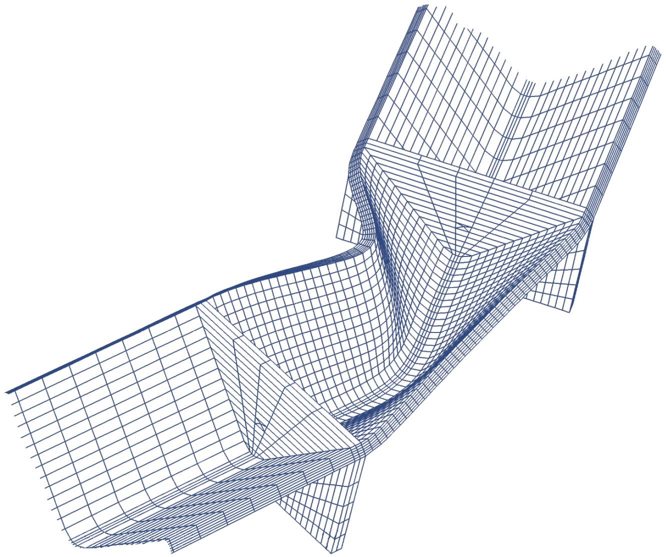
The Finite Element Method for the Analysis of Linear Systems



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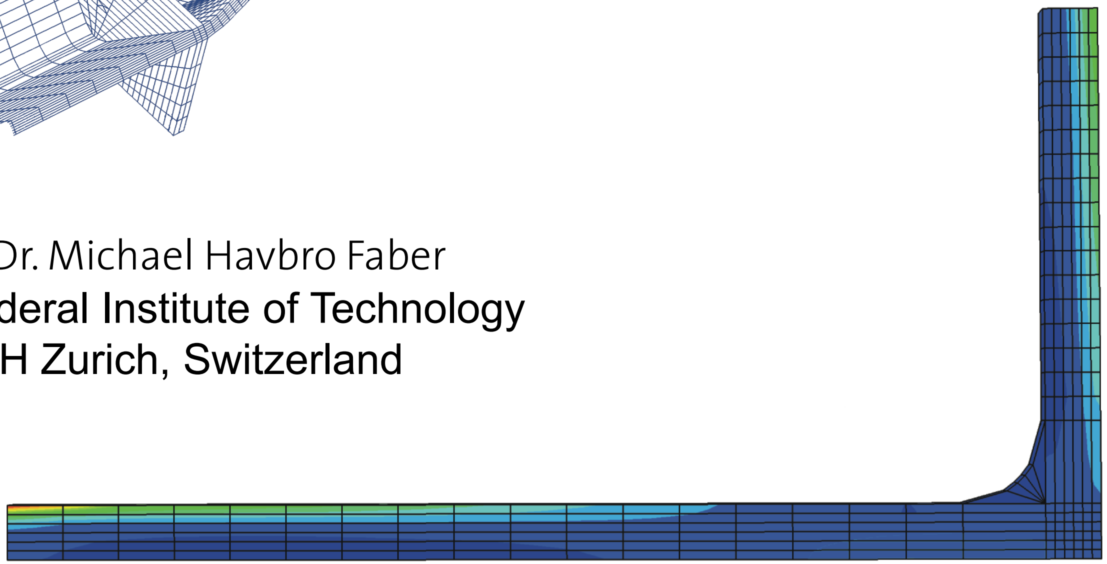

Contents of Today's Lecture

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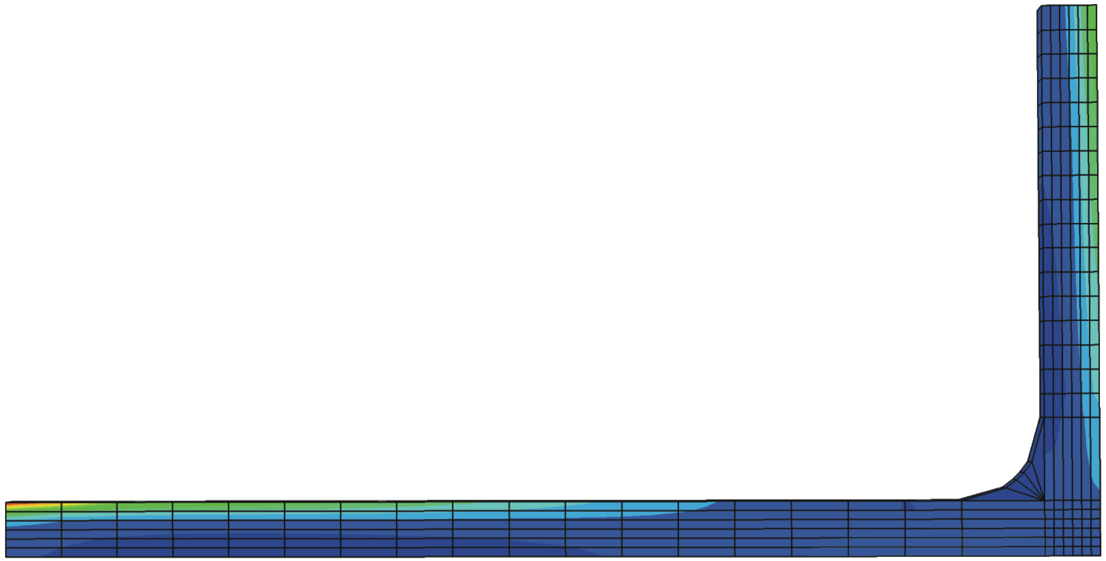
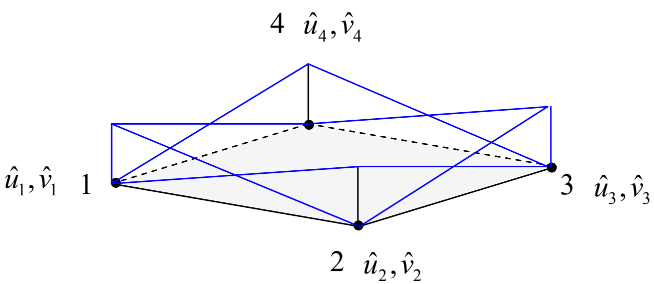


