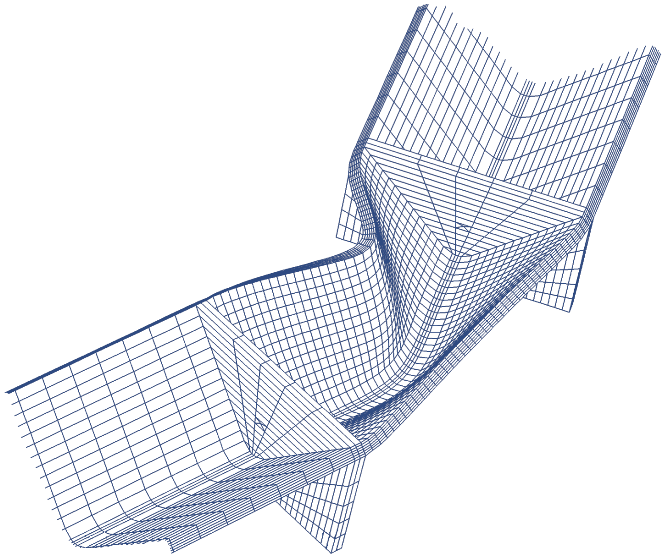
The Finite Element Method for the Analysis of Linear Systems



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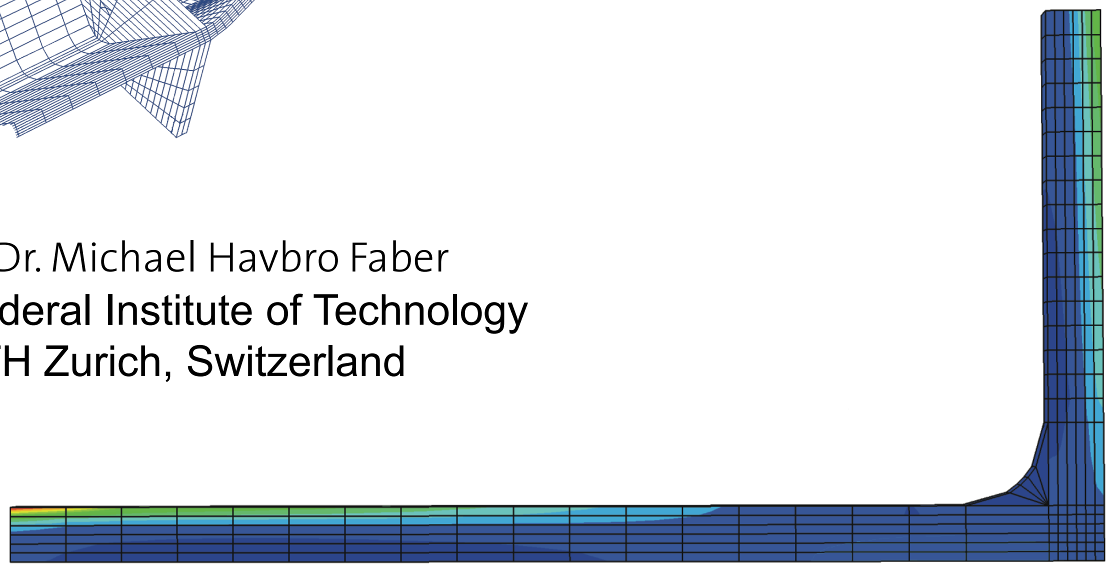


