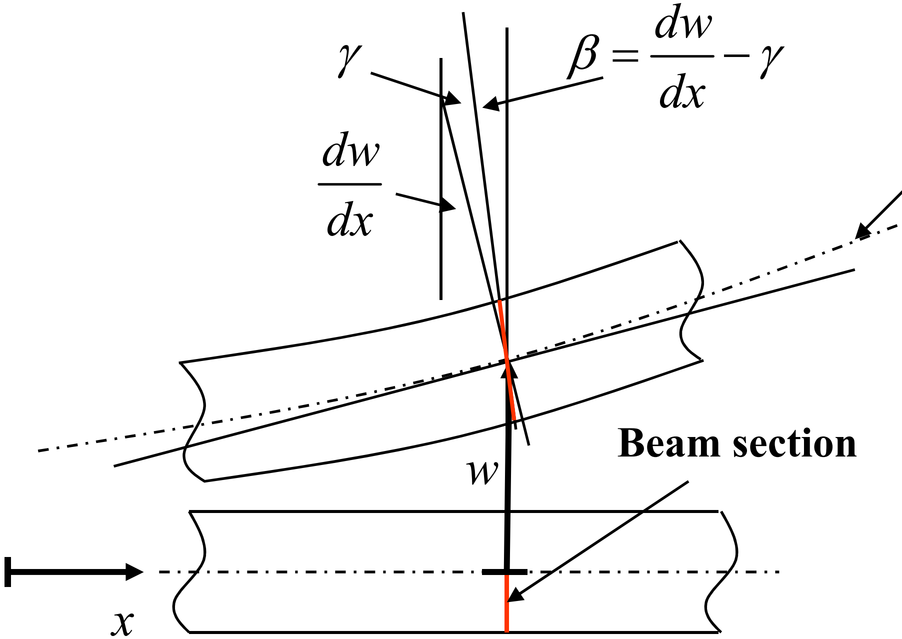
Boundary conditions between elements

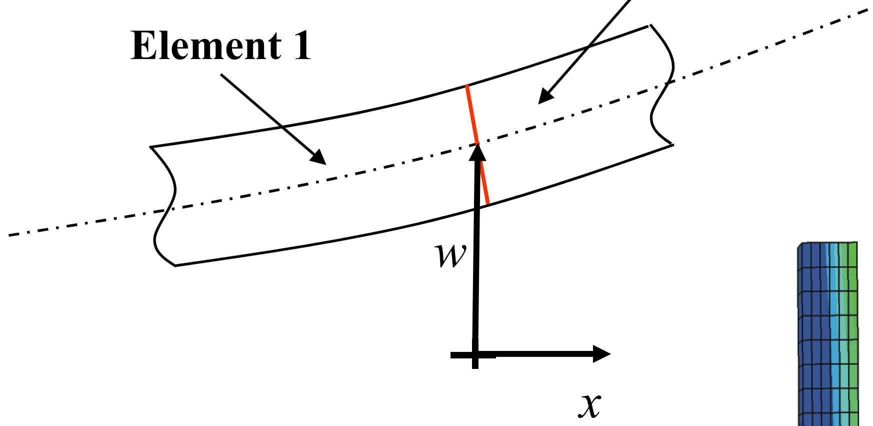
$$w\big|_{x^{-0}} = w\big|_{x^{+0}} \quad ; \quad \frac{dw}{dx}\bigg|_{x^{-0}} = \frac{dw}{dx}\bigg|_{x^{+0}}$$



Formulation of Structural Elements

Straight beam elements: Including shear effects

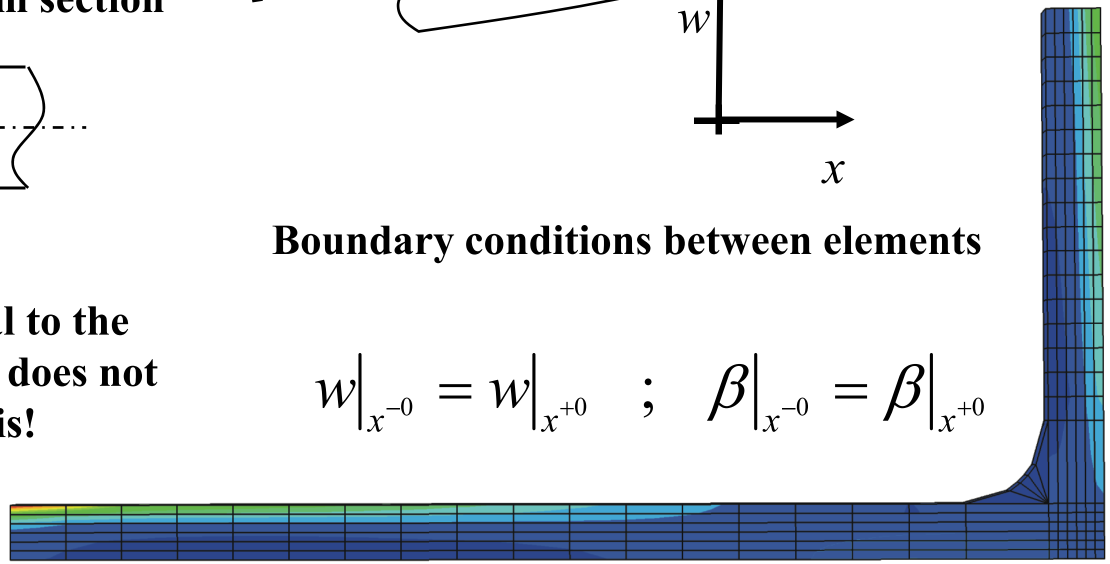




Boundary conditions between elements

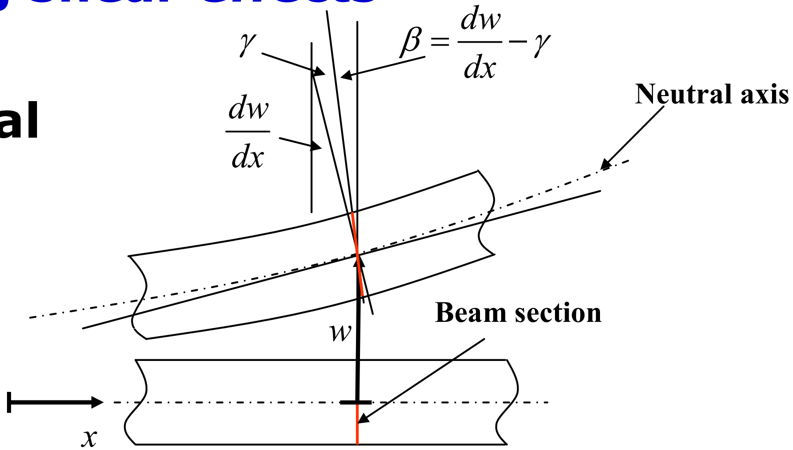
A plane section originally normal to the neutral axis remains plane – but does not remain normal to the neutral axis!

$$w\big|_{x^{-0}} = w\big|_{x^{+0}} \quad ; \quad \beta\big|_{x^{-0}} = \beta\big|_{x^{+0}}$$



Formulation of Structural Elements

Straight beam elements: Including shear effects

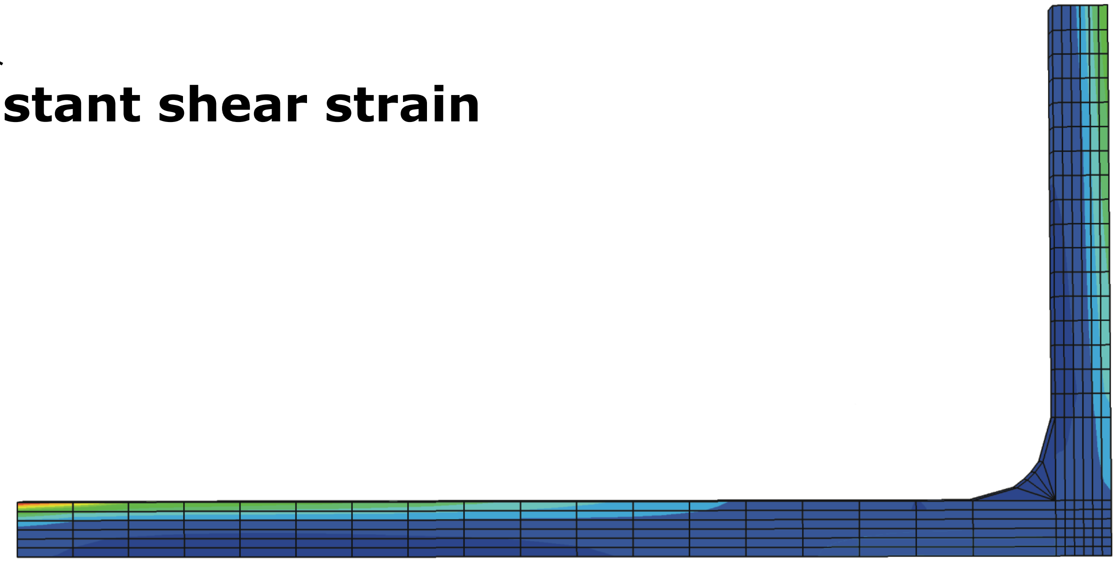


The total rotation of a plane normal to the neutral axis becomes

$$\beta = \frac{dw}{dx} - \gamma$$

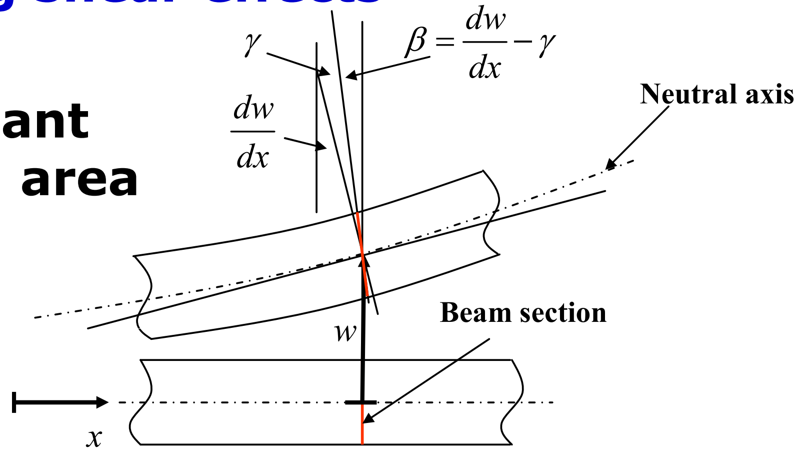
Constant shear strain

Tangent rotation



Formulation of Structural Elements

Straight beam elements: Including shear effects

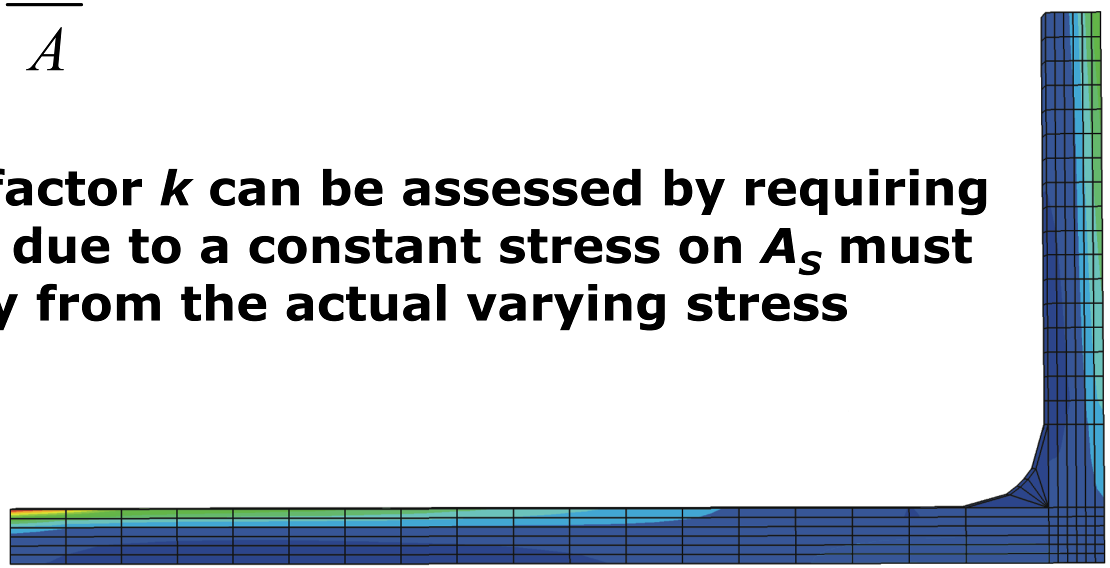


The shear strain is assumed constant over an equivalent cross sectional area

Equivalent shear area

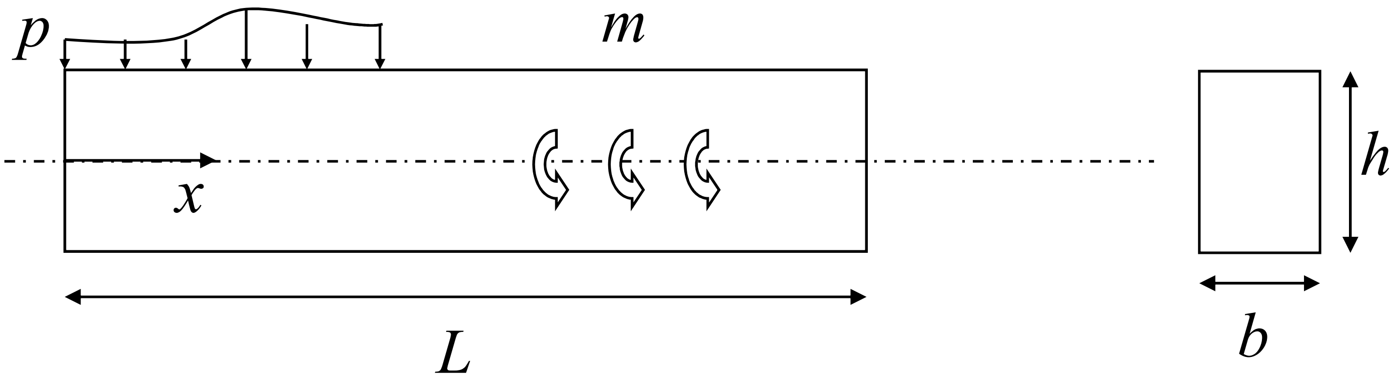
$$\tau = \frac{V}{A}; \quad \gamma = \frac{\tau}{G}; \quad k = \frac{A_S}{A}$$

The shear correction factor *k* can be assessed by requiring that the shear energy due to a constant stress on A_S must be equal to the energy from the actual varying stress (from beam theory)