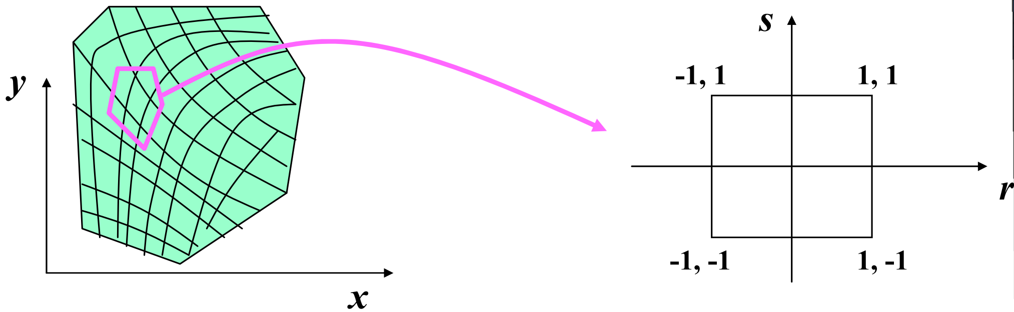
Quadrilateral elements

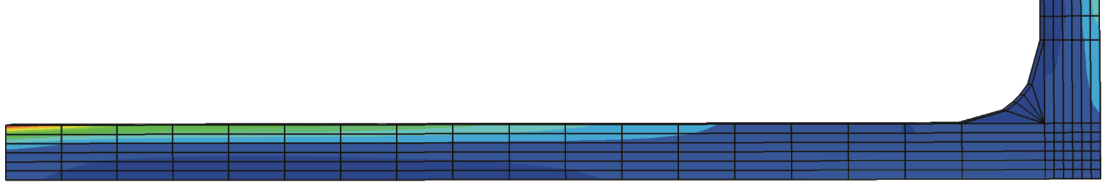
Bi-linear four node element:

In the following we will for matters of convenience consider the **iso-parametric** representation:

Displacement fields as well as the geometrical representation of the finite elements are approximated using the same approximating functions – shape functions







Quadrilateral elements

Bi-linear four node element:

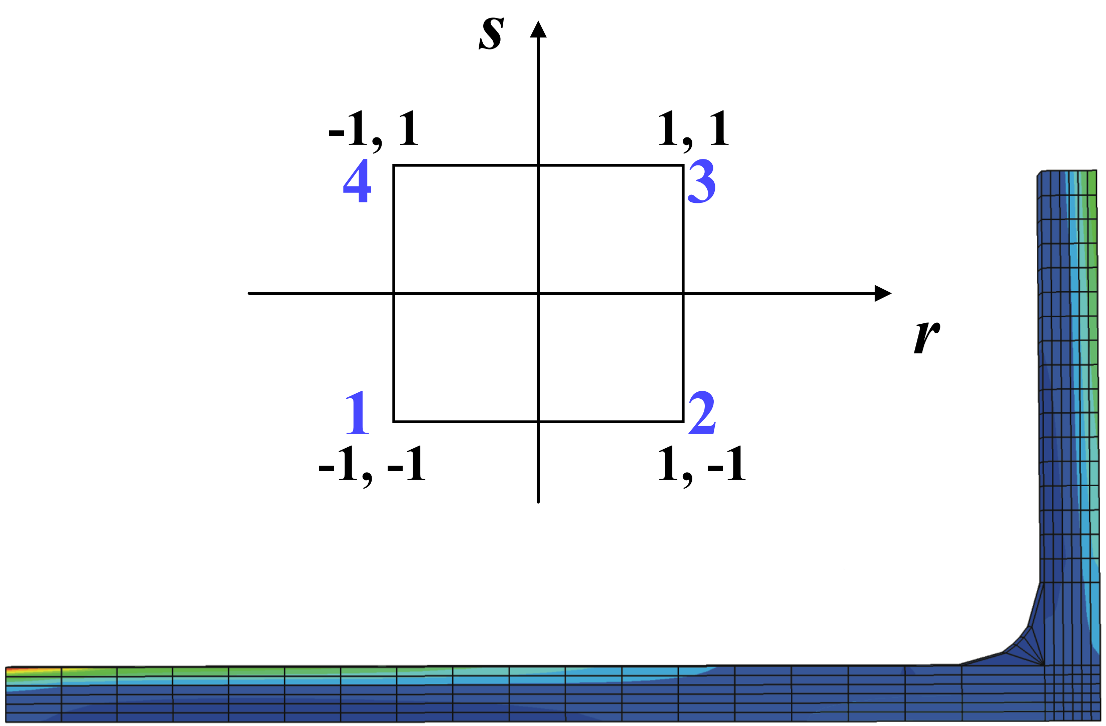
For the **bi-linear four node element** the shape functions in this coordinate system become:

$$h_1 = \frac{1}{2}(1-r)\frac{1}{2}(1-s)$$

$$h_2 = \frac{1}{2}(1+r)\frac{1}{2}(1-s)$$

$$h_3 = \frac{1}{2}(1+r)\frac{1}{2}(1+s)$$

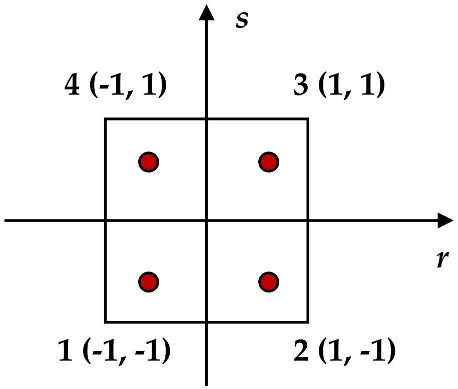
$$h_4 = \frac{1}{2}(1-r)\frac{1}{2}(1+s)$$



Quadrilateral elements

Bi-linear four node element:

Numerical integration: Gauss rule (2x2)

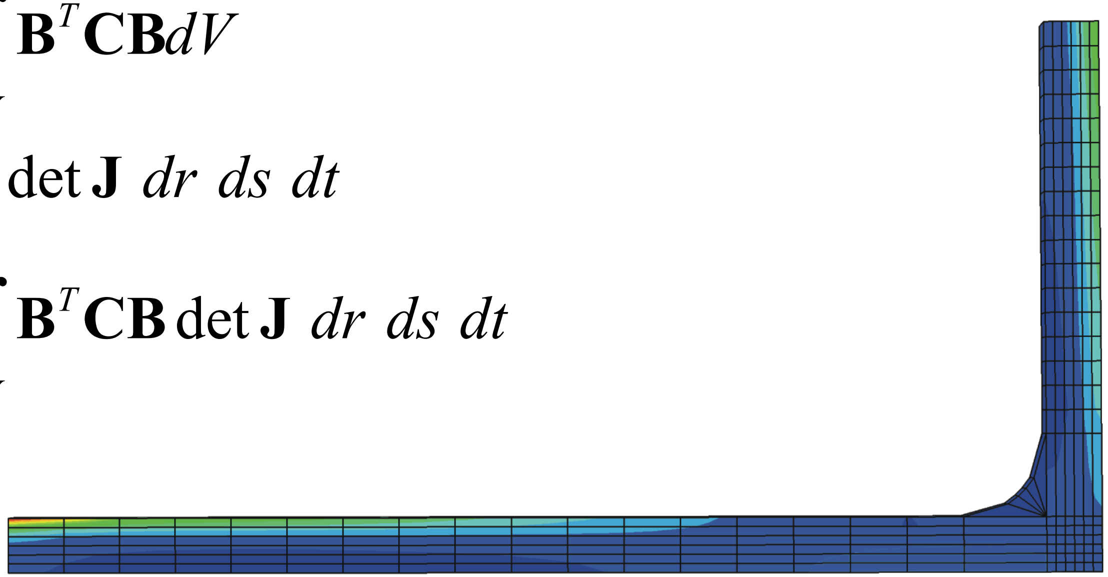


Stiffness matrix:

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} dV$$

$$dV = \det \mathbf{J} \; dr \; ds \; dt$$

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} \det \mathbf{J} \; dr \; ds \; dt$$



Quadrilateral elements

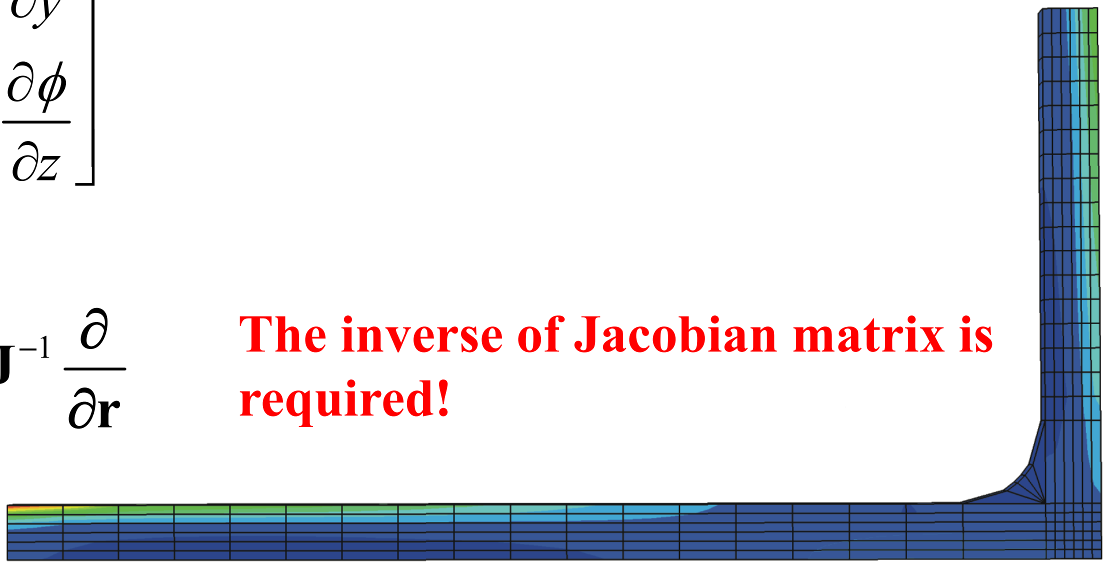
Singularity of Jacobian matrix:

Considering the general three-dimensional case there is:

$$\begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial s} \\ \frac{\partial \phi}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix}$$

$$\frac{\partial}{\partial \mathbf{r}} = \mathbf{J} \frac{\partial}{\partial \mathbf{x}} \quad \Rightarrow \quad \frac{\partial}{\partial \mathbf{x}} = \mathbf{J}^{-1} \frac{\partial}{\partial \mathbf{r}}$$

The inverse of Jacobian matrix is required!

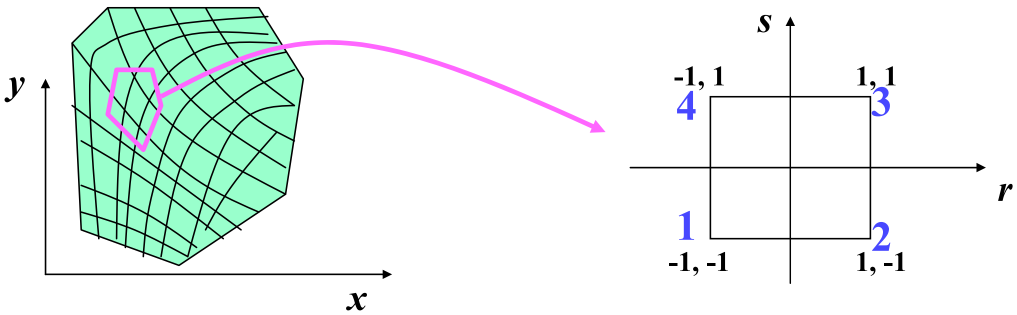


Quadrilateral elements

Singularity of Jacobian matrix:

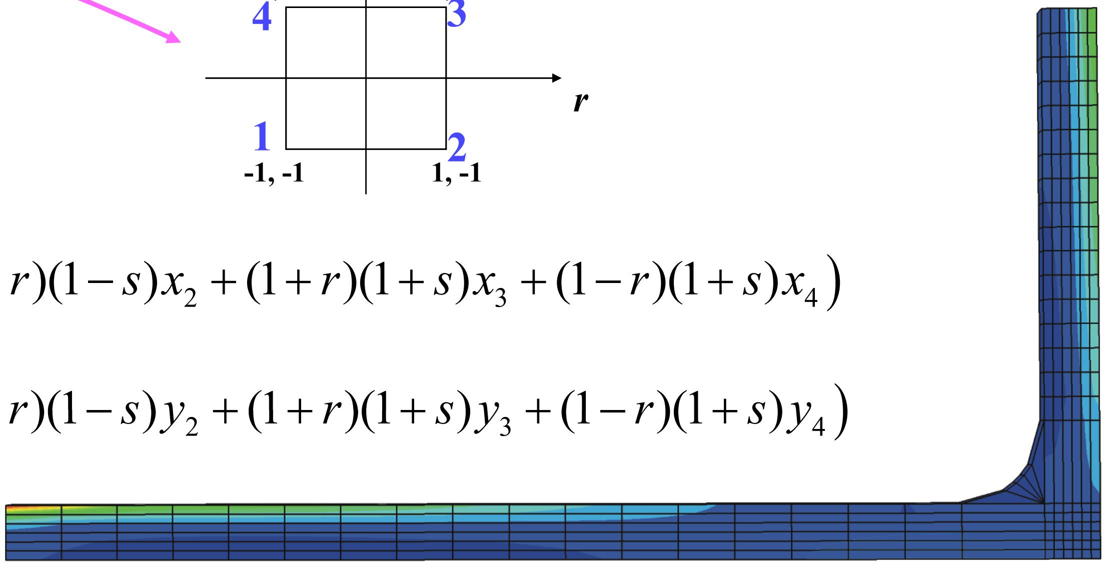
The existence of the inverse of Jacobian matrix provides the unique correspondence between natural and local coordinates.