Contents of Today's Lecture

- **Iso-parametric element formulations**

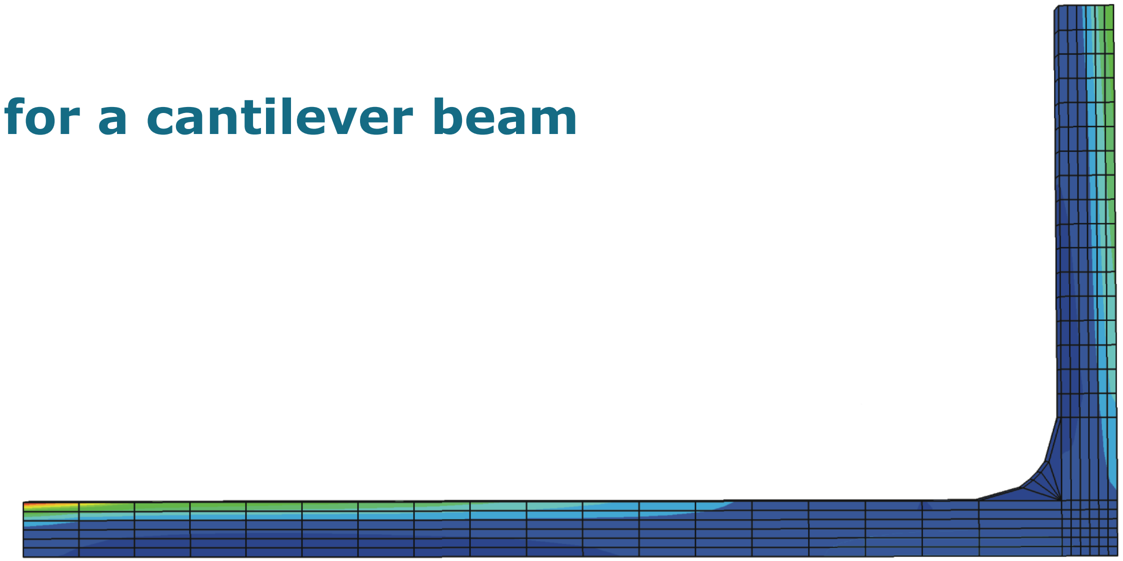
  - Bar elements

  - Quadrilateral elements

  - Triangular elements

- **Assignment 3**

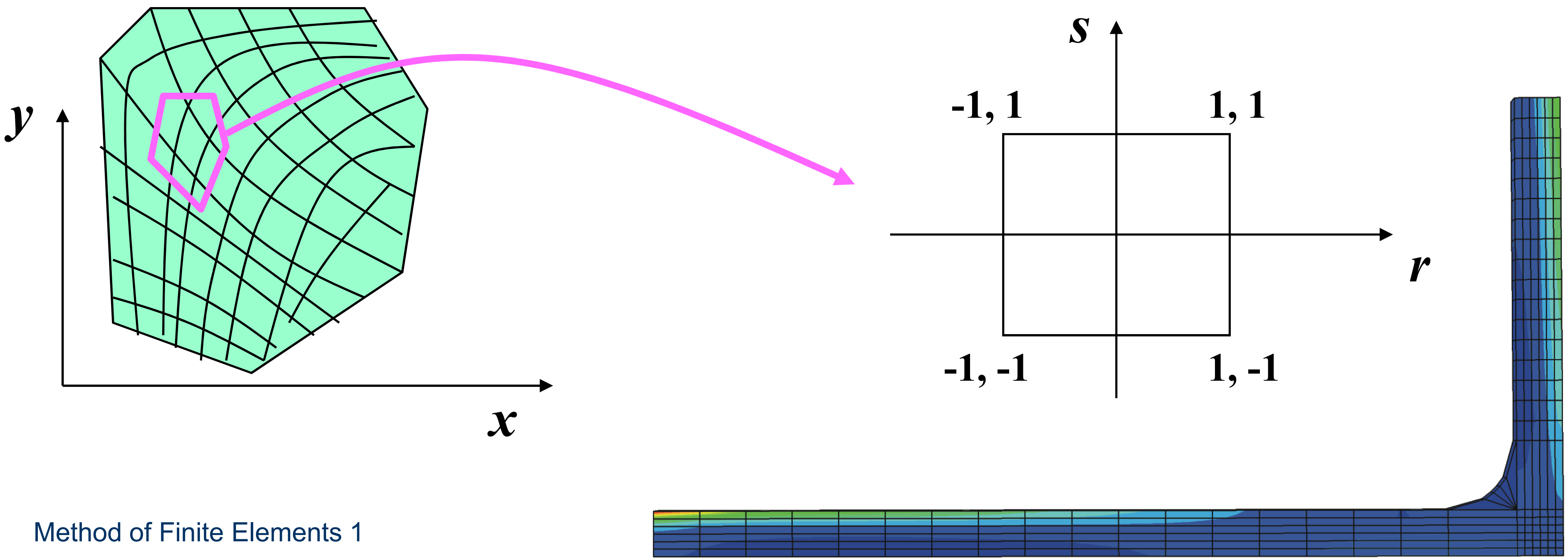
  - Finite element code for a cantilever beam



Iso-parametric Elements

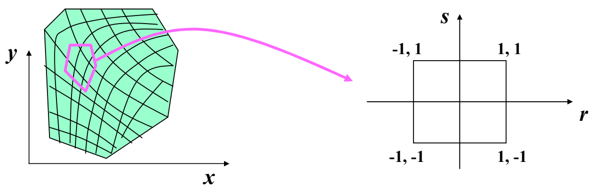
Shape functions – Natural coordinates:

As we have seen we are able to establish shape functions in global or local coordinate systems as we please. However, for the purpose of standardizing the process of developing the element matrixes it is convenient to introduce the so-called natural coordinate system.