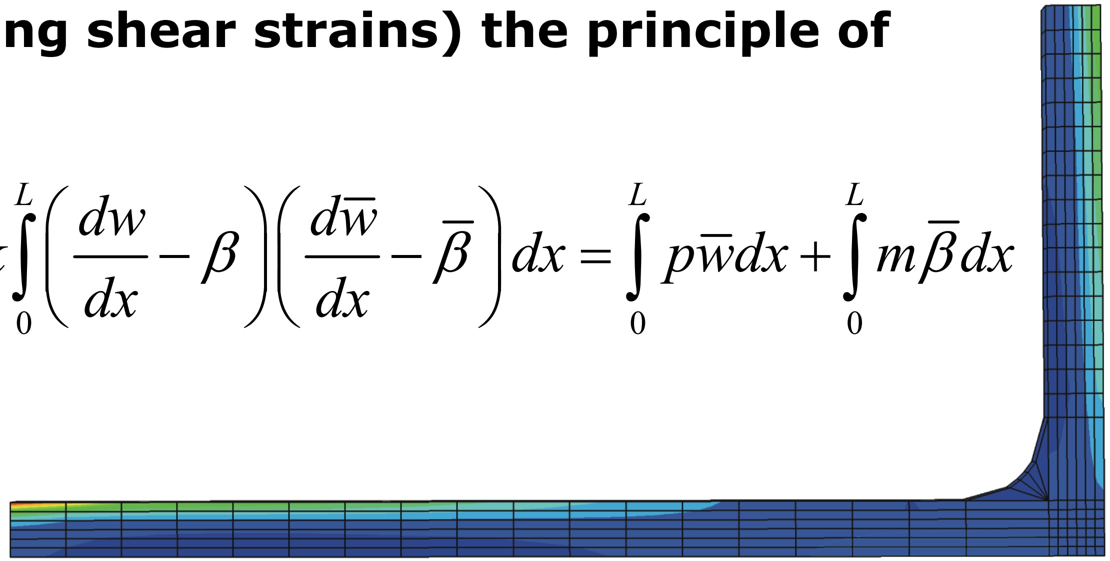
Formulation of Structural Elements

Two-dimensional beam elements:



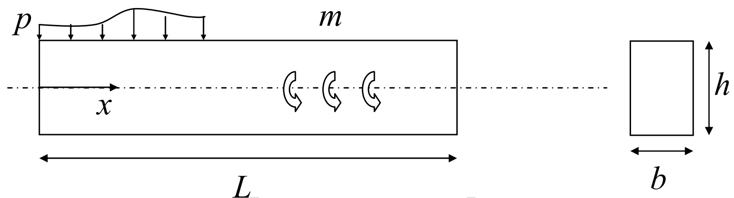
For this beam (including shear strains) the principle of virtual work gives:

$$EI\int_0^L \left(\frac{d\beta}{dx}\right)\left(\frac{d\bar{\beta}}{dx}\right)dx + GAk\int_0^L \left(\frac{dw}{dx}-\beta\right)\left(\frac{d\bar{w}}{dx}-\bar{\beta}\right)dx = \int_0^L p\bar{w}\,dx + \int_0^L m\bar{\beta}\,dx$$



Formulation of Structural Elements

Two-dimensional beam elements:



$$EI\int_0^L\left(\frac{d\beta}{dx}\right)\left(\frac{d\overline{\beta}}{dx}\right)dx+GAk\int_0^L\left(\frac{dw}{dx}-\beta\right)\left(\frac{d\overline{w}}{dx}-\overline{\beta}\right)dx=\int_0^L p\overline{w}\,dx+\int_0^L m\overline{\beta}\,dx$$

Using now the interpolations: $w=\sum_{i=1}^{q}h_i w_i;\quad \beta=\sum_{i=1}^{q}h_i\beta_i$

we can write:

$$w=\mathbf{H}_w\hat{\mathbf{u}};\qquad \beta=\mathbf{H}_\beta\hat{\mathbf{u}}$$

$$\frac{\partial w}{\partial x}=\mathbf{B}_w\hat{\mathbf{u}};\qquad \frac{\partial\beta}{\partial x}=\mathbf{B}_\beta\hat{\mathbf{u}}$$

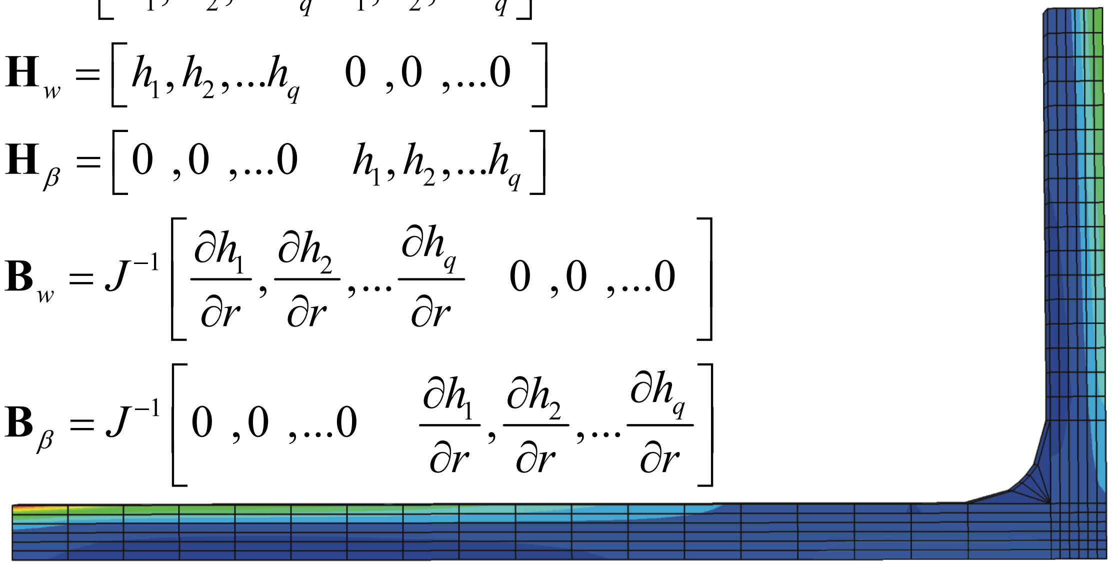
$$\hat{\mathbf{u}}^T=\left[w_1,w_2,\ldots w_q\ \ \theta_1,\theta_2,\ldots\theta_q\right]$$

$$\mathbf{H}_w=\left[h_1,h_2,\ldots h_q\ \ \ 0\ ,0\ ,\ldots 0\ \right]$$

$$\mathbf{H}_\beta=\left[0\ ,0\ ,\ldots 0\ \ \ h_1,h_2,\ldots h_q\right]$$

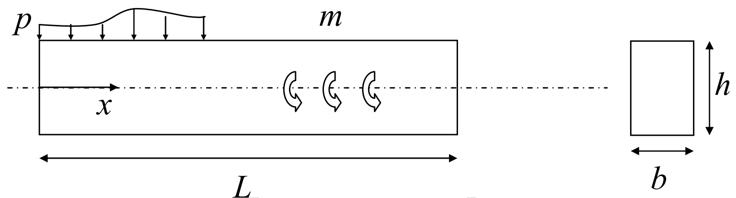
$$\mathbf{B}_w=J^{-1}\left[\frac{\partial h_1}{\partial r},\frac{\partial h_2}{\partial r},\ldots\frac{\partial h_q}{\partial r}\ \ \ 0\ ,0\ ,\ldots 0\ \right]$$

$$\mathbf{B}_\beta=J^{-1}\left[0\ ,0\ ,\ldots 0\ \ \ \frac{\partial h_1}{\partial r},\frac{\partial h_2}{\partial r},\ldots\frac{\partial h_q}{\partial r}\right]$$



Formulation of Structural Elements

Two-dimensional beam elements:

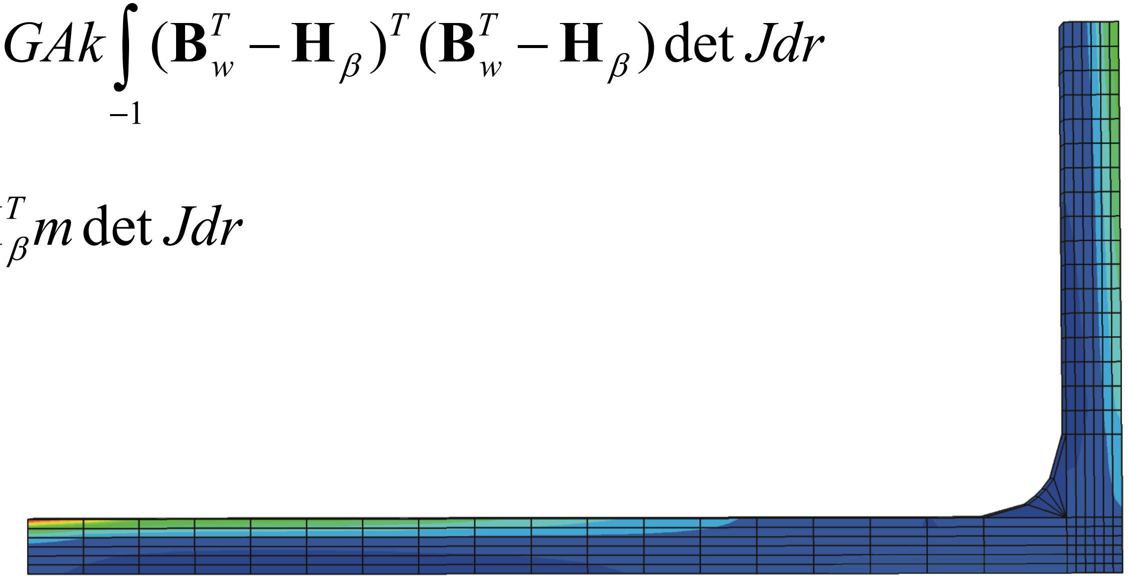


$$EI\int_0^L\left(\frac{d\beta}{dx}\right)\left(\frac{d\overline{\beta}}{dx}\right)dx+GAk\int_0^L\left(\frac{dw}{dx}-\beta\right)\left(\frac{d\overline{w}}{dx}-\overline{\beta}\right)dx=\int_0^L p\overline{w}dx+\int_0^L m\overline{\beta}\,dx$$

We can now assemble the element matrixes:

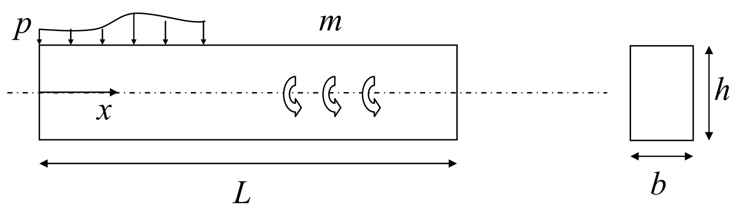
$$\mathbf{K}=EI\int_{-1}^{1}\mathbf{B}_\beta^T\mathbf{B}_\beta^T\det Jdr+GAk\int_{-1}^{1}(\mathbf{B}_w^T-\mathbf{H}_\beta)^T(\mathbf{B}_w^T-\mathbf{H}_\beta)\det Jdr$$

$$\mathbf{R}=\int_{-1}^{1}\mathbf{H}_W^T p\det Jdr+\int_{-1}^{1}\mathbf{H}_\beta^T m\det Jdr$$



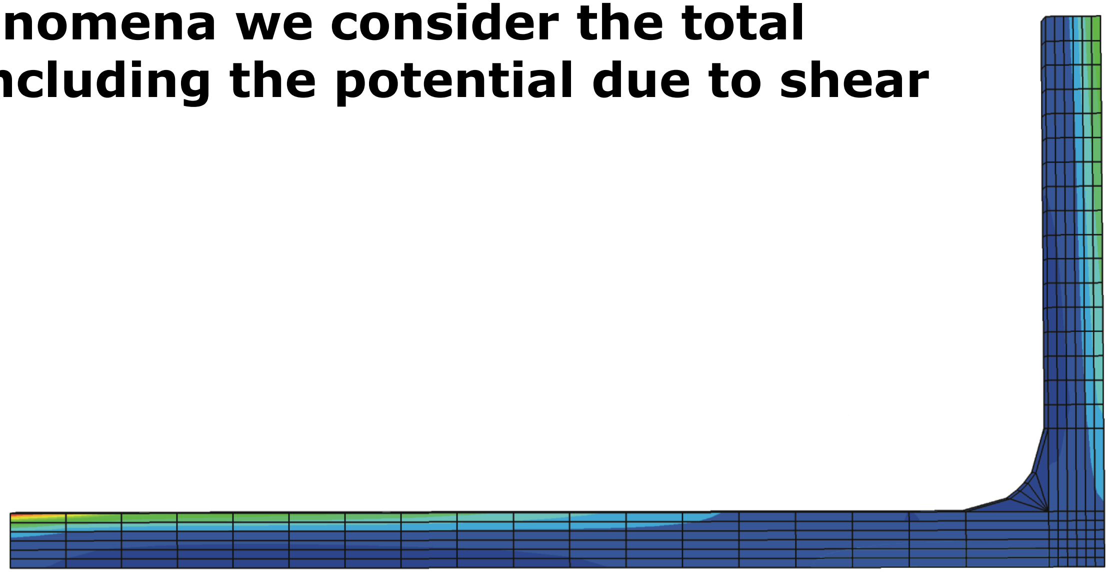
Formulation of Structural Elements

Two-dimensional beam elements:



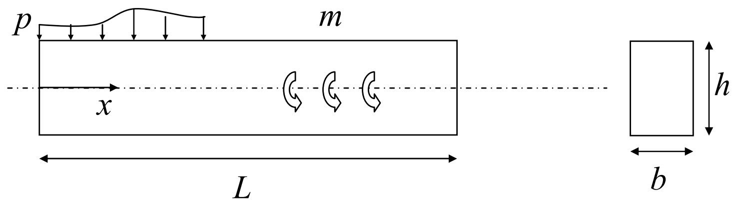
The element we just derived is however only recommendable if we use 3 of 4 node elements and provided that the interior nodes are located at the midpoint or third points respectively – the reason for this is shear locking

To appreciate this phenomena we consider the total potential for a beam including the potential due to shear stresses



Formulation of Structural Elements

Two-dimensional beam elements:

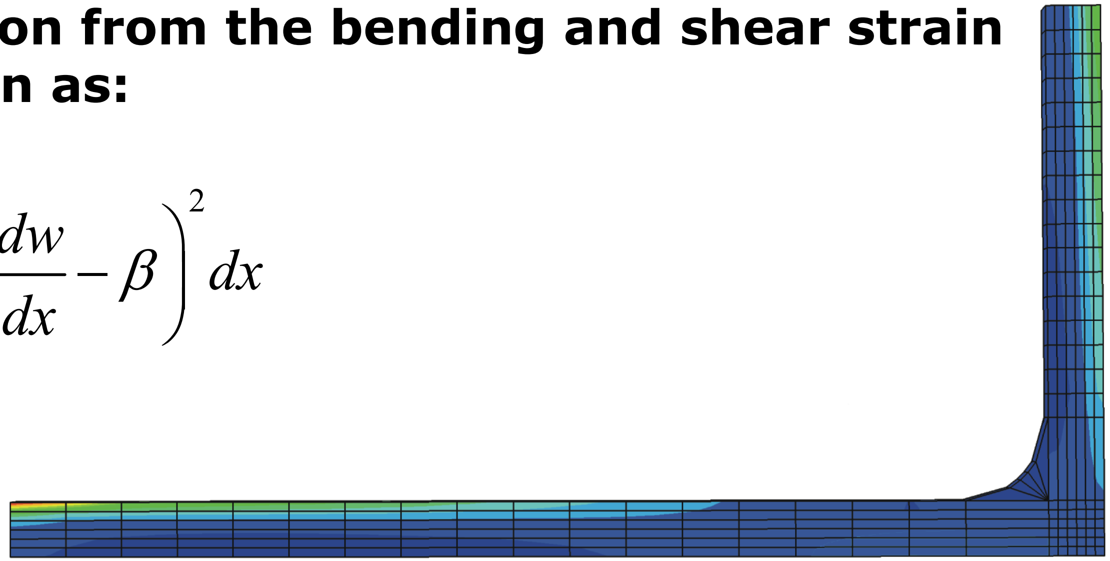


The total potential can be written as:

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d\beta}{dx}\right)^2 dx + \frac{GAk}{2}\int_0^L \left(\frac{dw}{dx} - \beta\right)^2 dx - \int_0^L pw dx + \int_0^L m\beta dx$$

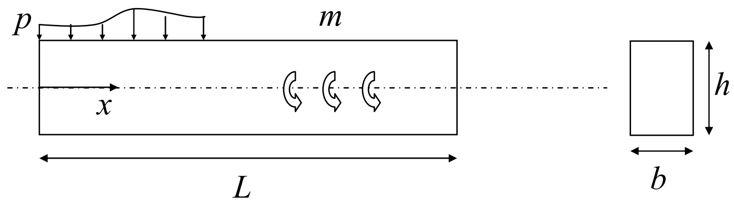
The relative contribution from the bending and shear strain energies can be written as:

$$\tilde{\Pi} = \int_0^L \left(\frac{d\beta}{dx}\right)^2 dx + \frac{GAk}{EI}\int_0^L \left(\frac{dw}{dx} - \beta\right)^2 dx$$



Formulation of Structural Elements

Two-dimensional beam elements:



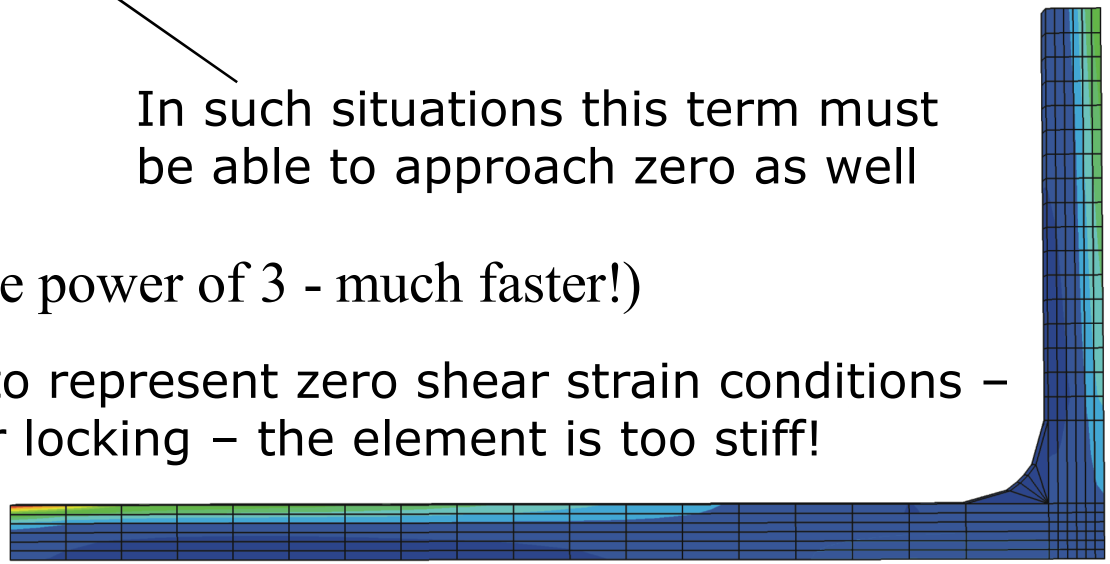
By study of this expression for different geometries it is evident that for small *h* the element must be able to represent zero shear strain conditions

$$\tilde{\Pi} = \int_0^L \left(\frac{d\beta}{dx}\right)^2 dx + \frac{GAk}{EI}\int_0^L \left(\frac{dw}{dx} - \beta\right)^2 dx$$

In such situations this term must be able to approach zero as well

For $h \rightarrow 0$ $I \rightarrow 0$ (but to the power of 3 - much faster!)

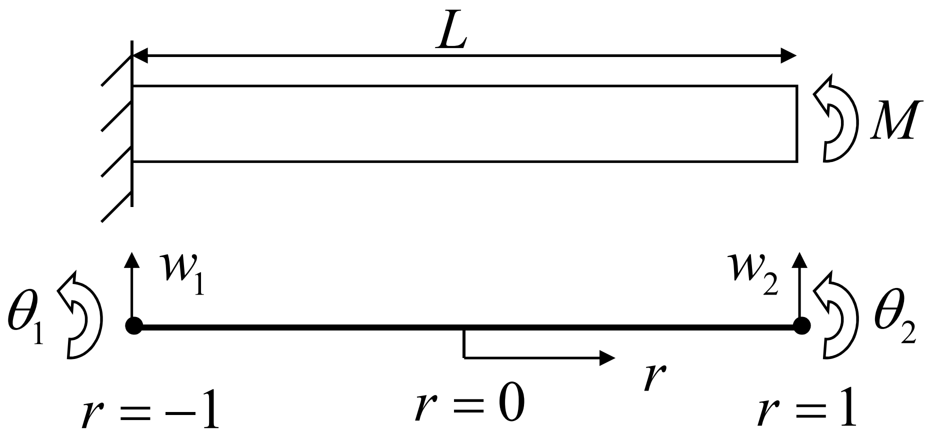
If elements are not able to represent zero shear strain conditions – we are dealing with shear locking – the element is too stiff!



Formulation of Structural Elements

Two-dimensional beam elements:

To study the phenomenon of shear locking lets consider a small example:



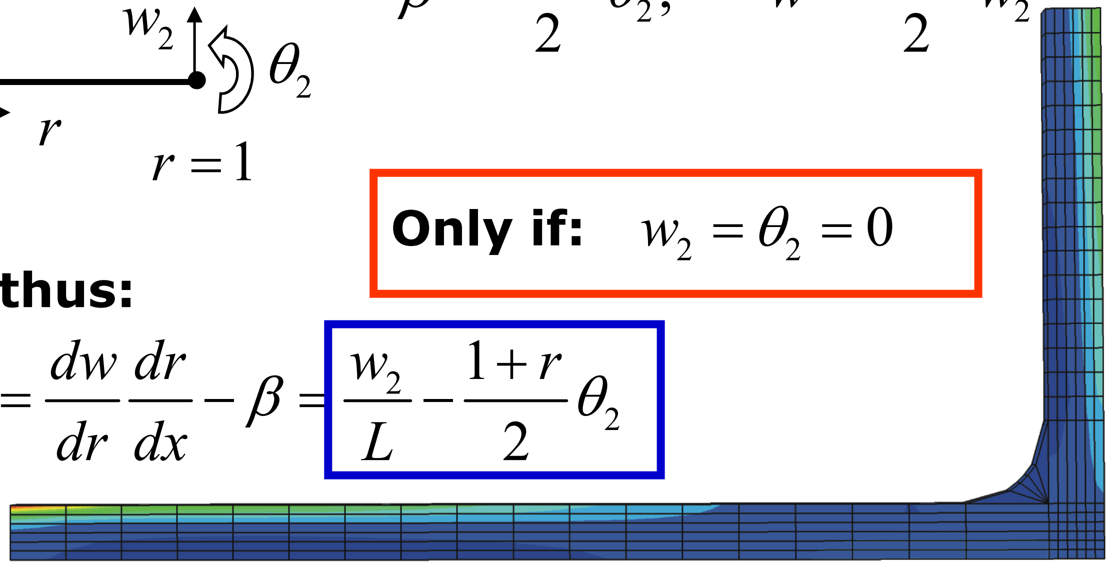
Linear interpolations may be established as:

$$\beta = \frac{1+r}{2}\theta_2, \qquad w = \frac{1+r}{2}w_2$$

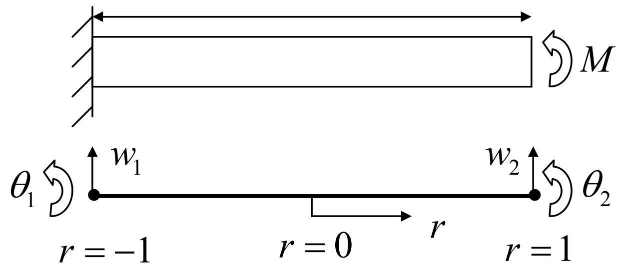
Only if: $w_2 = \theta_2 = 0$

The shear strains are thus:

$$\gamma = \frac{dw}{dx} - \beta = \frac{dw}{dr}J^{-1} - \beta = \frac{dw}{dr}\frac{dr}{dx} - \beta = \frac{w_2}{L} - \frac{1+r}{2}\theta_2$$



Formulation of Structural Elements



Two-dimensional beam elements:

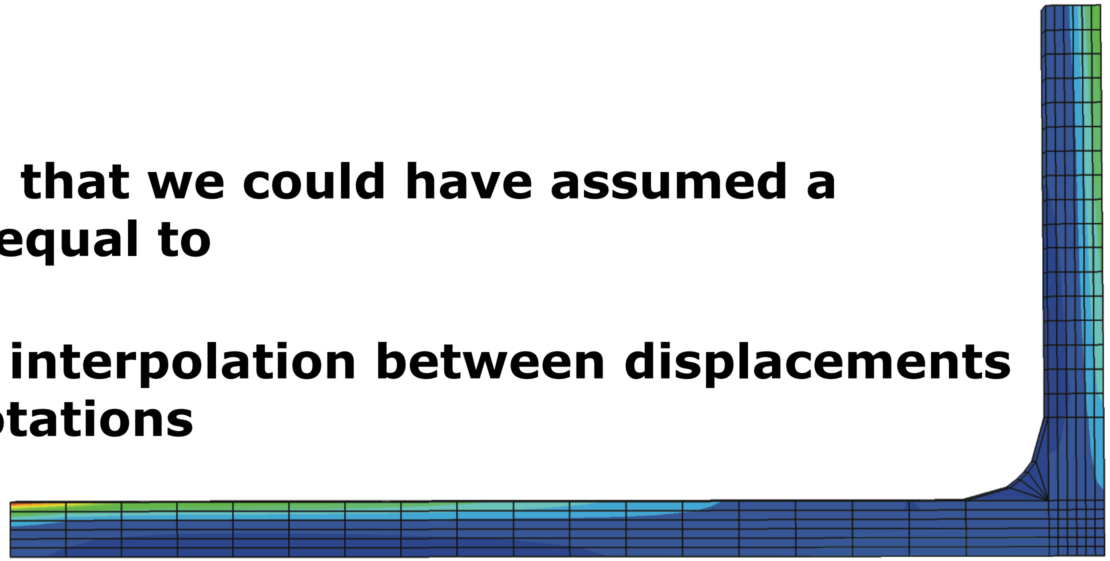
However, we may impose the requirement that the shear strains are zero only at mid point:

$$\gamma = 0 = \frac{w_2}{L} - \frac{1}{2}\theta_2$$ **At midpoint !**

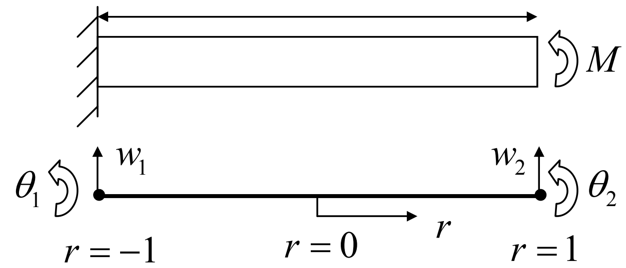
$$w_2 = \frac{L}{2}\theta_2$$

This points to the idea that we could have assumed a constant shear strain equal to

$$\gamma = \frac{w_2}{L} - \frac{1}{2}\theta_2$$ **Mixed interpolation between displacements and rotations**



Formulation of Structural Elements



Two-dimensional beam elements:

Thus more generally we might extend this approach:

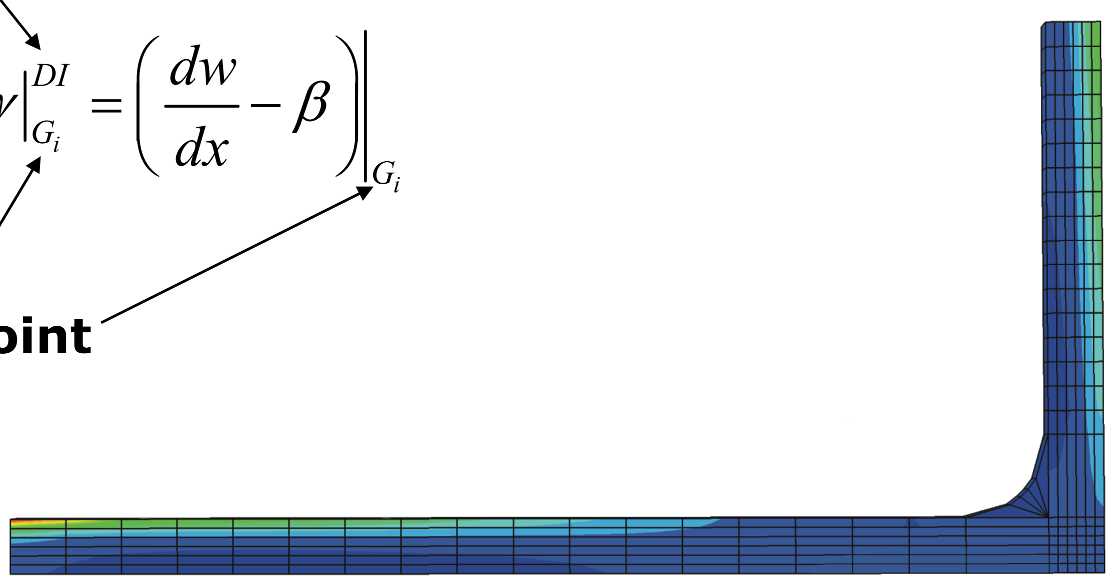
$$w = \sum_{i=1}^{q} h_i w_i$$

$$\beta = \sum_{i=1}^{q} h_i \theta_i$$

By displacement interpolation

$$\gamma = \sum_{i=1}^{q-1} h_i^* \, \gamma\Big|_{G_i}^{DI} \qquad \gamma\Big|_{G_i}^{DI} = \left(\frac{dw}{dx} - \beta \right)\Bigg|_{G_i}$$

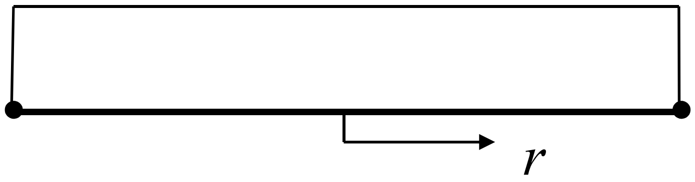
i'th Gauss-point



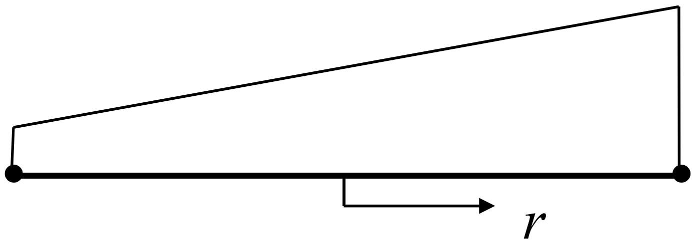
Formulation of Structural Elements

Two-dimensional beam elements:

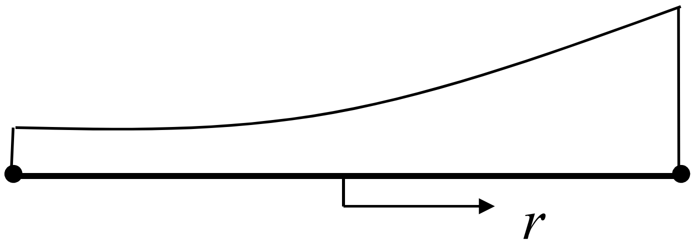
Thus we get for 2, 3 and 4 node elements:



γ constant, G_1 corresponds to $r = 0$

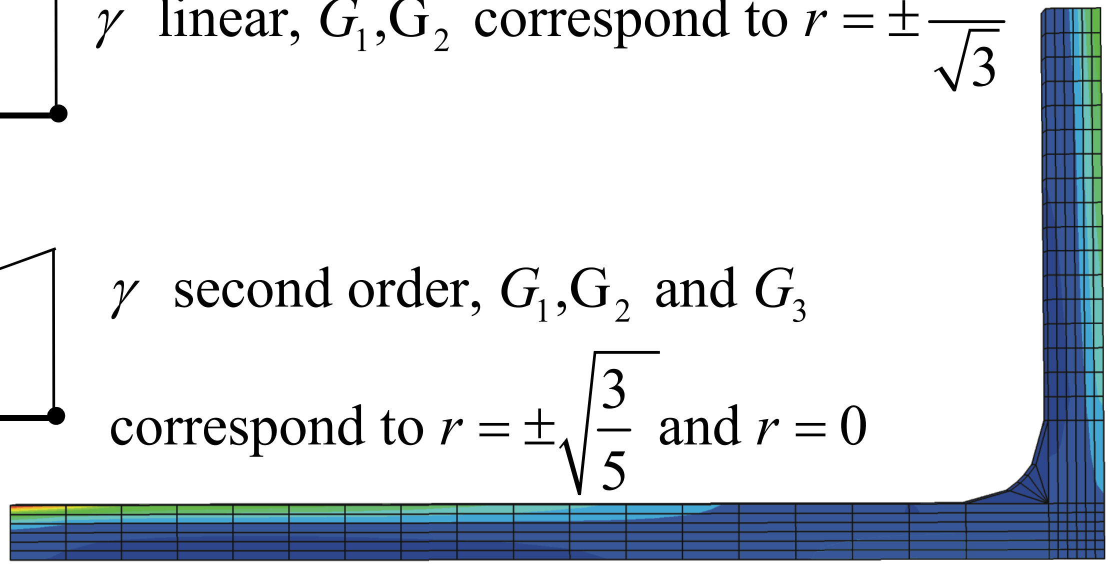


γ linear, G_1, G_2 correspond to $r = \pm\frac{1}{\sqrt{3}}$



γ second order, G_1, G_2 and G_3

correspond to $r = \pm\sqrt{\frac{3}{5}}$ and $r = 0$



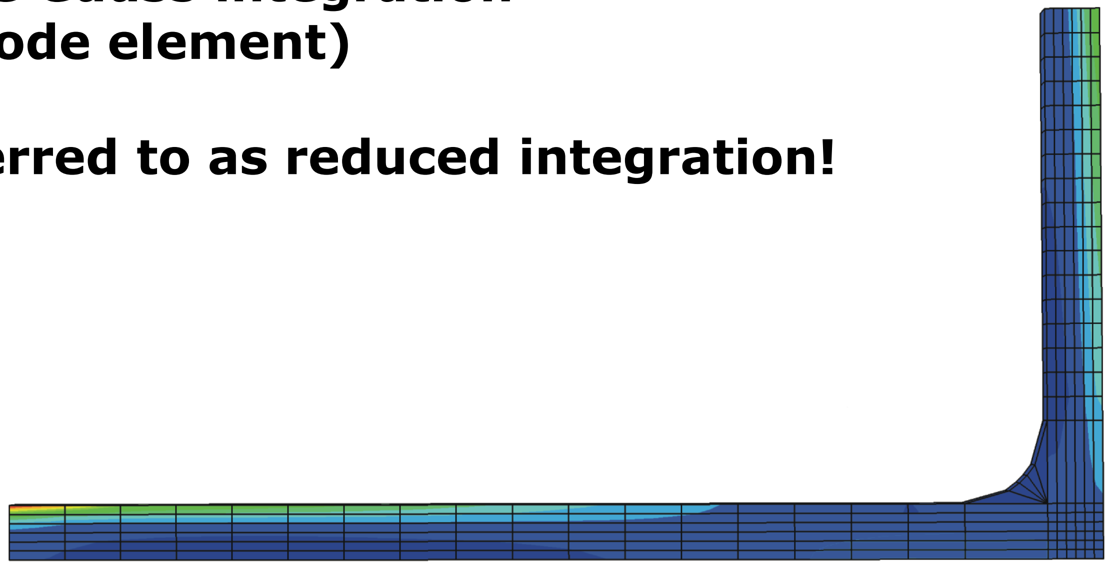
Formulation of Structural Elements

Two-dimensional beam elements:

Using mixed interpolation for the shear strains greatly improves the performance of the element:

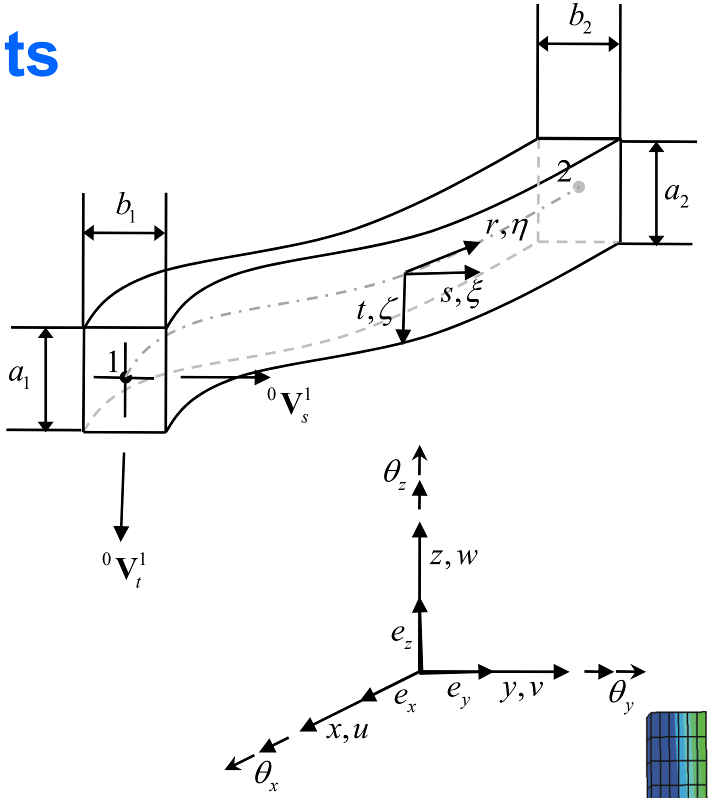
In addition – the stiffness matrix can still be calculated exactly for what concerns the bending strain contributions (full integration) and then simply add the terms from the shear strains using the Gauss integration (mid point for a two node element)

This is sometimes referred to as reduced integration!



Formulation of Structural Elements

General curved beam elements:



$$ {}^{\ell}x(r,s,t)=\sum_{k=1}^{q} h_k\,{}^{\ell}x_k+\frac{t}{2}\sum_{k=1}^{q} a_k h_k\,{}^{\ell}V_{t,x}^{k}+\frac{s}{2}\sum_{k=1}^{q} b_k h_k\,{}^{\ell}V_{s,x}^{k} $$

$$ {}^{\ell}y(r,s,t)=\sum_{k=1}^{q} h_k\,{}^{\ell}y_k+\frac{t}{2}\sum_{k=1}^{q} a_k h_k\,{}^{\ell}V_{t,y}^{k}+\frac{s}{2}\sum_{k=1}^{q} b_k h_k\,{}^{\ell}V_{s,y}^{k} $$

$$ {}^{\ell}z(r,s,t)=\sum_{k=1}^{q} h_k\,{}^{\ell}z_k+\frac{t}{2}\sum_{k=1}^{q} a_k h_k\,{}^{\ell}V_{t,z}^{k}+\frac{s}{2}\sum_{k=1}^{q} b_k h_k\,{}^{\ell}V_{s,z}^{k} $$

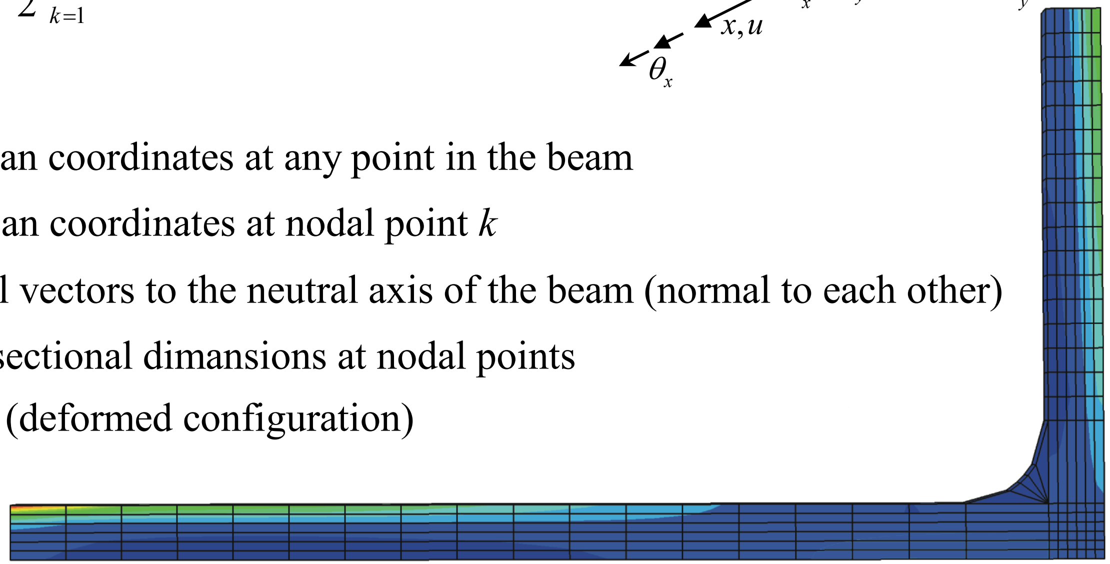
$^{\ell}x(r,s,t),\ {}^{\ell}y(r,s,t),\ {}^{\ell}z(r,s,t)$: Cartesian coordinates at any point in the beam

$^{\ell}x_k,\ {}^{\ell}y_k,\ {}^{\ell}z_k$: Cartesian coordinates at nodal point k

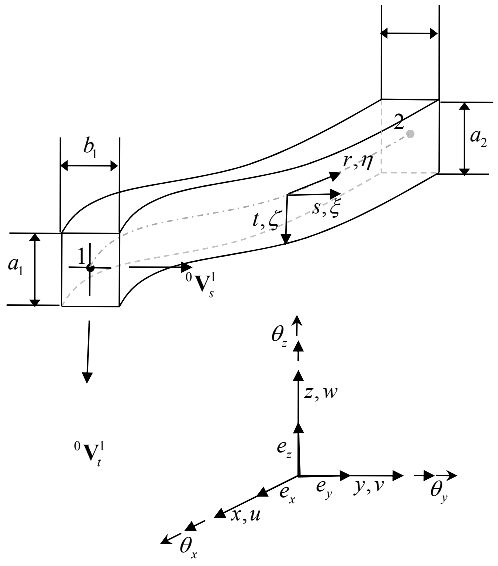
$^{\ell}V_{t,z}^{k},\ {}^{\ell}V_{s,z}^{k}$: Normal vectors to the neutral axis of the beam (normal to each other)

a_k, b_k: Cross sectional dimansions at nodal points

ℓ=0 (undeformed configuration), ℓ=1 (deformed configuration)



Formulation of Structural Elements



General curved beam elements:

The displacement components are found as:

$$u(r,s,t) = {}^1x - {}^0x$$
$$v(r,s,t) = {}^1y - {}^0y$$
$$w(r,s,t) = {}^1z - {}^0z$$

$$\Rightarrow$$

$$u(r,s,t) = \sum_{k=1}^{q} h_k\,{}^{\ell}u_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{t,x}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k V_{s,x}^k$$

$$v(r,s,t) = \sum_{k=1}^{q} h_k\,{}^{\ell}v_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{t,y}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k V_{s,y}^k$$

$$w(r,s,t) = \sum_{k=1}^{q} h_k\,{}^{\ell}w_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{t,z}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k V_{s,z}^k$$

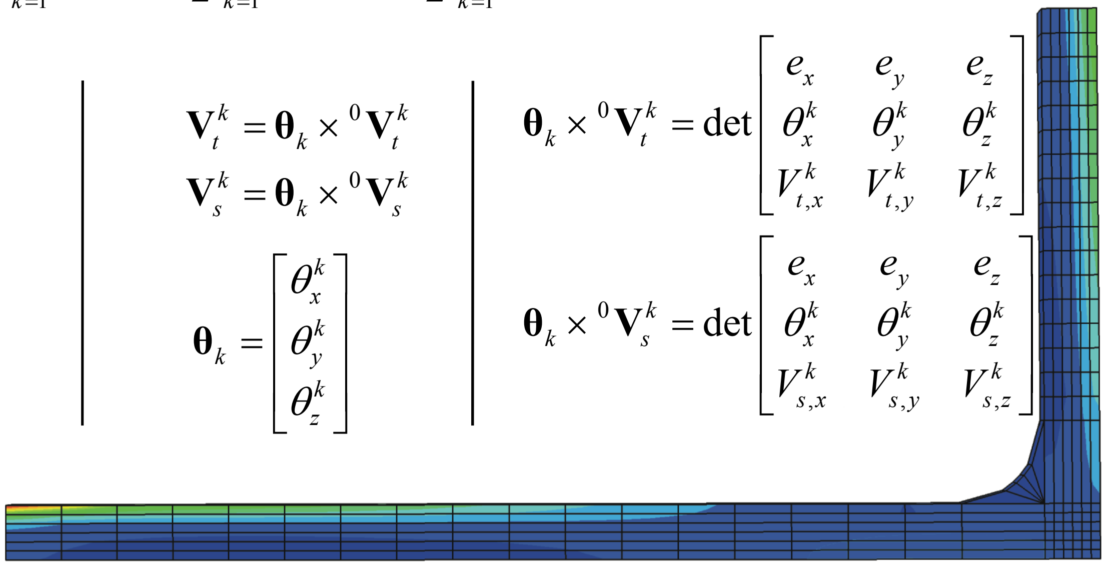
$$V_{t,x}^k = {}^1V_{t,x}^k - {}^0V_{t,x}^k \qquad V_{s,x}^k = {}^1V_{s,x}^k - {}^0V_{s,x}^k$$
$$V_{t,y}^k = {}^1V_{t,y}^k - {}^0V_{t,y}^k \qquad V_{s,y}^k = {}^1V_{s,y}^k - {}^0V_{s,y}^k$$
$$V_{t,z}^k = {}^1V_{t,z}^k - {}^0V_{t,z}^k \qquad V_{s,z}^k = {}^1V_{s,z}^k - {}^0V_{s,z}^k$$

$$\mathbf{V}_t^k = \boldsymbol{\theta}_k \times {}^0\mathbf{V}_t^k$$
$$\mathbf{V}_s^k = \boldsymbol{\theta}_k \times {}^0\mathbf{V}_s^k$$