Let us consider a quadrilateral element



$$x(r,s) = \frac{1}{4}\left((1-r)(1-s)x_1 + (1+r)(1-s)x_2 + (1+r)(1+s)x_3 + (1-r)(1+s)x_4\right)$$

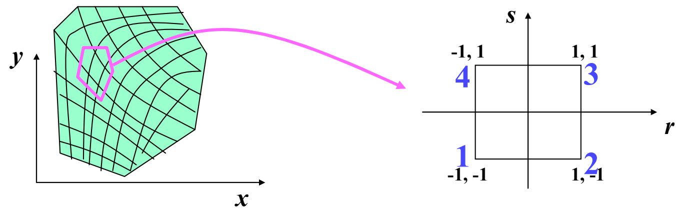
$$y(r,s) = \frac{1}{4}\left((1-r)(1-s)y_1 + (1+r)(1-s)y_2 + (1+r)(1+s)y_3 + (1-r)(1+s)y_4\right)$$



Quadrilateral elements

Singularity of Jacobian matrix:

Let us consider a quadrilateral element

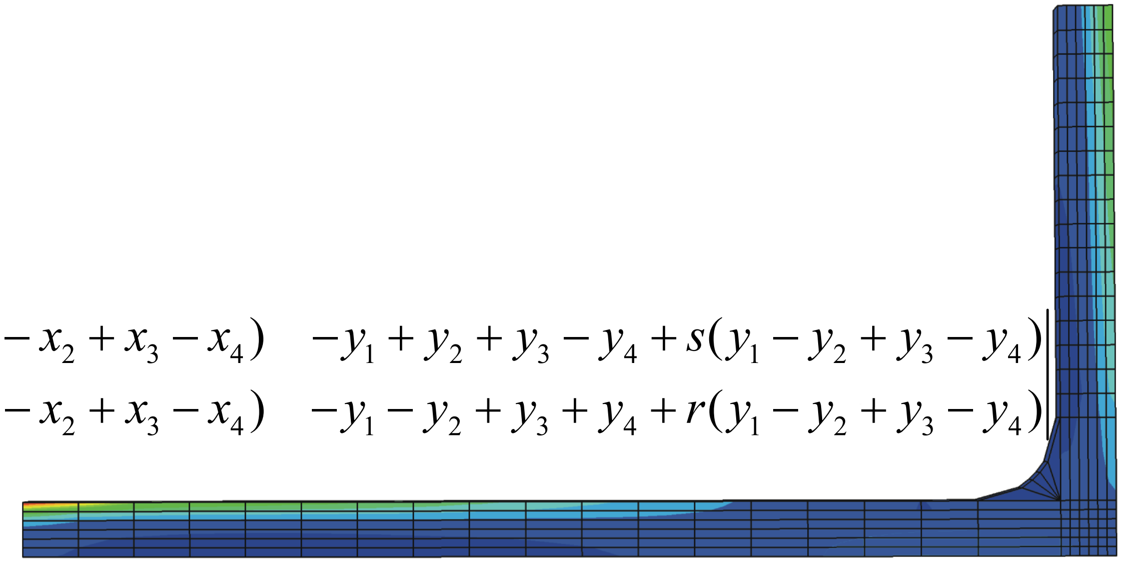


$$x(r,s)=\frac{1}{4}\big((1-r)(1-s)x_1+(1+r)(1-s)x_2+(1+r)(1+s)x_3+(1-r)(1+s)x_4\big)$$

$$y(r,s)=\frac{1}{4}\big((1-r)(1-s)y_1+(1+r)(1-s)y_2+(1+r)(1+s)y_3+(1-r)(1+s)y_4\big)$$

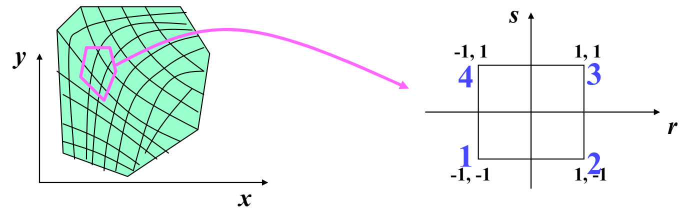
$$\det\mathbf{J}=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r}\\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s}\end{vmatrix}$$

$$=\frac{1}{4}\begin{vmatrix}-x_1+x_2+x_3-x_4+s(x_1-x_2+x_3-x_4) & -y_1+y_2+y_3-y_4+s(y_1-y_2+y_3-y_4)\\ -x_1-x_2+x_3+x_4+r(x_1-x_2+x_3-x_4) & -y_1-y_2+y_3+y_4+r(y_1-y_2+y_3-y_4)\end{vmatrix}$$



Quadrilateral elements

Singularity of Jacobian matrix:



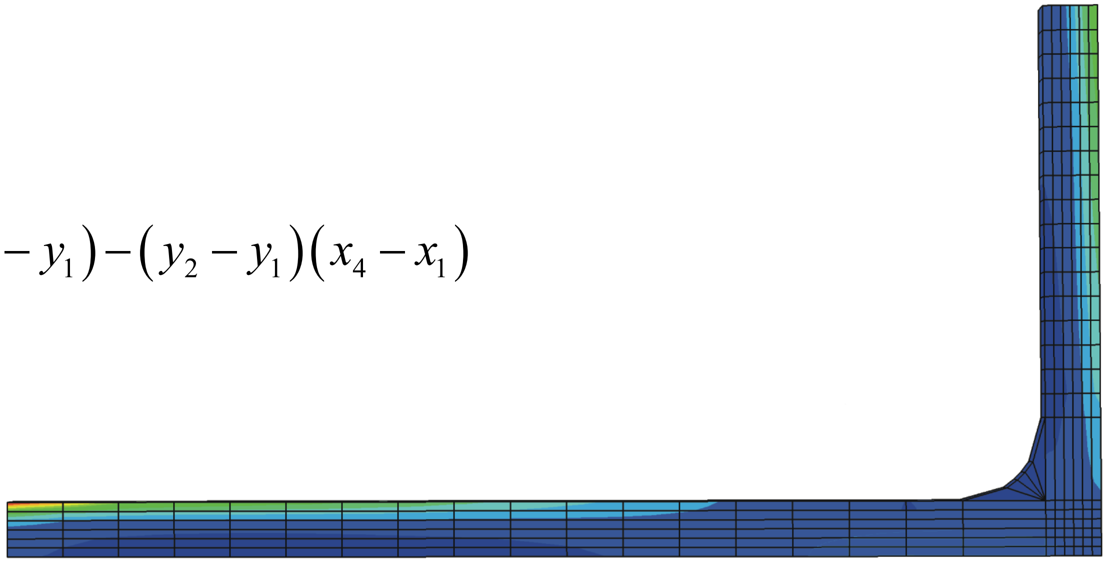
Let us consider a quadrilateral element

det J is a linear function of the coordinates r and s. Therefore, det J≠0 in the element only if its values in all nodes are positive or negative.

We have to inspect the values at the nodes.

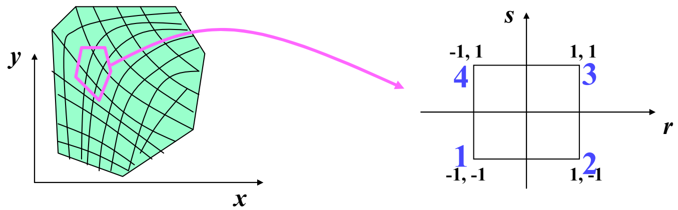
node 1:

$$\det \mathbf{J} = (x_2 - x_1)(y_4 - y_1) - (y_2 - y_1)(x_4 - x_1)$$



Quadrilateral elements

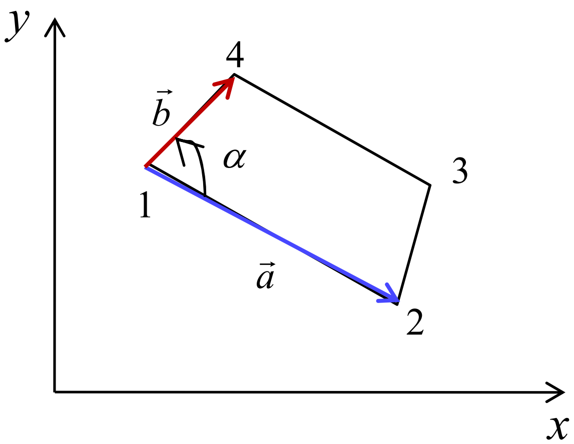
Singularity of Jacobian matrix:



Let us consider a quadrilateral element

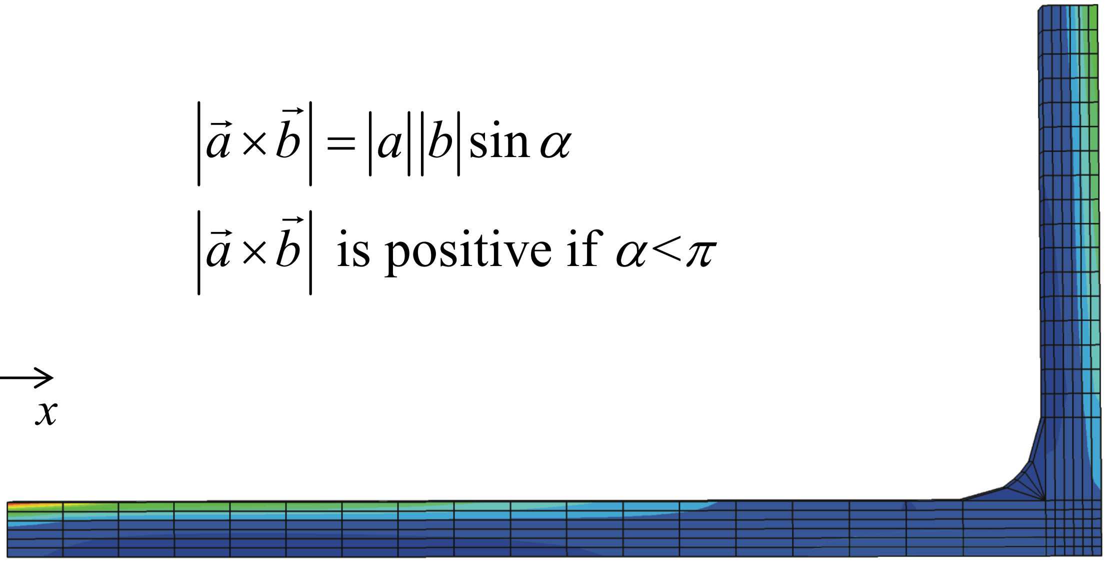
$$\det \mathbf{J} = (x_2 - x_1)(y_4 - y_1) - (y_2 - y_1)(x_4 - x_1)$$

It is connected with the value of the product of the vectors:

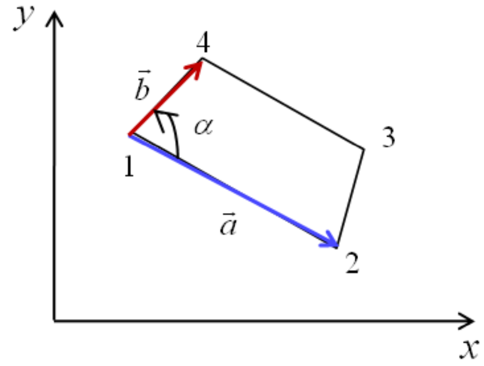


$$\left|\vec{a} \times \vec{b}\right| = |a||b| \sin\alpha$$

$$\left|\vec{a} \times \vec{b}\right| \text{ is positive if } \alpha < \pi$$



Quadrilateral elements

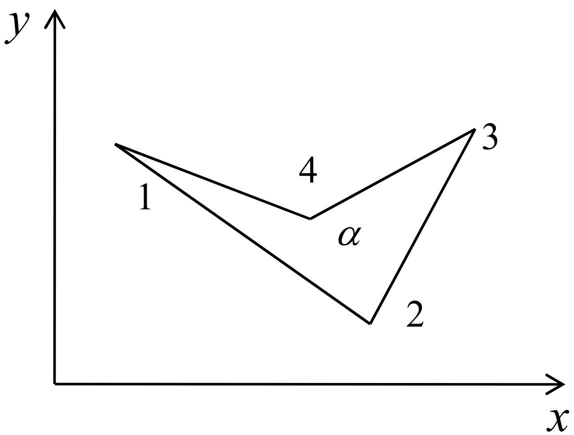


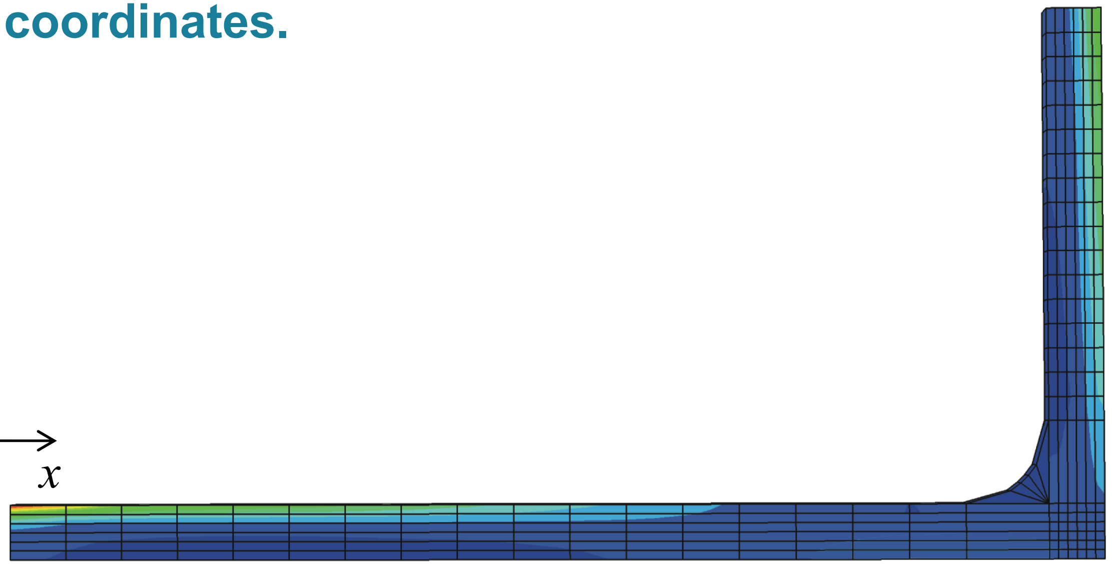
Singularity of Jacobian matrix:

Let us consider a quadrilateral element

If all angles of the quadrilateral element are smaller than 180 degrees, det J will not be 0.