Iso-parametric Elements

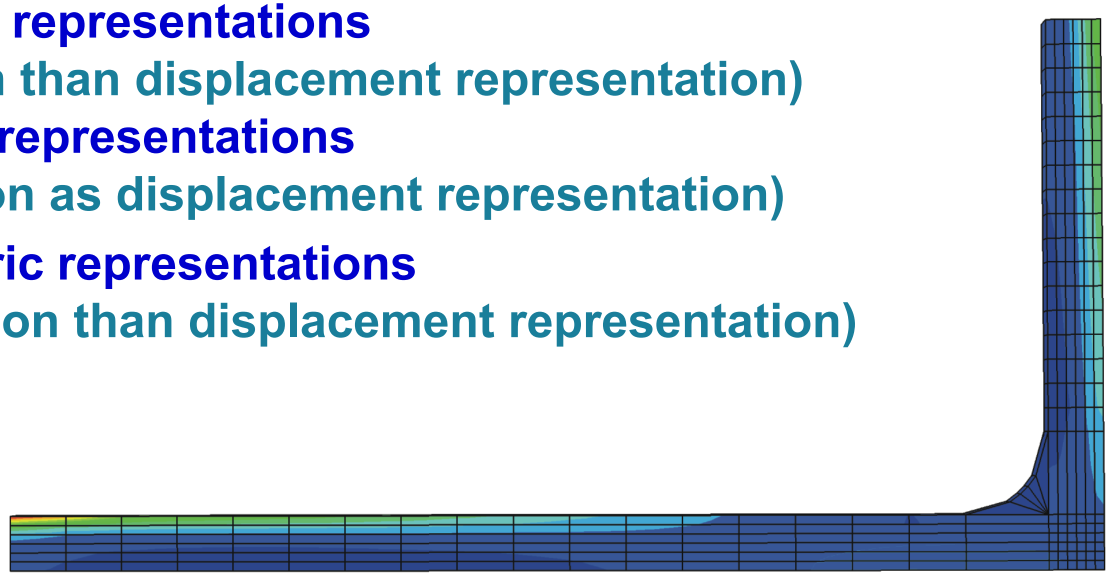
Shape functions – Natural coordinates :



In general we would like to be able to represent any element in a standardized manner – introducing a transformation between the natural coordinates and the real coordinates (global)

Different schemes exist for establishing such transformations:

1. **sub-parametric representations**
   (less resolution than displacement representation)
2. **iso-parametric representations**
   (same resolution as displacement representation)
3. **super-parametric representations**
   (higher resolution than displacement representation)

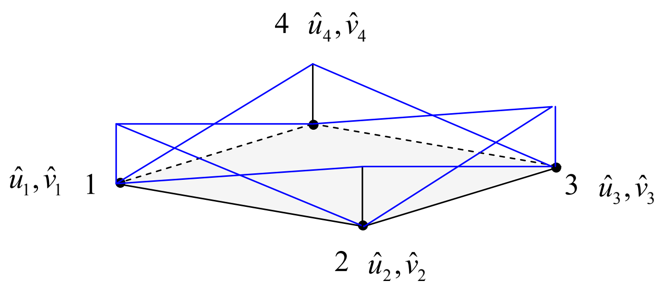


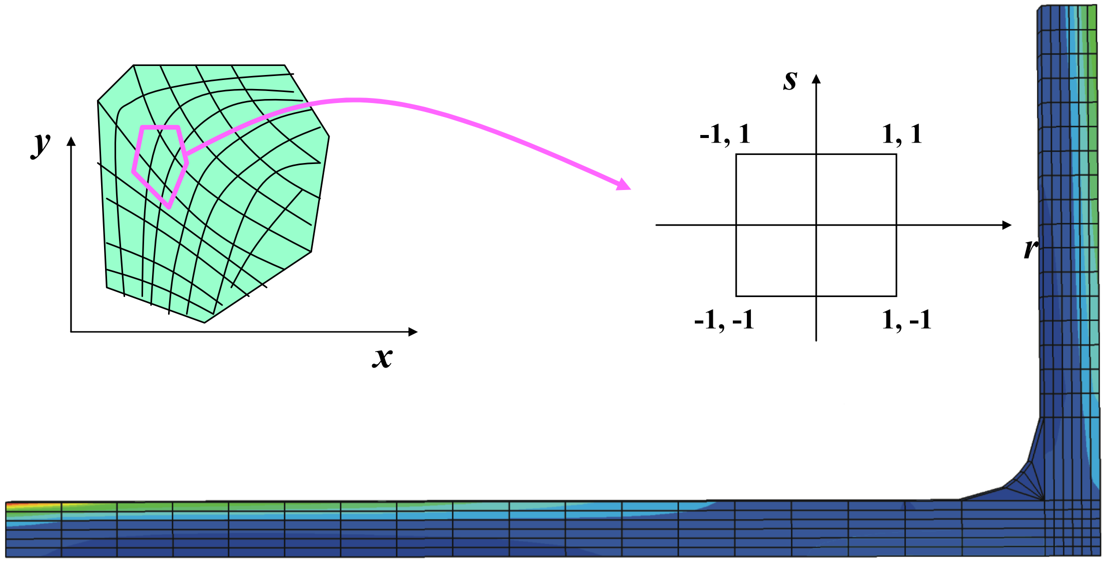
Iso-parametric Elements

Shape functions – Natural coordinates:

In the following we will for matters of convenience consider the **iso-parametric** representation:

Displacement fields as well as the geometrical representation of the finite elements are approximated using the same approximating functions – shape functions