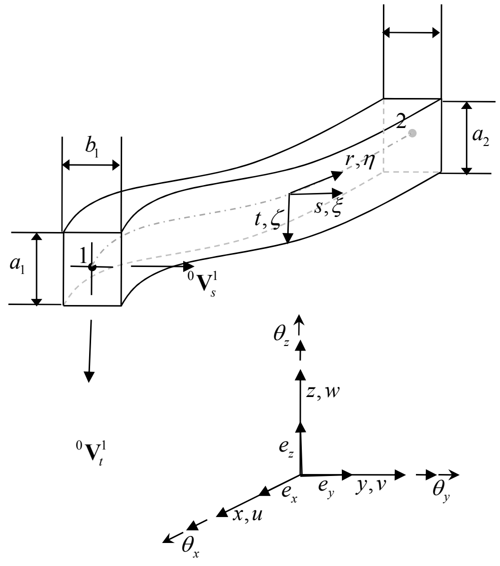
$$\boldsymbol{\theta}_k = \begin{bmatrix} \theta_x^k \\ \theta_y^k \\ \theta_z^k \end{bmatrix}$$

$$\boldsymbol{\theta}_k \times {}^0\mathbf{V}_t^k = \det\begin{bmatrix} e_x & e_y & e_z \\ \theta_x^k & \theta_y^k & \theta_z^k \\ V_{t,x}^k & V_{t,y}^k & V_{t,z}^k \end{bmatrix}$$

$$\boldsymbol{\theta}_k \times {}^0\mathbf{V}_s^k = \det\begin{bmatrix} e_x & e_y & e_z \\ \theta_x^k & \theta_y^k & \theta_z^k \\ V_{s,x}^k & V_{s,y}^k & V_{s,z}^k \end{bmatrix}$$



Formulation of Structural Elements



General curved beam elements:

The displacement interpolation matrix H is found by inserting:

$$\mathbf{V}_t^k = \boldsymbol{\theta}_k \times {}^0\mathbf{V}_t^k \qquad \boldsymbol{\theta}_k = \begin{bmatrix} \theta_x^k \\ \theta_y^k \\ \theta_z^k \end{bmatrix}$$

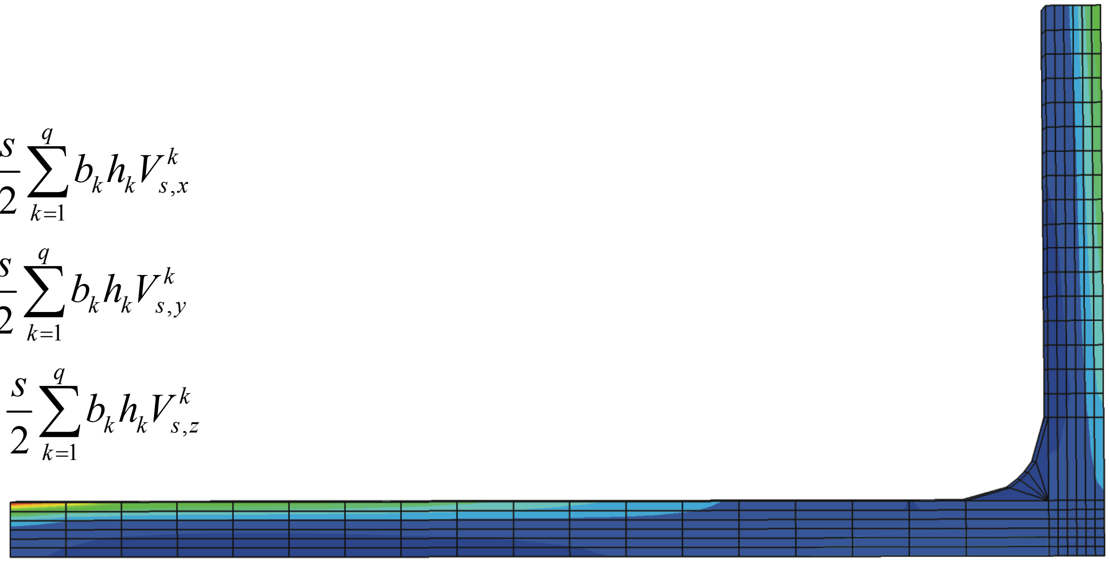
$$\mathbf{V}_s^k = \boldsymbol{\theta}_k \times {}^0\mathbf{V}_s^k$$

into:

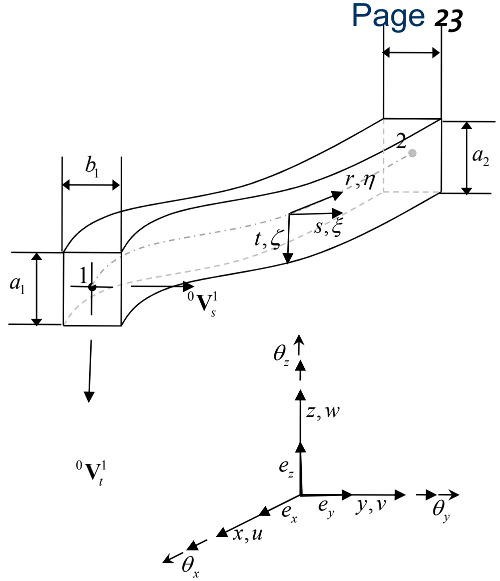
$$u(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{\ell}u_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{t,x}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k V_{s,x}^k$$

$$v(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{\ell}v_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{t,y}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k V_{s,y}^k$$

$$w(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{\ell}w_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{t,z}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k V_{s,z}^k$$



Formulation of Structural Elements



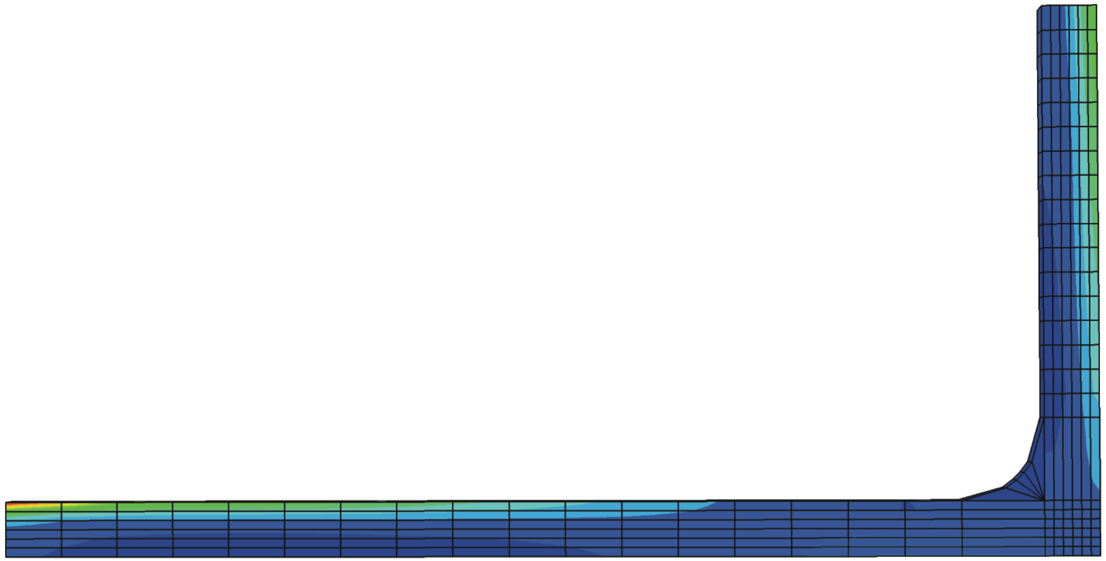
General curved beam elements:

For the beam only the longitudinal strain and the transversal shear strains are required:

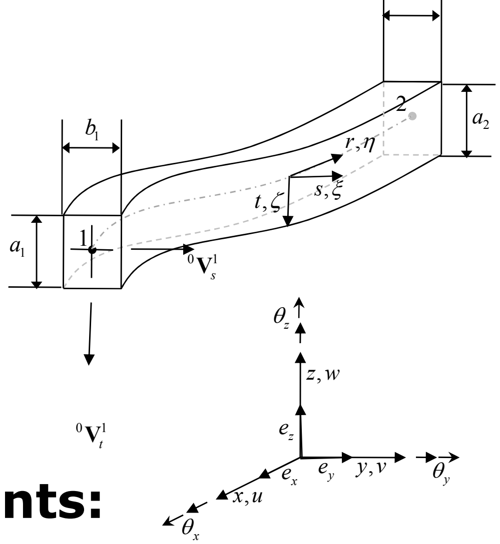
$$\begin{bmatrix} \varepsilon_{\eta\eta} \\ \gamma_{\eta\xi} \\ \gamma_{\eta\zeta} \end{bmatrix} = \sum_{k=1}^{q} \mathbf{B}_k \hat{\mathbf{u}}_k \qquad \hat{\mathbf{u}}_k^T = \begin{bmatrix} u_k & v_k & w_k & \theta_k & \theta_k & \theta_k \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{B}_q \end{bmatrix}$$



Formulation of Structural Elements



General curved beam elements:

In order to evaluate the components of B we must take the derivatives of H following the usual procedure for iso-parametric elements:

$$\begin{bmatrix} \dfrac{\partial(u,v,w)}{\partial r} \\ \dfrac{\partial(u,v,w)}{\partial s} \\ \dfrac{\partial(u,v,w)}{\partial t} \end{bmatrix} = \sum_{k=1}^{q} \begin{bmatrix} \dfrac{\partial h_k}{\partial r} & \begin{bmatrix} 1 & (g)_{1i}^k & (g)_{2i}^k & (g)_{3i}^k \end{bmatrix} \\ h_k & \begin{bmatrix} 0 & (\hat{g})_{1i}^k & (\hat{g})_{2i}^k & (\hat{g})_{3i}^k \end{bmatrix} \\ h_k & \begin{bmatrix} 0 & (\overline{g})_{1i}^k & (\overline{g})_{2i}^k & (\overline{g})_{3i}^k \end{bmatrix} \end{bmatrix} \begin{bmatrix} (u,v,w)_k \\ \theta_x^k \\ \theta_y^k \\ \theta_z^k \end{bmatrix},$$

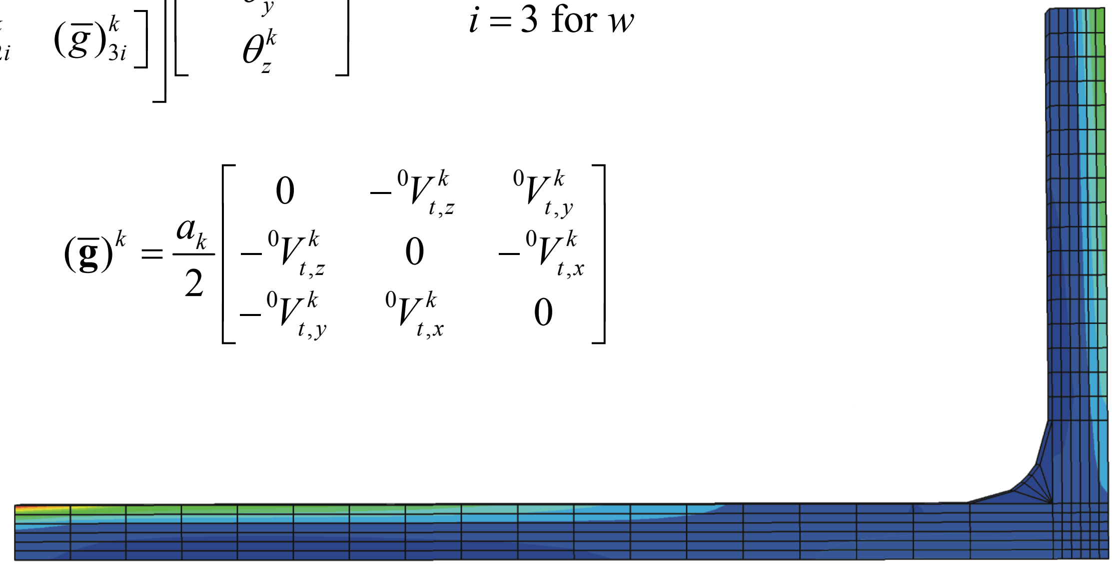
$i = 1$ for u

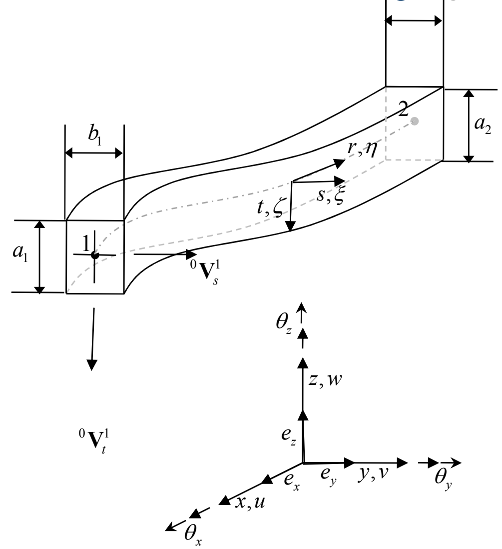
$i = 2$ for v

$i = 3$ for w

$$(\hat{\mathbf{g}})^k = \frac{b_k}{2} \begin{bmatrix} 0 & -{}^0V_{s,z}^k & {}^0V_{s,y}^k \\ -{}^0V_{s,z}^k & 0 & -{}^0V_{s,x}^k \\ -{}^0V_{s,y}^k & {}^0V_{s,x}^k & 0 \end{bmatrix}, \qquad (\overline{\mathbf{g}})^k = \frac{a_k}{2} \begin{bmatrix} 0 & -{}^0V_{t,z}^k & {}^0V_{t,y}^k \\ -{}^0V_{t,z}^k & 0 & -{}^0V_{t,x}^k \\ -{}^0V_{t,y}^k & {}^0V_{t,x}^k & 0 \end{bmatrix}$$

$$(g)_{ij}^k = s(\hat{g})_{ij}^k + t(\overline{g})_{ij}^k$$





Formulation of Structural Elements

General curved beam elements:

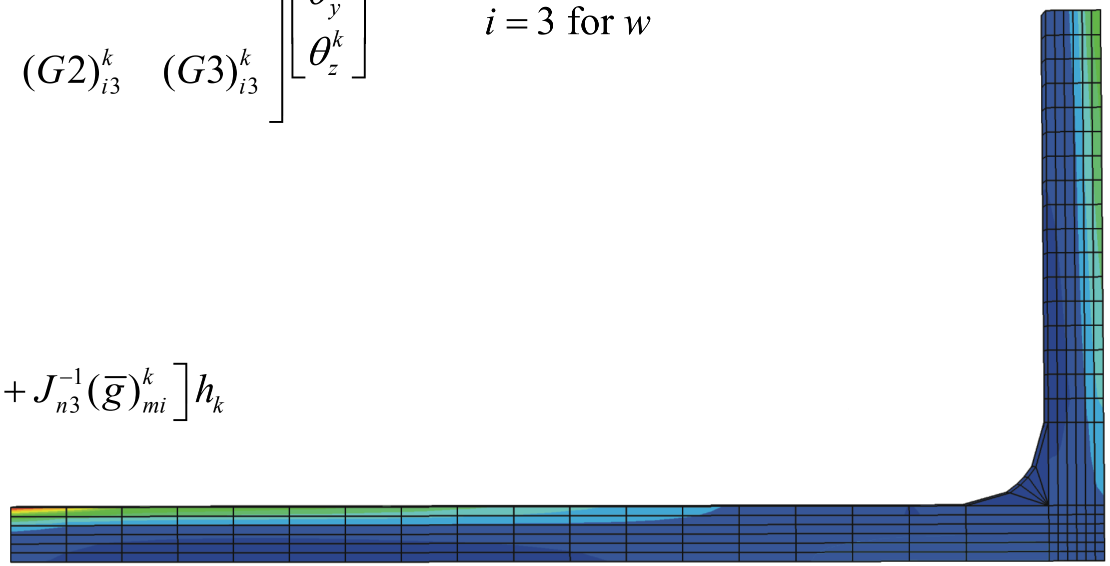
Now we may transform into the global coordinate system:

$$\begin{bmatrix} \dfrac{\partial u(u,v,w)}{\partial x} \\ \dfrac{\partial (u,v,w)}{\partial y} \\ \dfrac{\partial (u,v,w)}{\partial z} \end{bmatrix} = \sum_{k=1}^{q} \begin{bmatrix} J_{11}^{-1}\dfrac{\partial h_k}{\partial r} & (G1)_{i1}^k & (G2)_{i1}^k & (G3)_{i1}^k \\ J_{21}^{-1}\dfrac{\partial h_k}{\partial r} & (G1)_{i2}^k & (G2)_{i2}^k & (G3)_{i2}^k \\ J_{31}^{-1}\dfrac{\partial h_k}{\partial r} & (G1)_{i3}^k & (G2)_{i3}^k & (G3)_{i3}^k \end{bmatrix} \begin{bmatrix} u_k \\ \theta_x^k \\ \theta_y^k \\ \theta_z^k \end{bmatrix},$$

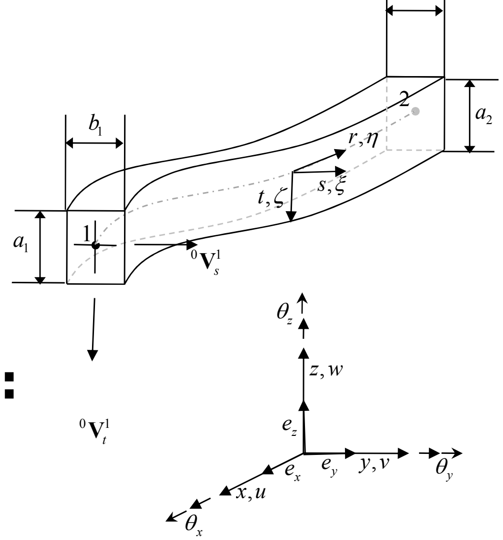
$i = 1$ for u
$i = 2$ for v
$i = 3$ for w

where

$$(Gm)_{in}^k = \left[J_{n1}^{-1} (g)_{mi}^k \right] \frac{\partial h_k}{\partial r} + \left[J_{n2}^{-1} (\hat{g})_{mi}^k + J_{n3}^{-1} (\bar{g})_{mi}^k \right] h_k$$



Formulation of Structural Elements

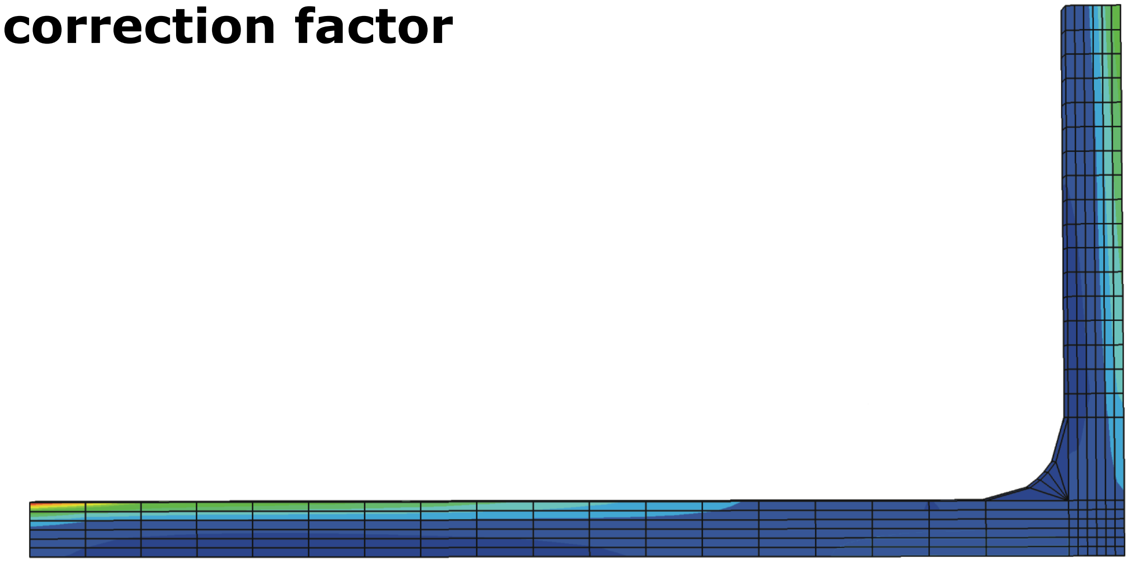


General curved beam elements:

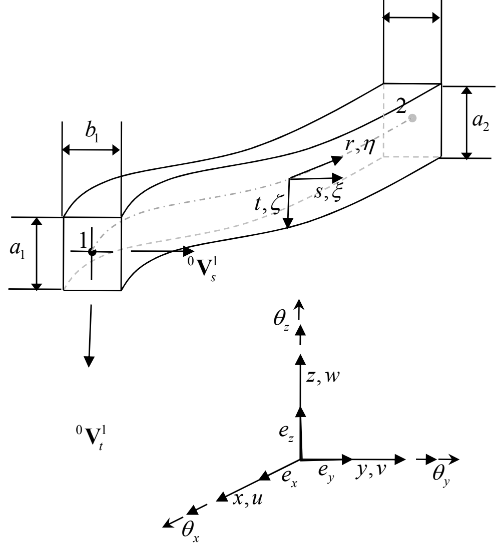
Finally all we need is the strain-stress matrix:

$$\begin{bmatrix}\tau_{\eta\eta}\\ \tau_{\eta\xi}\\ \tau_{\eta\zeta}\end{bmatrix}=\begin{bmatrix}E & 0 & 0\\ 0 & Gk & 0\\ 0 & 0 & Gk\end{bmatrix}\begin{bmatrix}\varepsilon_{\eta\eta}\\ \gamma_{\eta\xi}\\ \gamma_{\eta\zeta}\end{bmatrix}$$

where *k* is the shear correction factor

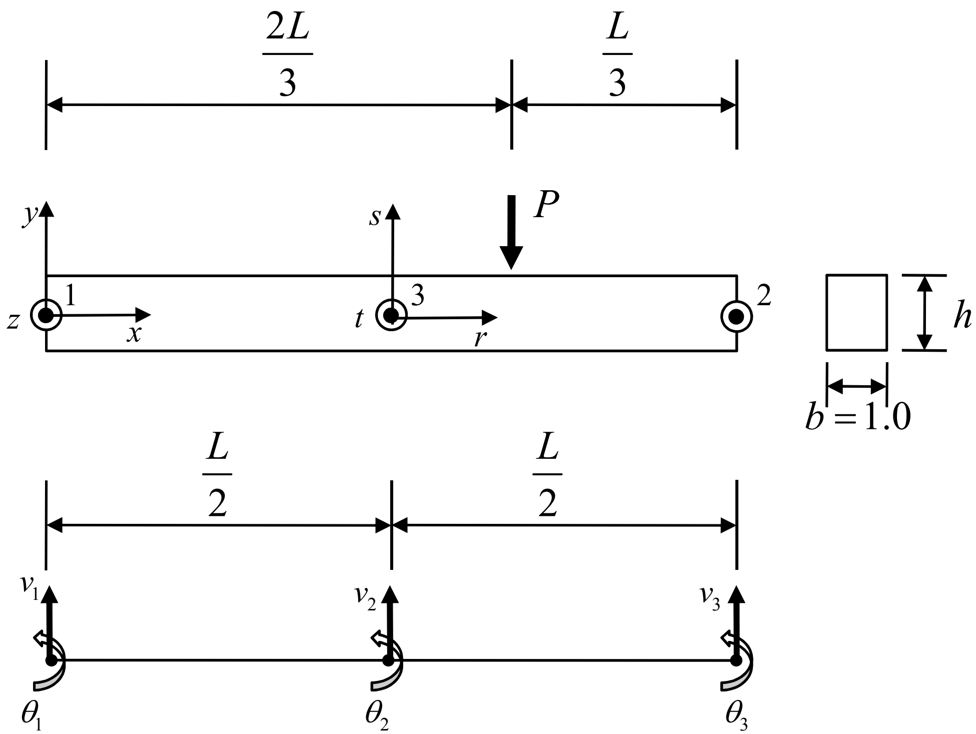


Formulation of Structural Elements



General curved beam elements: Example

We consider the following simple beam:

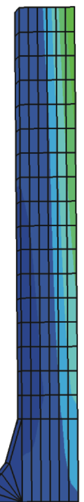


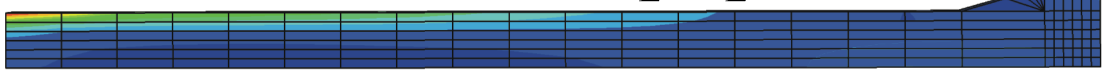
$$x = \sum_{i=1}^{3} h_i x_i, \qquad x = \frac{L}{2}(1+r)$$

$$h_1 = \frac{2x^2}{L^2} - \frac{3x}{L} + 1$$

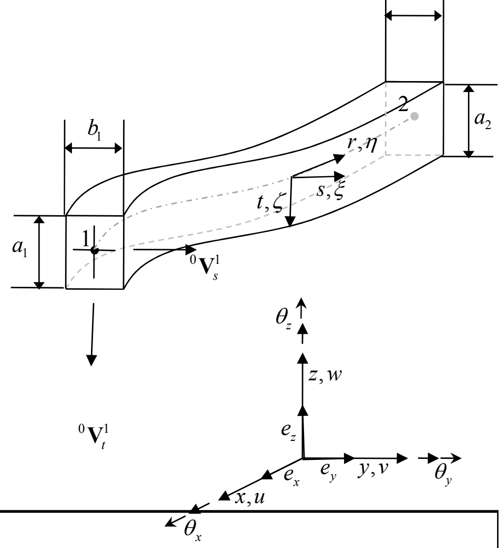
$$h_2 = \frac{2x^2}{L^2} - \frac{x}{L}$$

$$h_3 = \frac{4x}{L} - \frac{4x^2}{L^2}$$



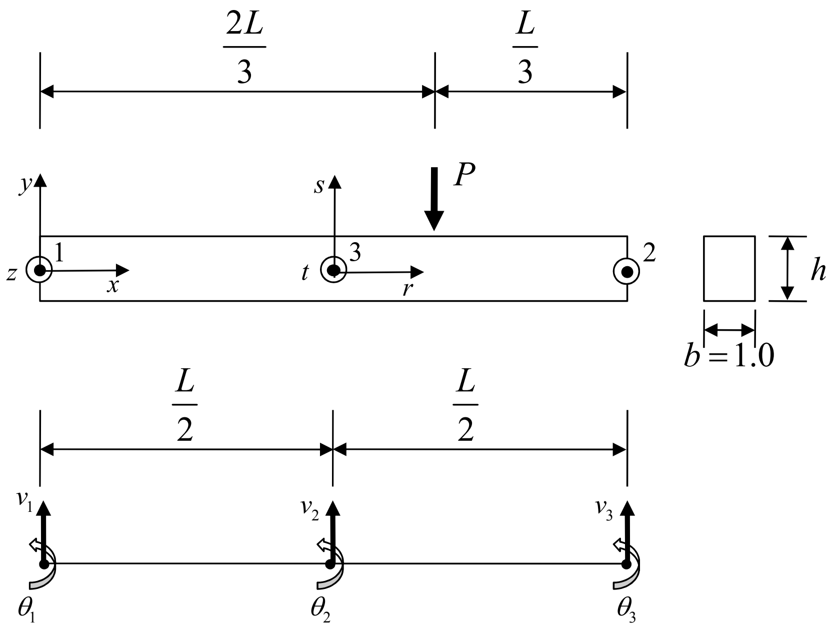


Formulation of Structural Elements



General curved beam elements: Example

We consider the following simple beam:



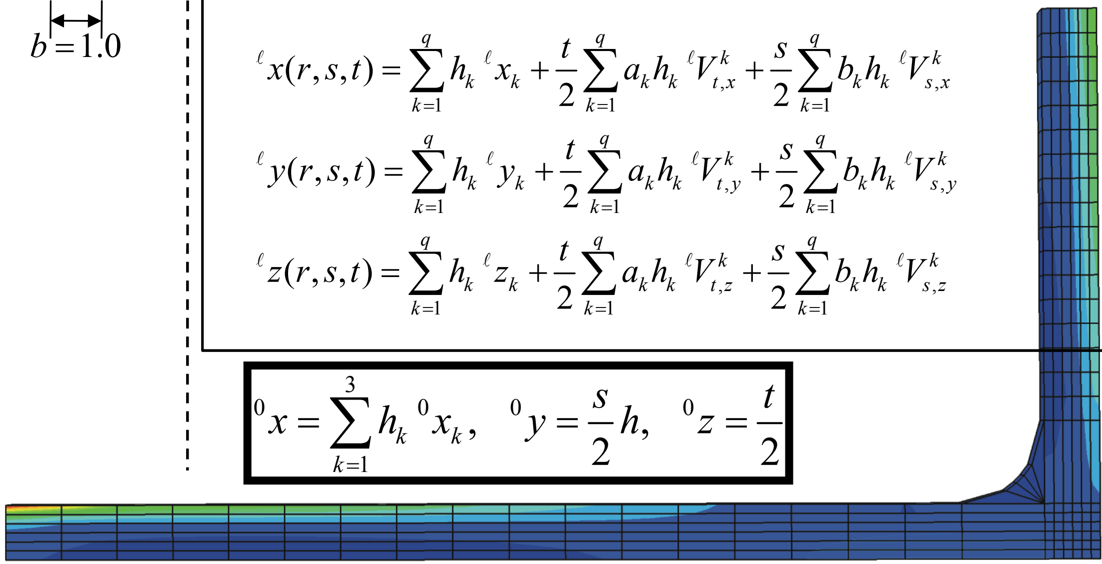
$$ {}^0\mathbf{V}_s^k = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad {}^0\mathbf{V}_t^k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad a_k = 1; \; b_k = h; \quad k = 1, 2, 3 $$

$$ {}^\ell x(r,s,t) = \sum_{k=1}^{q} h_k \, {}^\ell x_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^\ell V_{t,x}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^\ell V_{s,x}^k $$