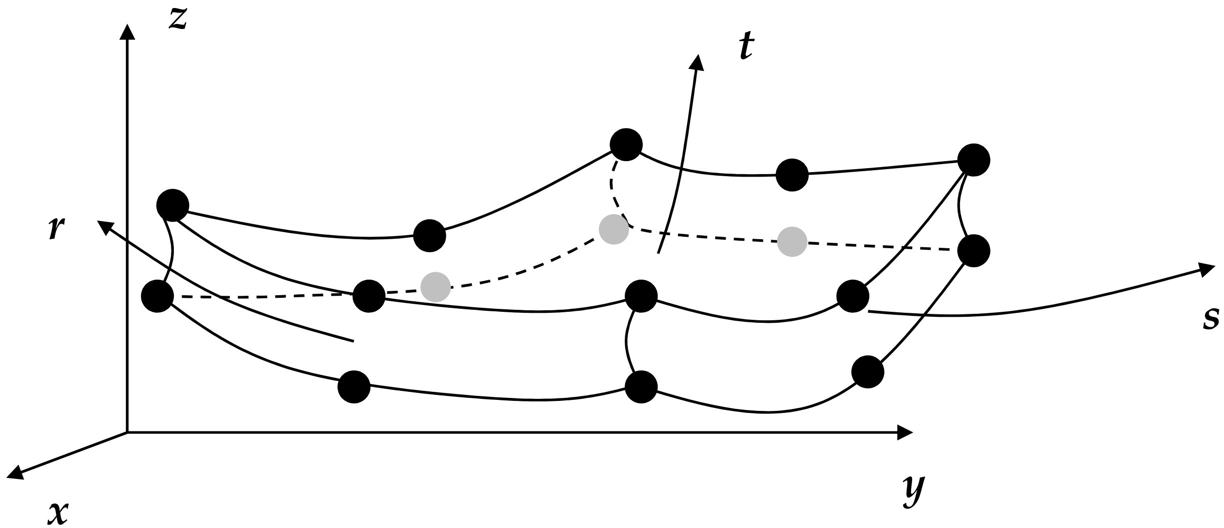
If this is not the case, there will be singularity somewhere in the element and it will not be possible to establish unique relation between natural and local coordinates.





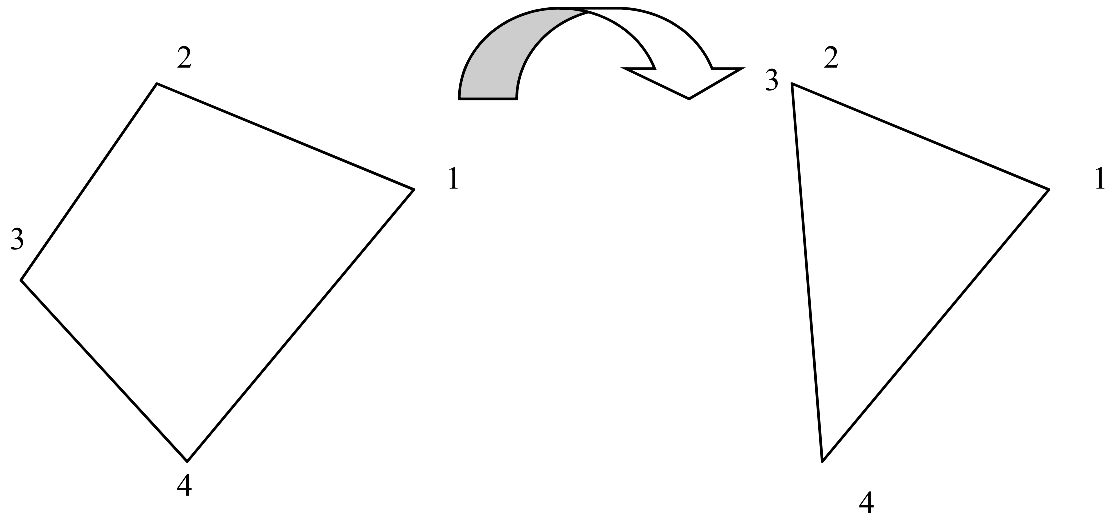
Quadrilateral elements

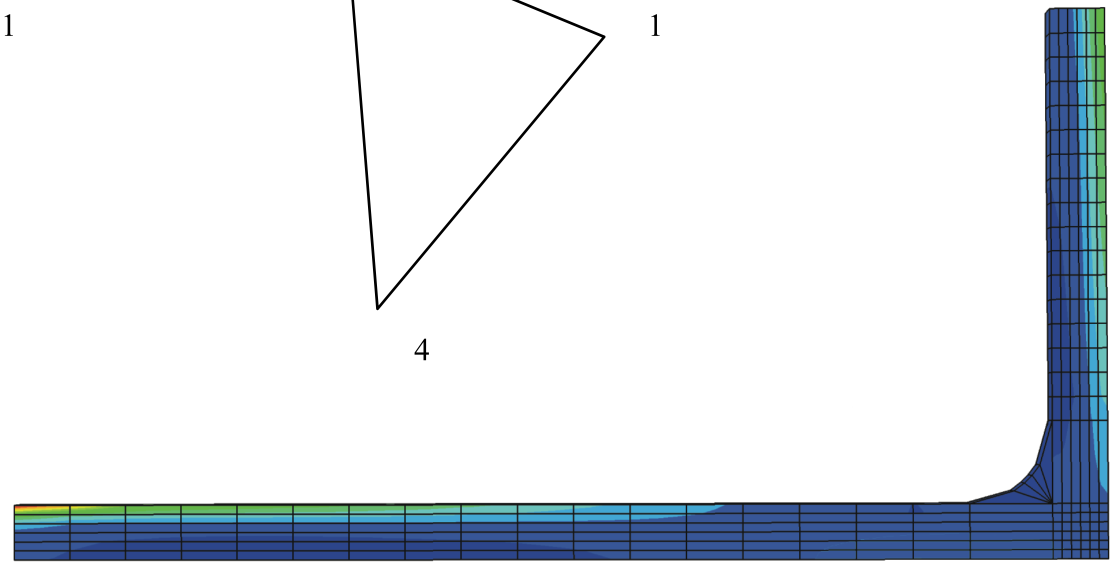
Following the same principle one may define isoparametric shape functions for three-dimensional quadrilateral elements (see Bathe pp. 344-345.)



Quadrilateral elements

We can also construct the triangular element directly from the quadrilateral element – by so-called collapsing:





Quadrilateral elements

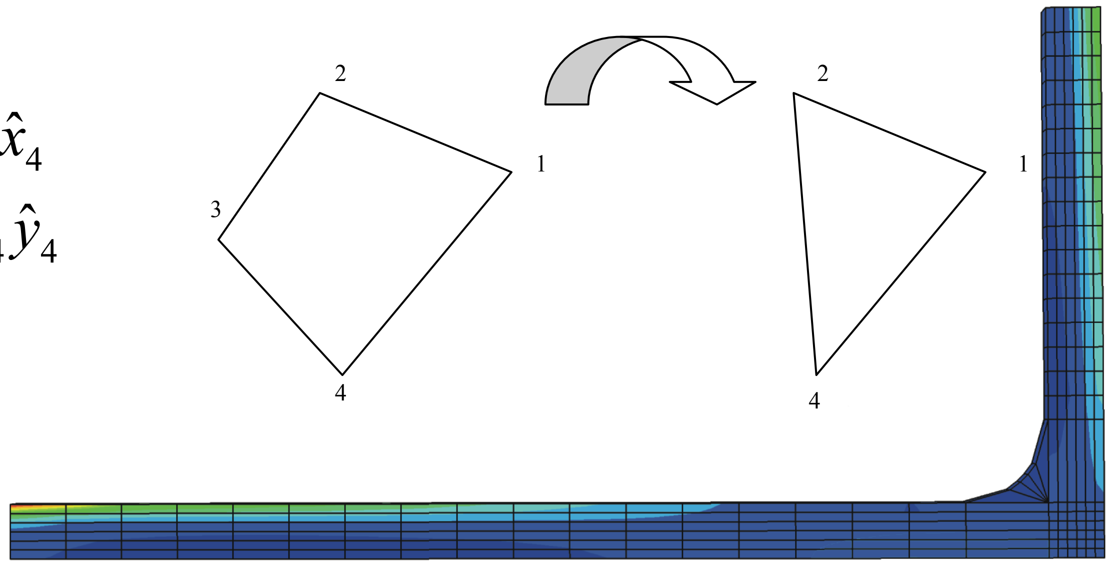
We can also construct the triangular element directly from the quadrilateral element – by so-called collapsing:

$$x = h_1\hat{x}_1 + h_2\hat{x}_2 + h_3\hat{x}_3 + h_4\hat{x}_4 \qquad \hat{x}_3 = \hat{x}_2$$

$$y = h_1\hat{y}_1 + h_2\hat{y}_2 + h_3\hat{y}_3 + h_4\hat{y}_4 \qquad \hat{y}_3 = \hat{y}_2$$

$$x = h_1\hat{x}_1 + (h_2 + h_3)\hat{x}_2 + h_4\hat{x}_4$$

$$y = h_1\hat{y}_1 + (h_2 + h_3)\hat{y}_2 + h_4\hat{y}_4$$



Element matrices in global coordinate system

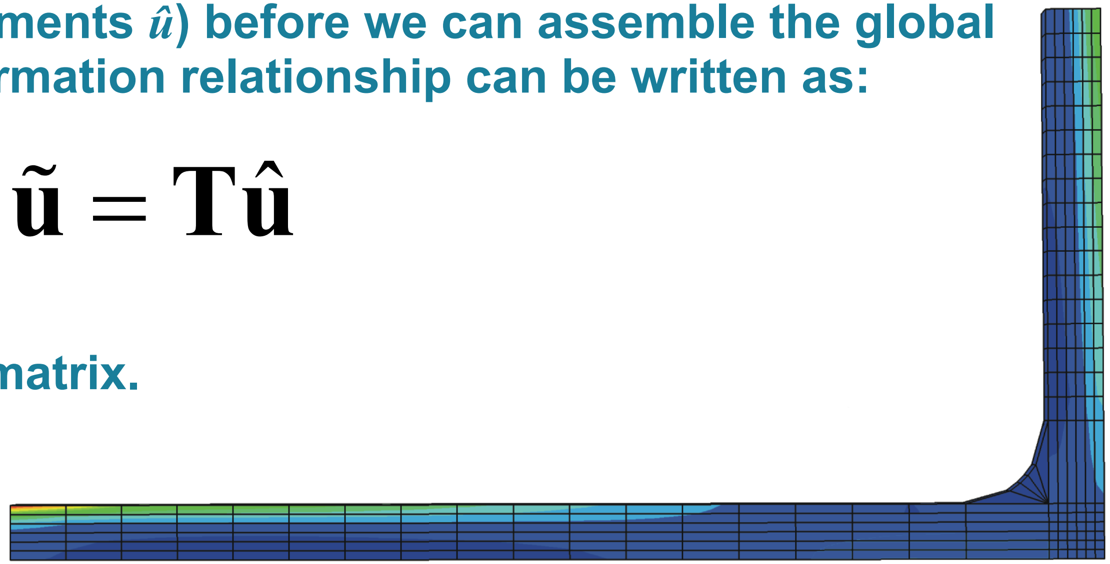
Local to global coordinate transformations:

It is often more convenient to define the element stiffness relations and to calculate their contributions to the load vector in a local coordinate system (e.g. displacements $\tilde{u}$) – this is often specific for each individual element.

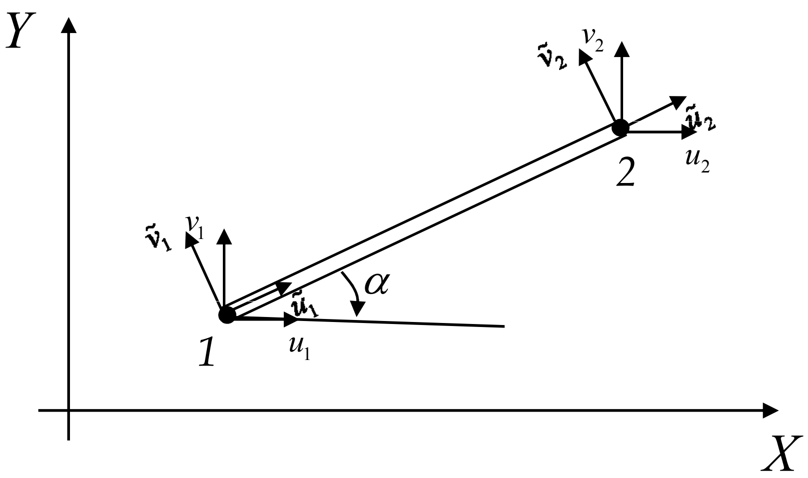
In this case we need to transform the element matrixes into global coordinates (e.g. displacements $\hat{u}$) before we can assemble the global stiffness relation. Transformation relationship can be written as:

$$\tilde{\mathbf{u}} = \mathbf{T}\hat{\mathbf{u}}$$

$\mathbf{T}$ being a transformation matrix.

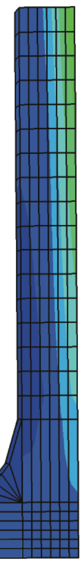
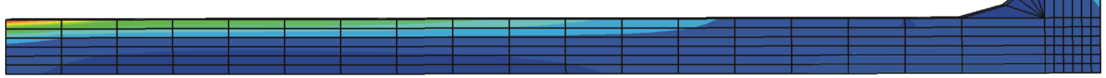


Element matrices in global coordinate system



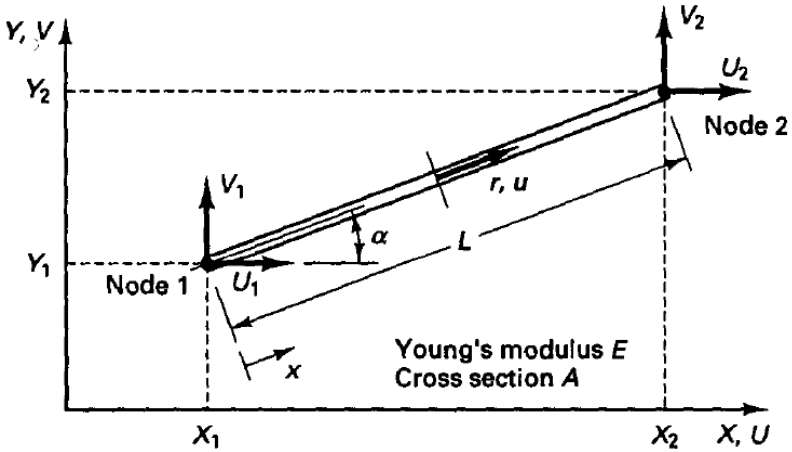
$$\tilde{\mathbf{u}} = \mathbf{T}\hat{\mathbf{u}}$$