Iso-parametric Elements

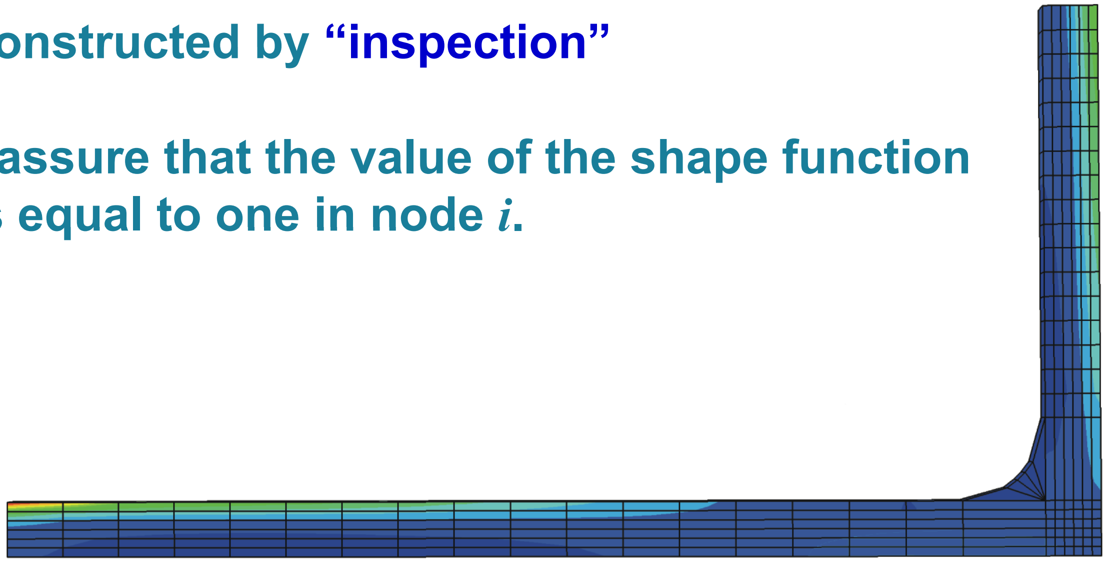
Shape functions – Natural coordinates:

Iso-parametric elements may be constructed for elements of one, two or three dimensions

$$x = \sum_{i=1}^{n} h_i \hat{x}_i; \quad y = \sum_{i=1}^{n} h_i \hat{y}_i; \quad z = \sum_{i=1}^{n} h_i \hat{z}_i$$

Shape functions may be constructed by "inspection"

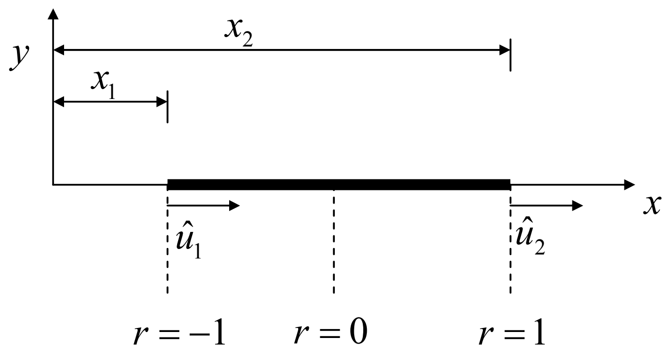
The principle is to always assure that the value of the shape function h_i in natural coordinates is equal to one in node i.



Iso-parametric Elements

Shape functions – Natural coordinates:

Let us consider the simple bar element



The relation between the x-coordinate and the r-coordinate is given as:

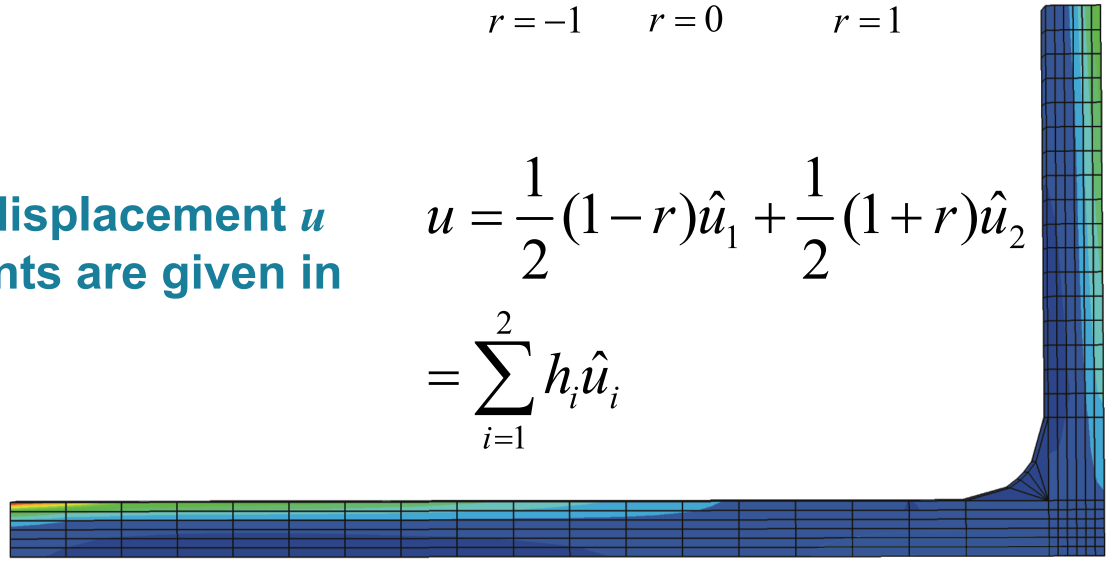
$$x = \frac{1}{2}(1-r)\hat{x}_1 + \frac{1}{2}(1+r)\hat{x}_2$$