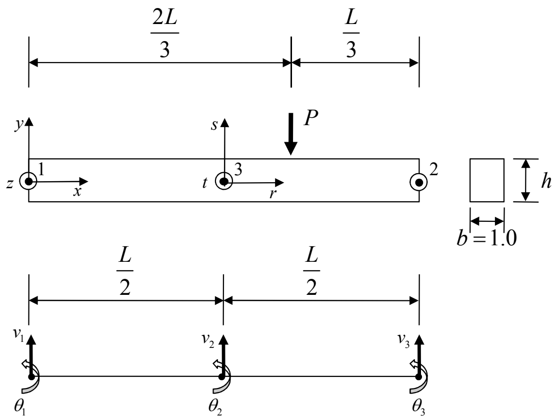
$$ {}^\ell y(r,s,t) = \sum_{k=1}^{q} h_k \, {}^\ell y_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^\ell V_{t,y}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^\ell V_{s,y}^k $$

$$ {}^\ell z(r,s,t) = \sum_{k=1}^{q} h_k \, {}^\ell z_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^\ell V_{t,z}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^\ell V_{s,z}^k $$

$$ {}^0 x = \sum_{k=1}^{3} h_k \, {}^0 x_k, \quad {}^0 y = \frac{s}{2} h, \quad {}^0 z = \frac{t}{2} $$



Formulation of Structural Elements



General curved beam elements: Example

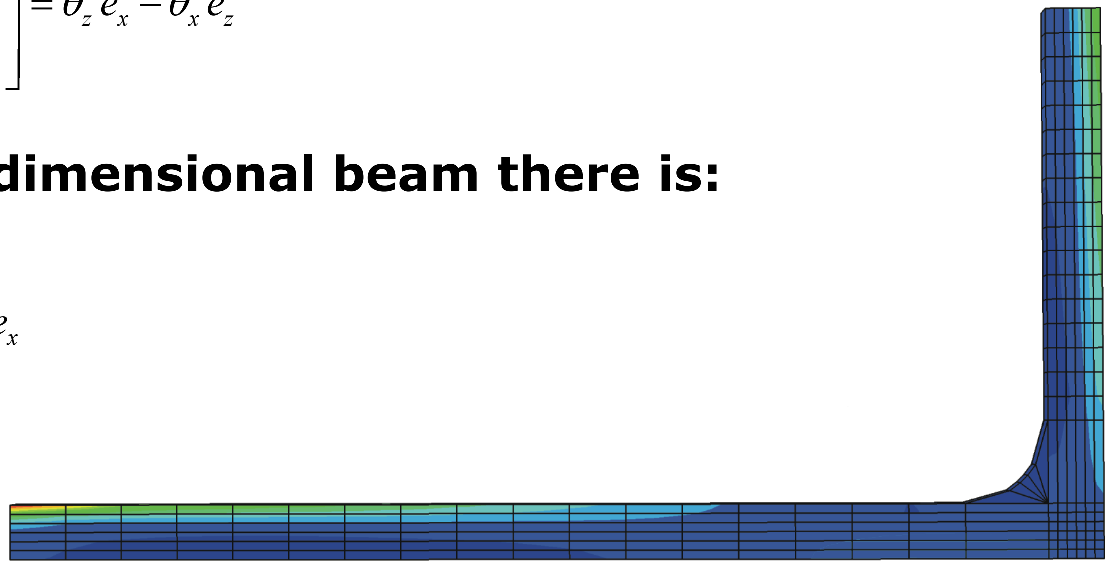
We consider the following simple beam:

$$\mathbf{V}_t^k = \boldsymbol{\theta}_k \times {}^0\mathbf{V}_t^k = \det\begin{bmatrix} e_x & e_y & e_z \\ \theta_x^k & \theta_y^k & \theta_z^k \\ 0 & 0 & 1 \end{bmatrix} = \theta_y^k e_x - \theta_x^k e_y$$

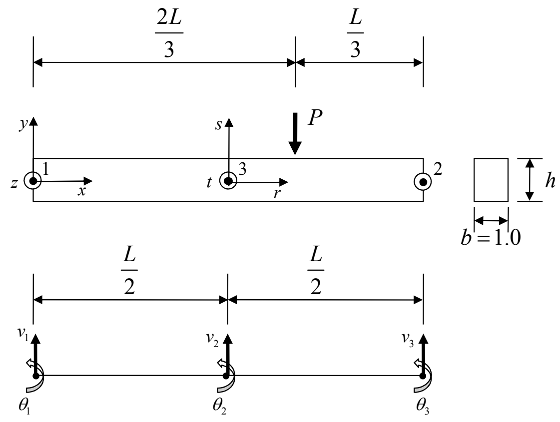
$$\mathbf{V}_s^k = \boldsymbol{\theta}_k \times {}^0\mathbf{V}_s^k = \det\begin{bmatrix} e_x & e_y & e_z \\ \theta_x^k & \theta_y^k & \theta_z^k \\ 0 & 1 & 0 \end{bmatrix} = \theta_z^k e_x - \theta_x^k e_z$$

as we consider a 2-dimensional beam there is:

$$\theta_x^k = \theta_y^k = 0 \Rightarrow \mathbf{V}_t^k = 0, \quad \mathbf{V}_s^k = \theta_z^k e_x$$



Formulation of Structural Elements



General curved beam elements: Example

Furthermore we have that only displacements in the y-direction are possible

$$u(r,s,t)=\sum_{k=1}^{q}h_k\,{}^{\ell}u_k+\frac{t}{2}\sum_{k=1}^{q}a_k h_k V_{t,x}^k+\frac{s}{2}\sum_{k=1}^{q}b_k h_k V_{s,x}^k$$

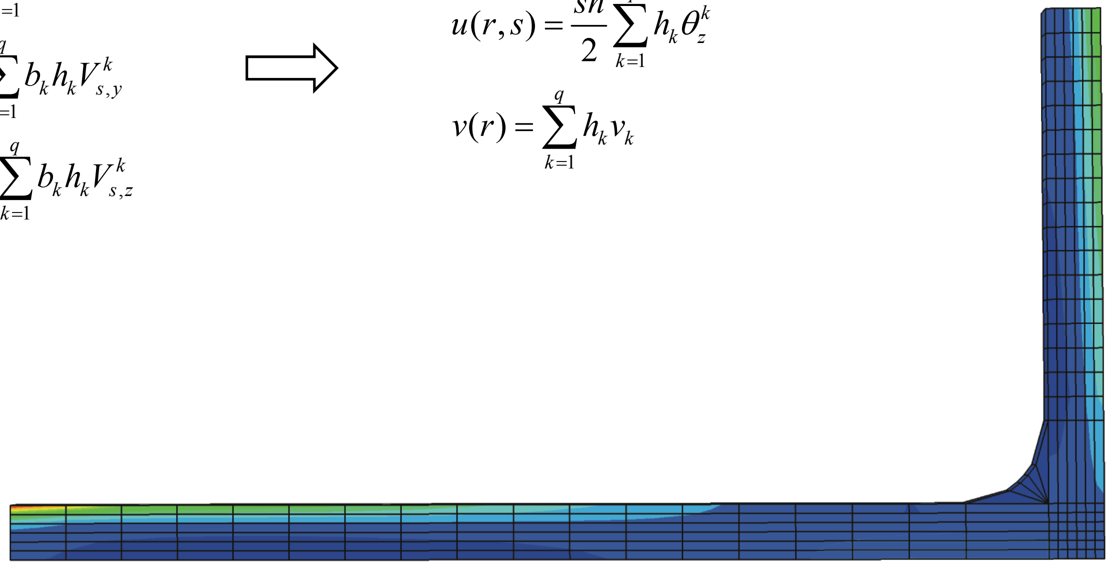
$$v(r,s,t)=\sum_{k=1}^{q}h_k\,{}^{\ell}v_k+\frac{t}{2}\sum_{k=1}^{q}a_k h_k V_{t,y}^k+\frac{s}{2}\sum_{k=1}^{q}b_k h_k V_{s,y}^k$$

$$w(r,s,t)=\sum_{k=1}^{q}h_k\,{}^{\ell}w_k+\frac{t}{2}\sum_{k=1}^{q}a_k h_k V_{t,z}^k+\frac{s}{2}\sum_{k=1}^{q}b_k h_k V_{s,z}^k$$

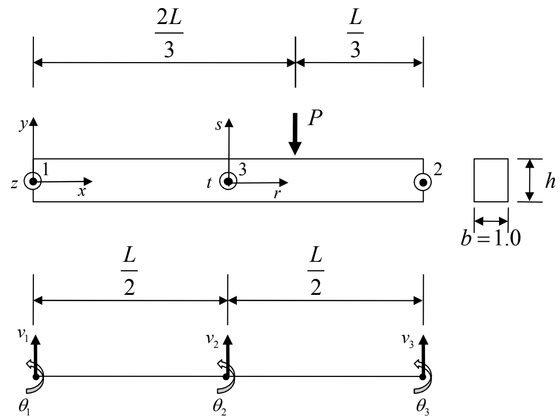
$\Rightarrow$

$$u(r,s)=\frac{sh}{2}\sum_{k=1}^{q}h_k\theta_z^k$$

$$v(r)=\sum_{k=1}^{q}h_k v_k$$



Formulation of Structural Elements



General curved beam elements: Example

Now we proceed to take the derivatives to develop the B matrix

$$\begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} \frac{\partial h_k}{\partial r} \begin{bmatrix} 1 & (g)_{31}^k \end{bmatrix} \\ h_k \begin{bmatrix} 0 & (\bar{g})_{31}^k \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_k \\ \theta_z^k \end{bmatrix}, \quad u_k = 0$$

$\Downarrow$

$$\begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} \frac{\partial h_k}{\partial r} (g)_{31}^k \\ h_k (\bar{g})_{31}^k \end{bmatrix} \theta_z^k$$

$\Downarrow$

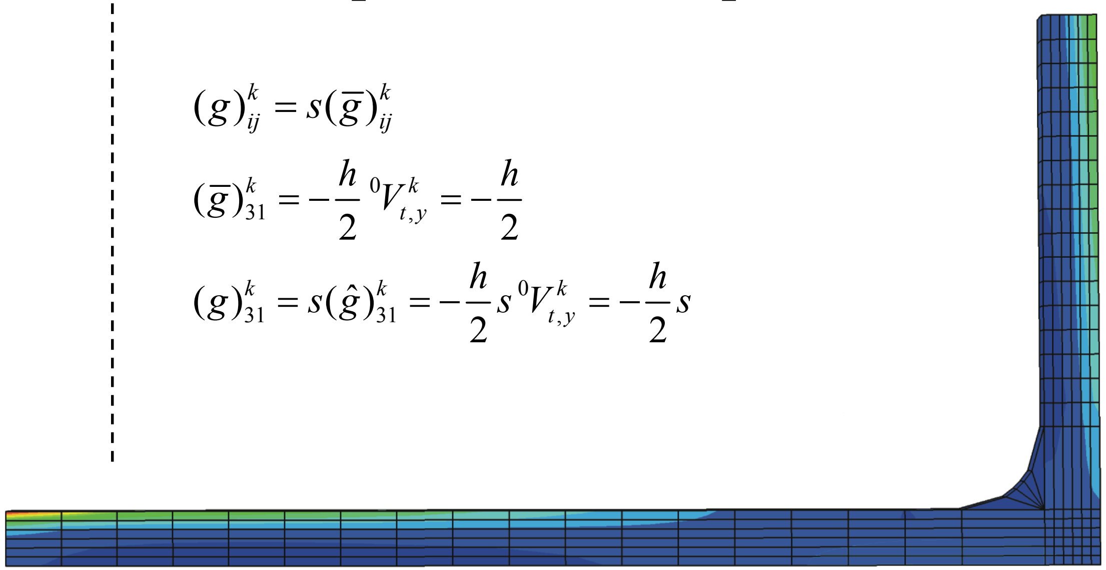
$$\begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} \frac{-sh}{2} \frac{\partial h_k}{\partial r} \\ \frac{-h}{2} h_k \end{bmatrix} \theta_z^k$$

$$(\mathbf{g})^k = \frac{a_k}{2} \begin{bmatrix} 0 & -{}^0V_{t,z}^k & {}^0V_{t,y}^k \\ -{}^0V_{t,z}^k & 0 & -{}^0V_{t,x}^k \\ -{}^0V_{t,y}^k & {}^0V_{t,x}^k & 0 \end{bmatrix}$$

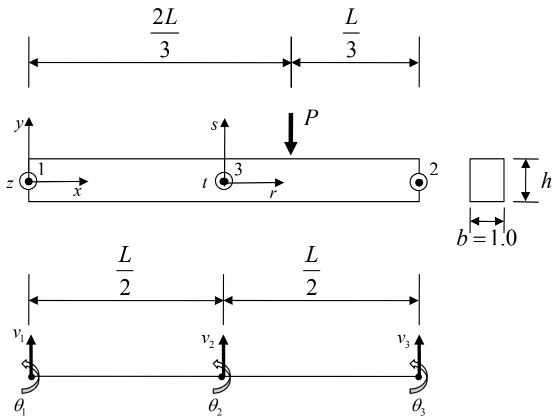
$$(g)_{ij}^k = s(\bar{g})_{ij}^k$$

$$(\bar{g})_{31}^k = -\frac{h}{2} {}^0V_{t,y}^k = -\frac{h}{2}$$

$$(g)_{31}^k = s(\hat{g})_{31}^k = -\frac{h}{2} s \, {}^0V_{t,y}^k = -\frac{h}{2} s$$