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{v}_1 \\ \tilde{u}_2 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

Element matrices in global coordinate system

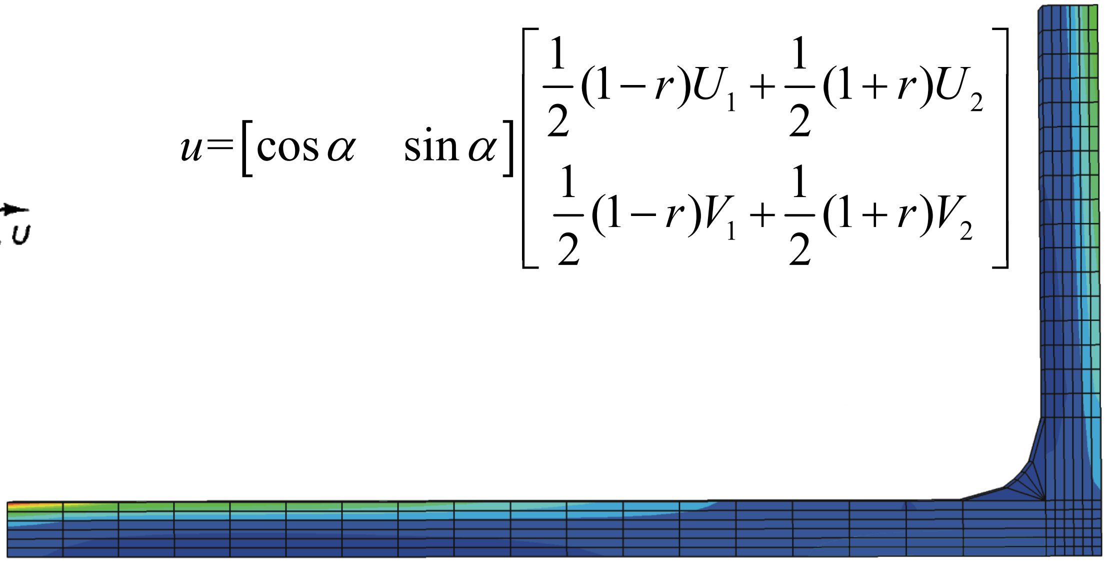
Let us try to establish the stiffness matrix of the truss element using directly global nodal point displacements. (see example 5.22, Bathe pp. 387-388)



$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} dV$$

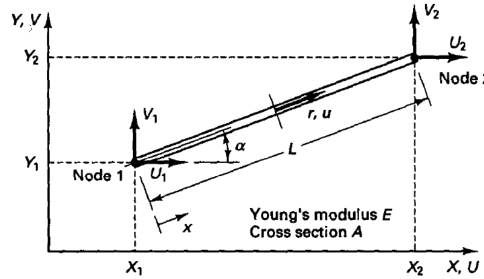
For the truss element considered we have

$$u = \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-r)U_1 + \frac{1}{2}(1+r)U_2 \\ \frac{1}{2}(1-r)V_1 + \frac{1}{2}(1+r)V_2 \end{bmatrix}$$



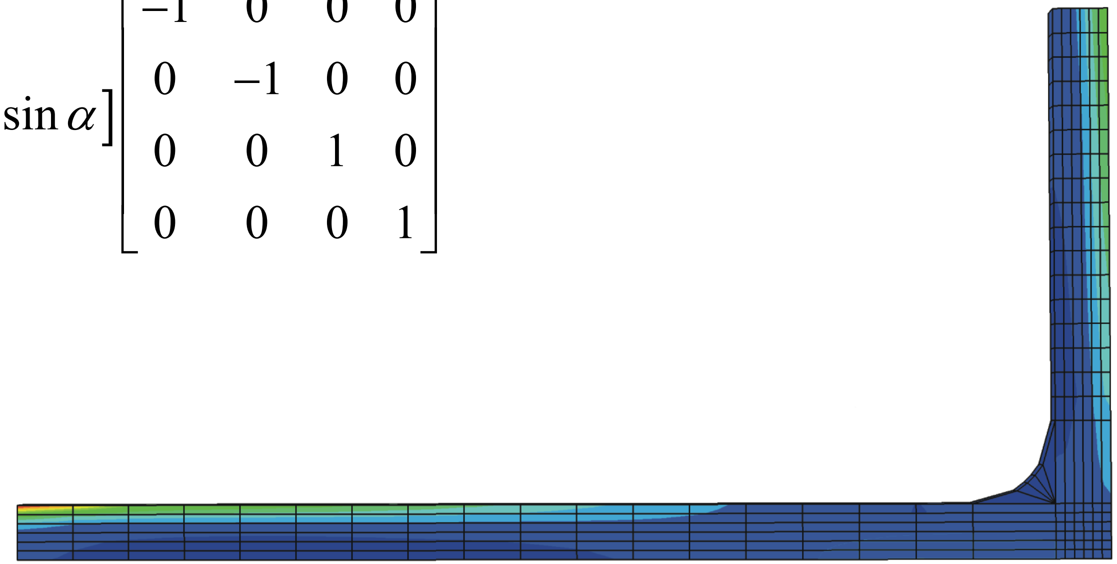
Element matrices in global coordinate system

Let us try to establish the stiffness matrix of the truss element using directly global nodal point displacements. (see example 5.22, Bathe pp. 387-388)



Using $\varepsilon = \dfrac{\partial u}{\partial x}, \ \varepsilon = \dfrac{2}{L}\dfrac{\partial u}{\partial r}$ in the natural coordinate system,

$$\mathbf{B} = \frac{1}{L}\begin{bmatrix} \cos\alpha & \sin\alpha & \cos\alpha & \sin\alpha \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Element matrices in global coordinate system

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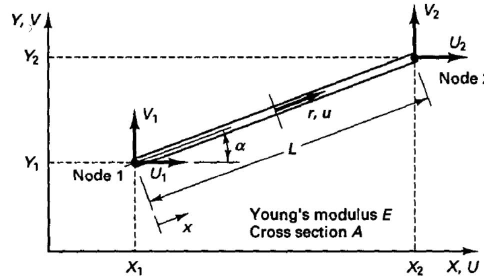

we have

$$dV = \frac{AL}{2} dr, \text{ and } \mathbf{C} = E$$

Finally, we obtain

$$\mathbf{K} = \frac{AE}{L} \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ & \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha \\ & & \cos^2\alpha & \cos\alpha\sin\alpha \\ \text{sym} & & & \sin^2\alpha \end{bmatrix}$$