$$= \sum_{i=1}^{2} h_i \hat{x}_i$$

The relation between the displacement u and the nodal displacements are given in the same way:

$$u = \frac{1}{2}(1-r)\hat{u}_1 + \frac{1}{2}(1+r)\hat{u}_2$$

$$= \sum_{i=1}^{2} h_i \hat{u}_i$$

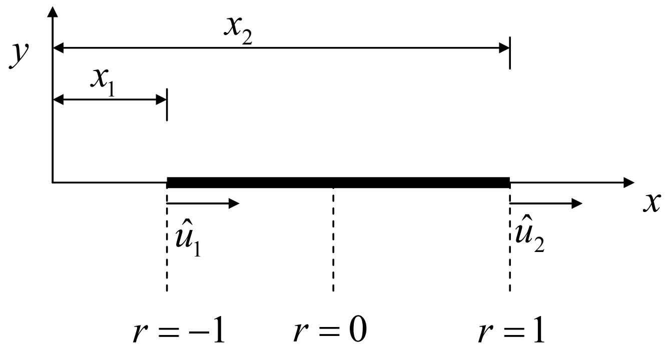


Iso-parametric Elements

Shape functions – Natural coordinates:

Let us consider the simple bar element

We need to be able to establish the strains – meaning we need to be able to take the derivatives of the displacement field in regard to the x-coordinate



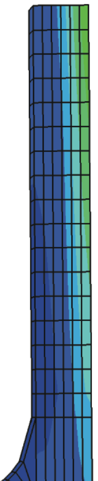
$$\varepsilon = \frac{du}{dx} = \frac{du}{dr}\frac{dr}{dx}$$

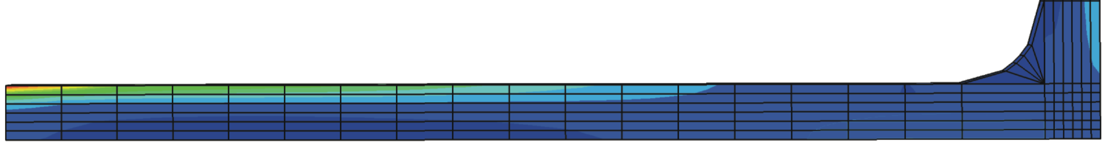
$$\frac{du}{dr} = \frac{d}{dr}\left(\frac{1}{2}(1-r)\hat{u}_1 + \frac{1}{2}(1+r)\hat{u}_2\right) = \frac{1}{2}(\hat{u}_2 - \hat{u}_1)$$

$$\frac{dx}{dr} = \frac{d}{dr}\left(\frac{1}{2}(1-r)\hat{x}_1 + \frac{1}{2}(1+r)\hat{x}_2\right) = \frac{1}{2}(\hat{x}_2 - \hat{x}_1)$$

$$\Downarrow$$

$$\frac{du}{dx} = \frac{(\hat{u}_2 - \hat{u}_1)}{(\hat{x}_2 - \hat{x}_1)} = \frac{(\hat{u}_2 - \hat{u}_1)}{L}$$





Iso-parametric Elements

Shape functions – Natural coordinates:

Let us consider the **simple bar element**

The strain-displacement matrix then becomes:

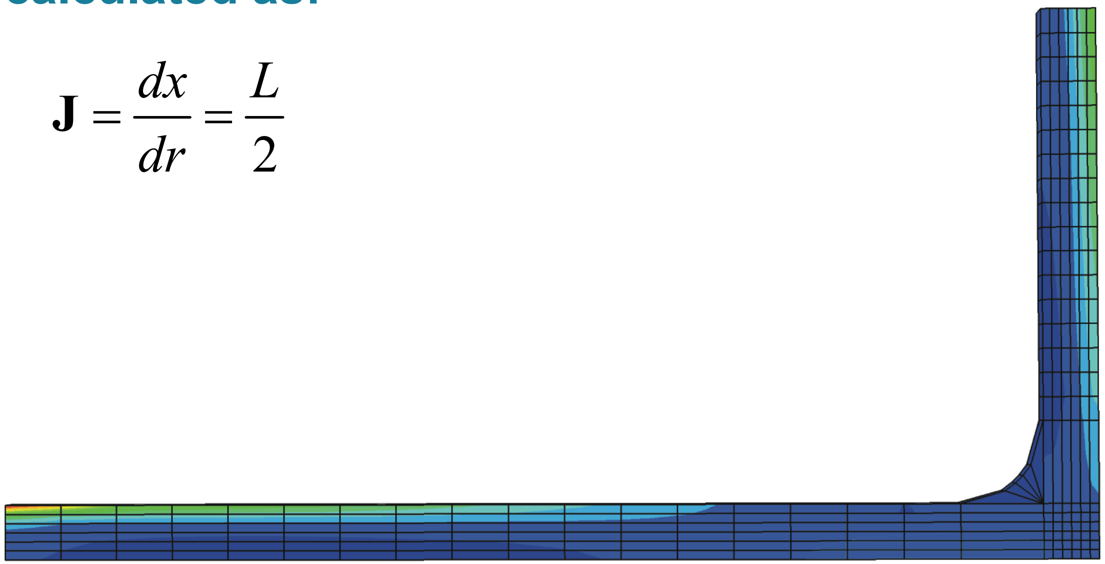
$$\mathbf{B} = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