Formulation of Structural Elements



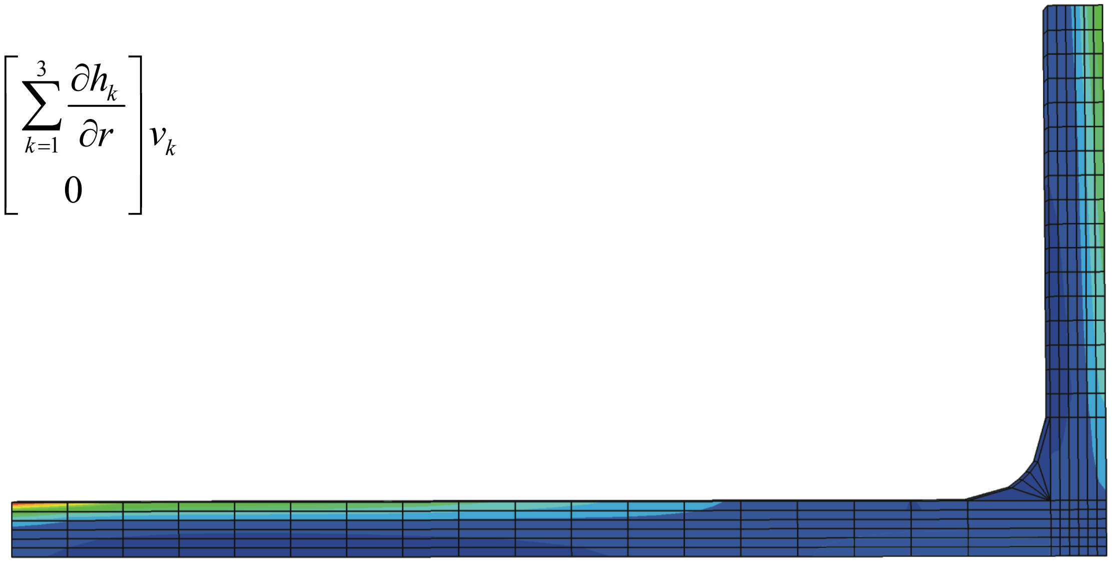
General curved beam elements: Example

Now we proceed to take the derivatives to develop the B matrix

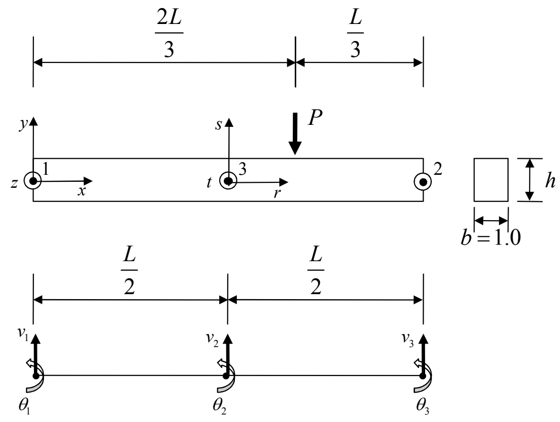
$$\begin{bmatrix} \dfrac{\partial v}{\partial r} \\ \dfrac{\partial v}{\partial s} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} \dfrac{\partial h_k}{\partial r} \begin{bmatrix} 1 & (g)_{32}^k \end{bmatrix} \\ h_k \begin{bmatrix} 0 & (\overline{g})_{32}^k \end{bmatrix} \end{bmatrix} \begin{bmatrix} v_k \\ \theta_z^k \end{bmatrix}, \qquad (\overline{g})_{32}^k = -{}^0V_{t,x}^k = 0, \quad (g)_{32}^k = 0$$

$$\Downarrow$$

$$\begin{bmatrix} \dfrac{\partial v}{\partial r} \\ \dfrac{\partial v}{\partial s} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} \dfrac{\partial h_k}{\partial r} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ h_k \begin{bmatrix} 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} v_k \\ \theta_z^k \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{k=1}^{3} \dfrac{\partial h_k}{\partial r} \\ 0 \end{bmatrix} v_k$$



Formulation of Structural Elements



General curved beam elements: Example

The next step is to transform into global coordinates

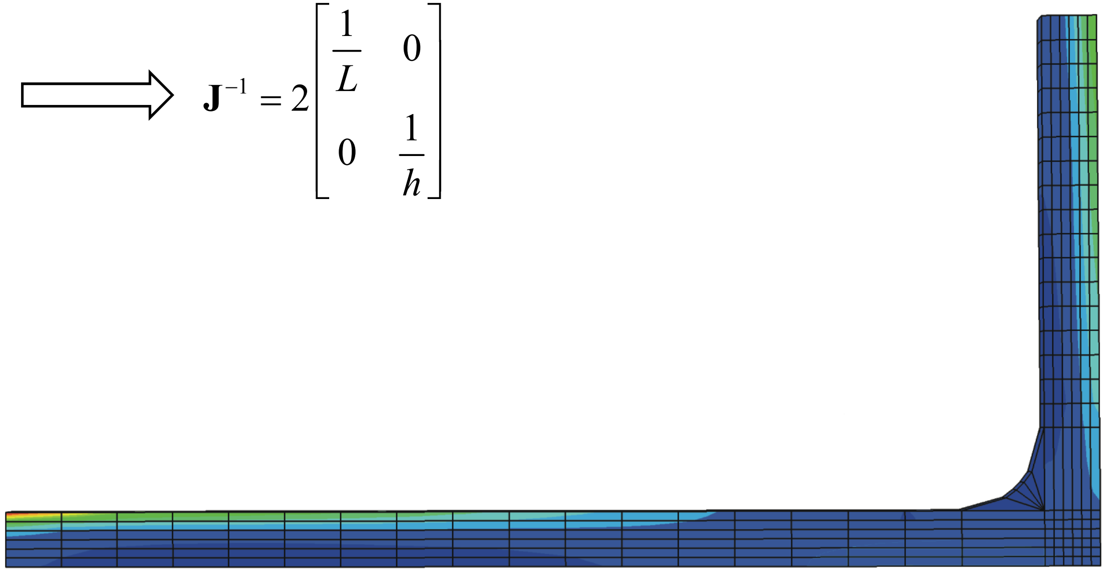
$$\frac{\partial}{\partial \mathbf{x}} = \mathbf{J}^{-1}\frac{\partial}{\partial \mathbf{r}}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}$$

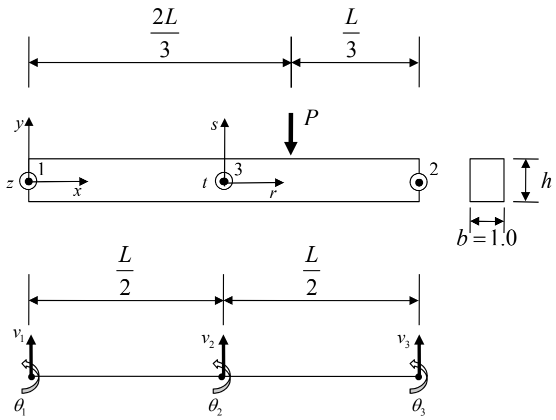
$$x = \frac{L}{2}(1+r)$$

$$y = \frac{h}{2}(1+s)$$

$$\mathbf{J} = \begin{bmatrix} \frac{L}{2} & 0 \\ 0 & \frac{h}{2} \end{bmatrix} \Longrightarrow \mathbf{J}^{-1} = 2\begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{h} \end{bmatrix}$$



Formulation of Structural Elements



General curved beam elements: Example

The next step is to transform into global coordinates

$$\begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} \frac{-sh}{2}\frac{\partial h_k}{\partial r} \\ \frac{-h}{2} h_k \end{bmatrix} \theta_z^k$$

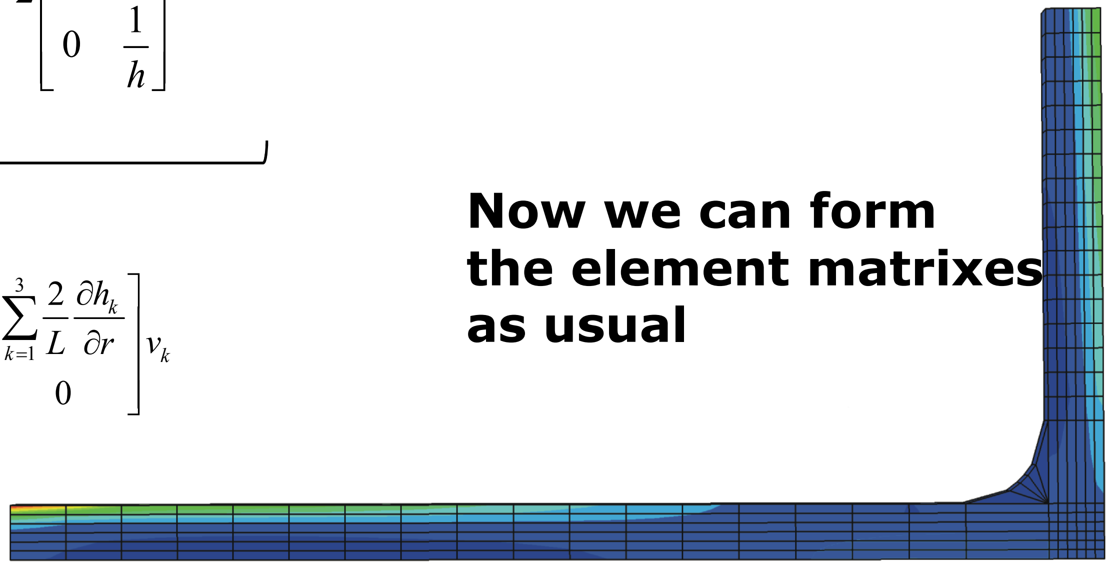
$$\begin{bmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial s} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{3} \frac{\partial h_k}{\partial r} \\ 0 \end{bmatrix} v_k$$

$$\frac{\partial}{\partial \mathbf{x}} = \mathbf{J}^{-1} \frac{\partial}{\partial \mathbf{r}}$$

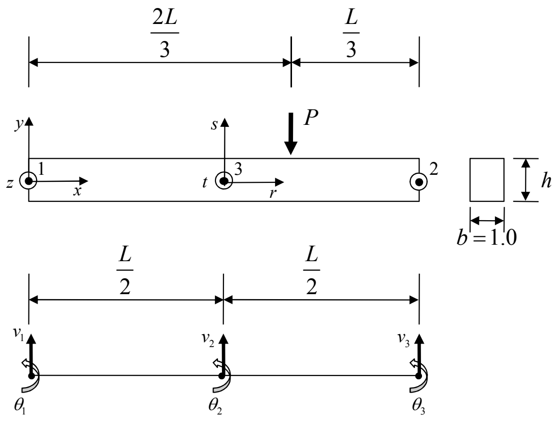
$$\mathbf{J}^{-1} = 2 \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{h} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} -\frac{sh}{L}\frac{\partial h_k}{\partial r} \\ -h_k \end{bmatrix} \theta_z^k, \qquad \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{3} \frac{2}{L}\frac{\partial h_k}{\partial r} \\ 0 \end{bmatrix} v_k$$

Now we can form the element matrixes as usual



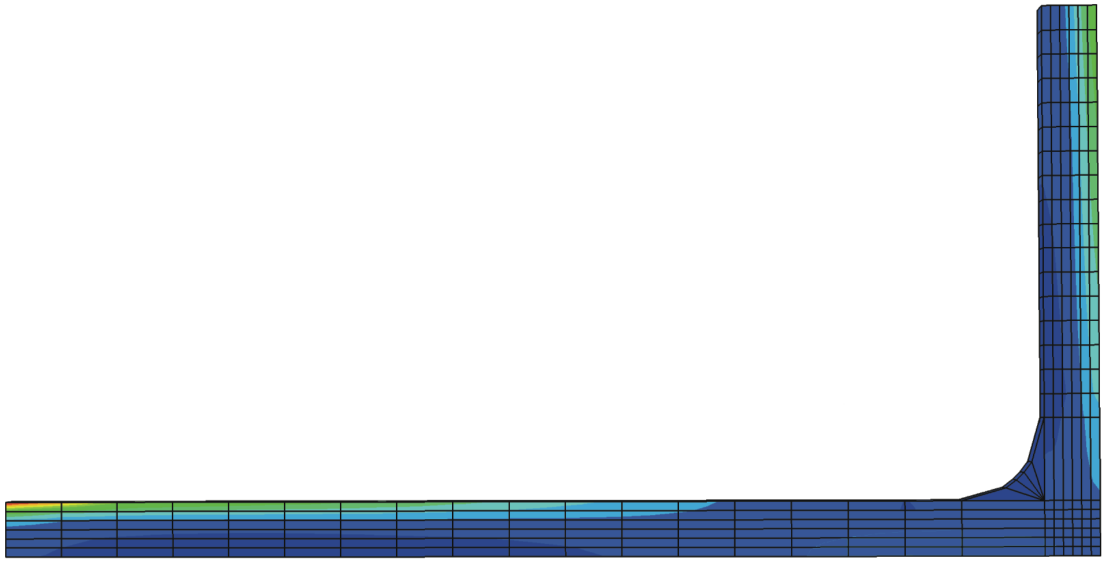
Formulation of Structural Elements



General curved beam elements: Example

The stiffness matrix is found from:

$$\mathbf{K} = \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} \varepsilon_{\eta\eta} & \gamma_{\eta\zeta} \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & Gk \end{bmatrix} \begin{bmatrix} \varepsilon_{\eta\eta} \\ \gamma_{\eta\zeta} \end{bmatrix} \det \mathbf{J} \, ds\,dr$$

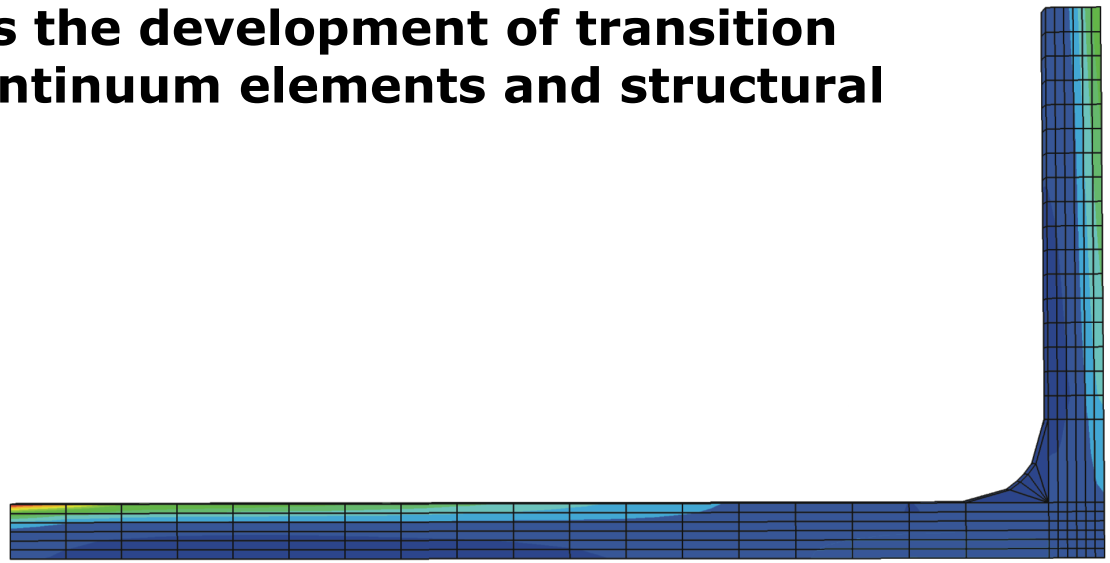


Formulation of Structural Elements

Transition element

It is interesting to notice that the structural elements can be derived from the continuum elements by imposition of the deformation assumptions (degeneration)

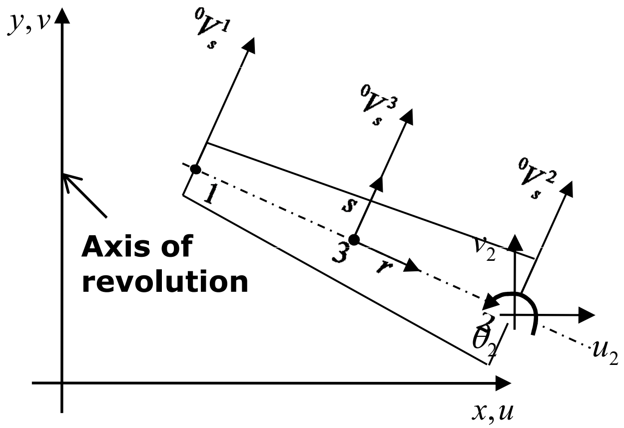
This result facilitates the development of transition element between continuum elements and structural elements



Formulation of Structural Elements

Axi-symmetric shell element

This element can be constructed from the iso-parametric beam element (the 2-dimensional case; movements in the x,y-plane)



The stress-strain matrix must then be ammended by a row corresponding to the hoop strain – and the stress-strain law must include a term coupling between the hoop and the *r*-direction and ensuring that the stress in the *s*-direction is equal to zero

K.J. Bathe (Example 5.9 and Exercise 5.41)