and the stiffness matrix is calculated as:

$$\mathbf{K} = \frac{AE}{L^2}\int_{-1}^{1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix}\mathbf{J}dr, \qquad \mathbf{J} = \frac{dx}{dr} = \frac{L}{2}$$

$$\Downarrow$$

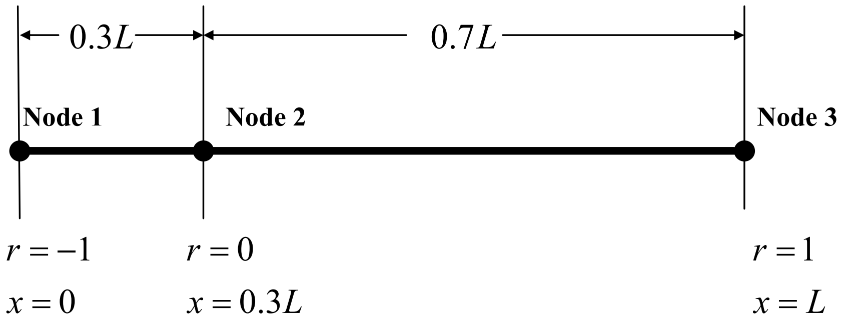
$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

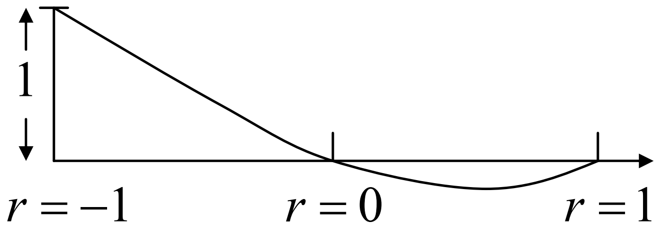


Iso-parametric Elements

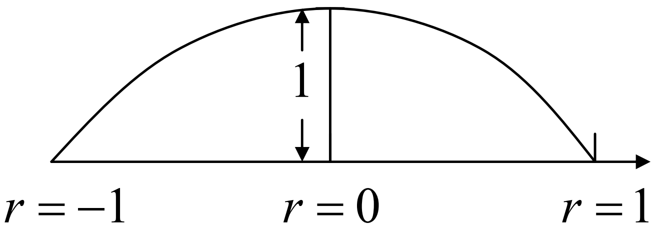
Shape functions – Natural coordinates:

3-node bar element:

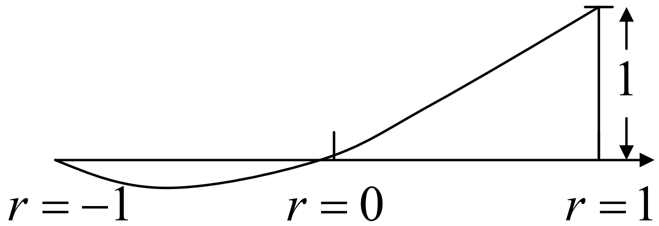




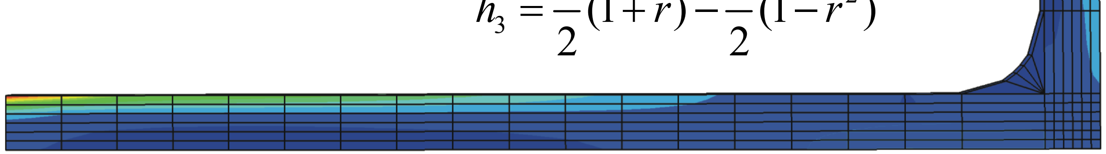
$$h_1 = \frac{1}{2}(1-r) - \frac{1}{2}(1-r^2)$$



$$h_2 = 1 - r^2$$



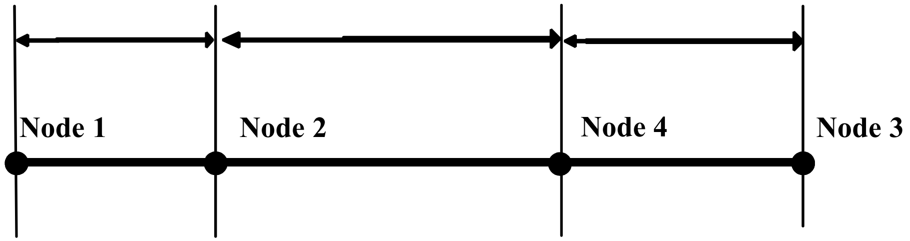
$$h_3 = \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2)$$



Iso-parametric Elements

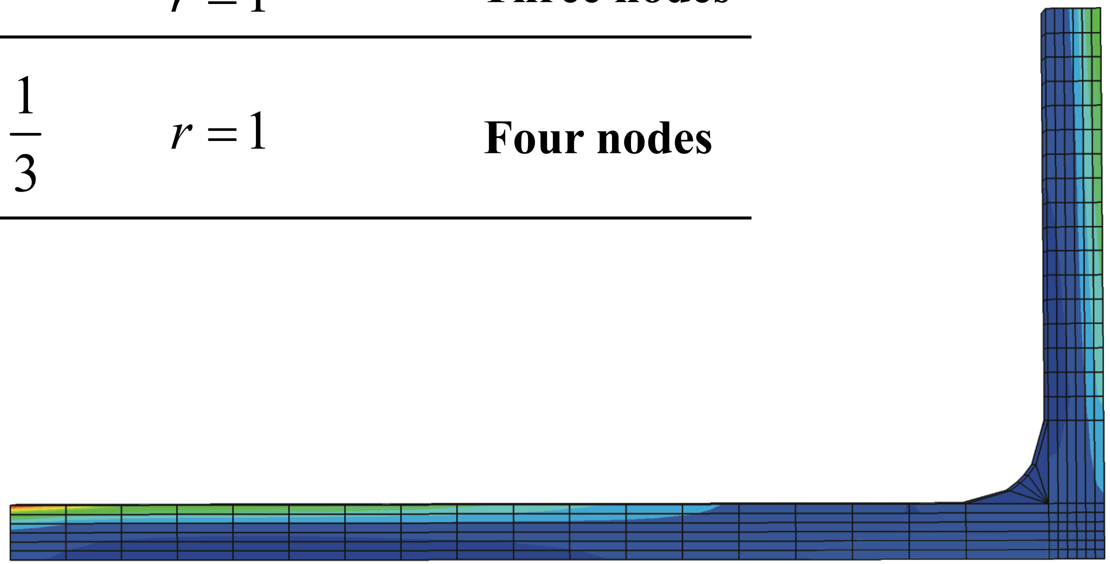
Shape functions – Natural coordinates:

We can generalize this approach (bar element):



$r=-1 \qquad r=0 \qquad\qquad r=1$ **Three nodes**

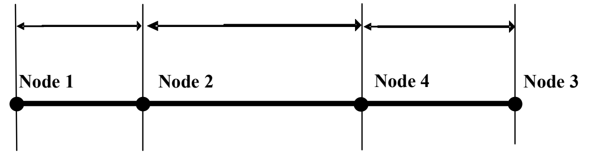
$r=-1 \qquad r=-\frac{1}{3} \qquad\qquad r=\frac{1}{3} \qquad r=1$ **Four nodes**



Iso-parametric Elements

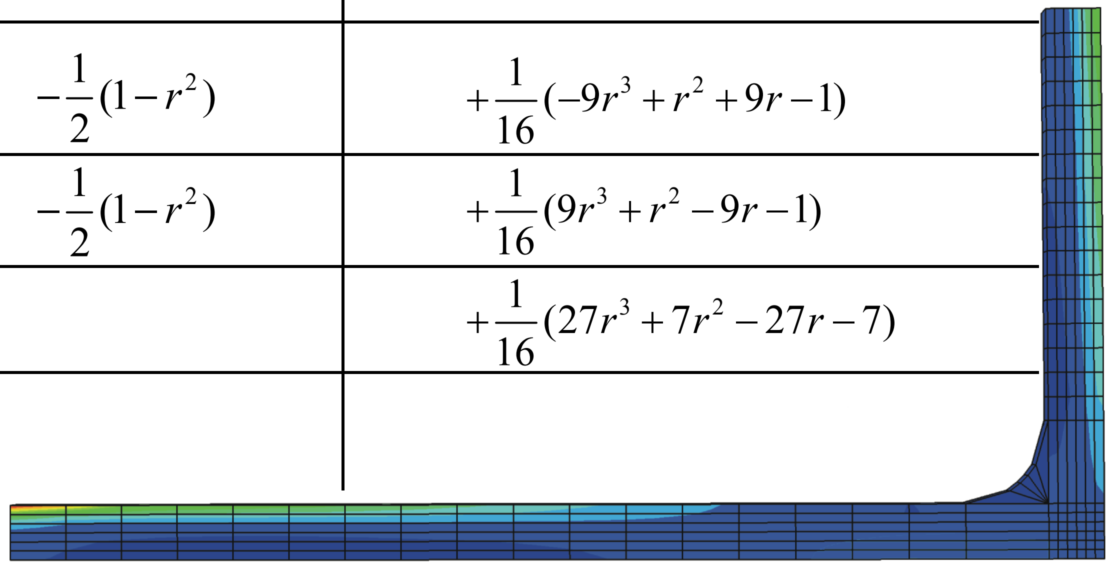
Shape functions – Natural coordinates:

We can generalize this approach (bar element):



$r=-1$	$r=0$		$r=1$	**Three nodes**
$r=-1$	$r=-\frac{1}{3}$	$r=\frac{1}{3}$	$r=1$	**Four nodes**

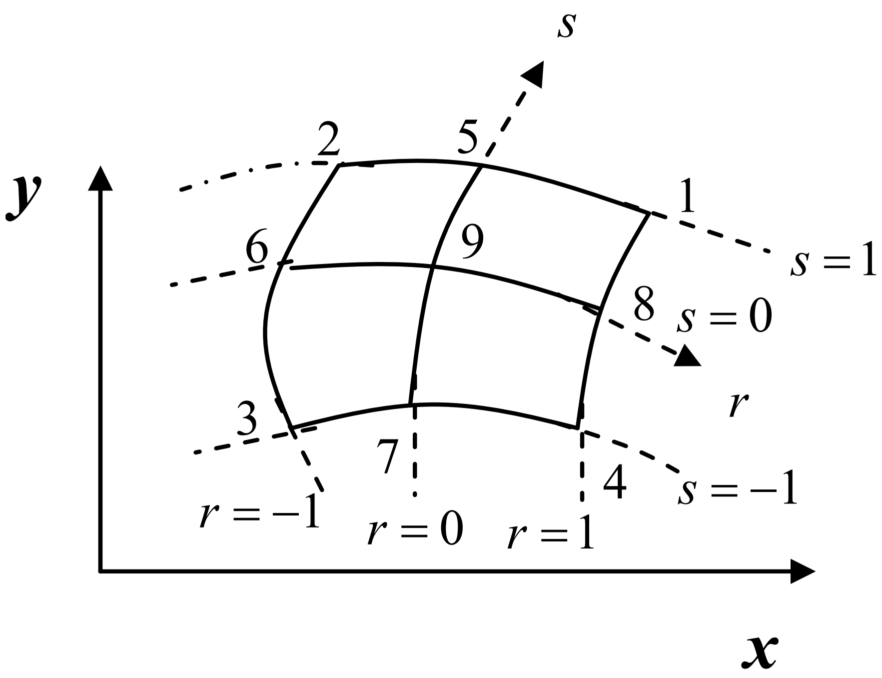
Shape functions	Include only if node 3 is present	Include only if node 3 and node 4 are present
$h_1=\frac{1}{2}(1-r)$	$-\frac{1}{2}(1-r^2)$	$+\frac{1}{16}(-9r^3+r^2+9r-1)$
$h_2=\frac{1}{2}(1+r)$	$-\frac{1}{2}(1-r^2)$	$+\frac{1}{16}(9r^3+r^2-9r-1)$
$h_3=(1-r^2)$		$+\frac{1}{16}(27r^3+7r^2-27r-7)$
$h_4=\frac{1}{16}(-27r^3-9r^2+27r+9)$		

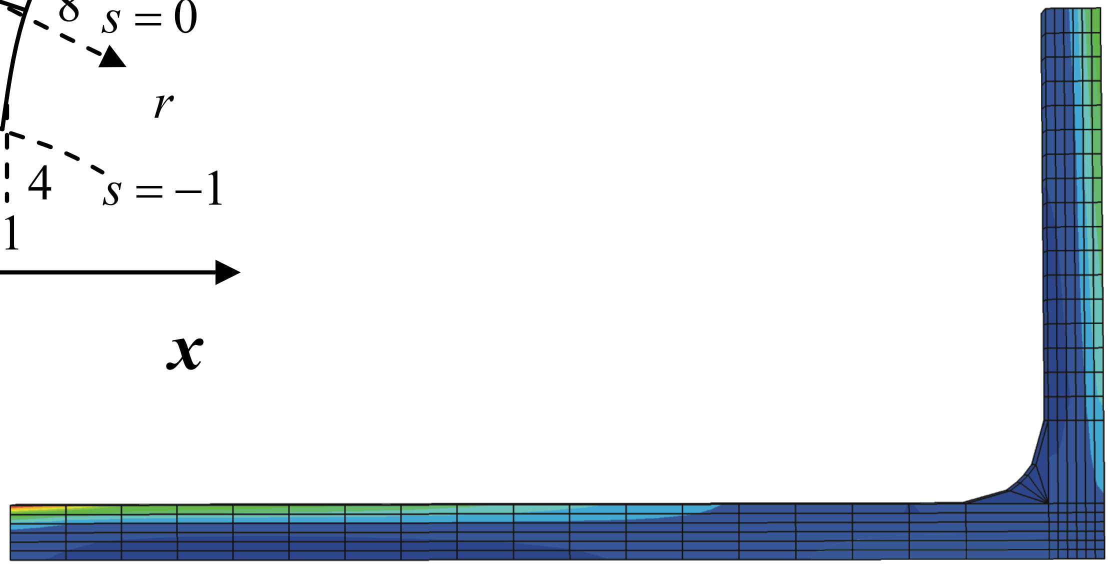


Iso-parametric Elements

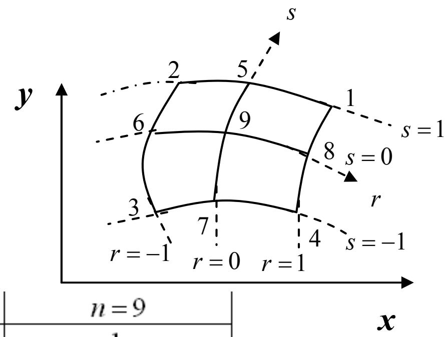
Shape functions – Natural coordinates:

We can generalize to quadrilateral elements (4-9 nodes):





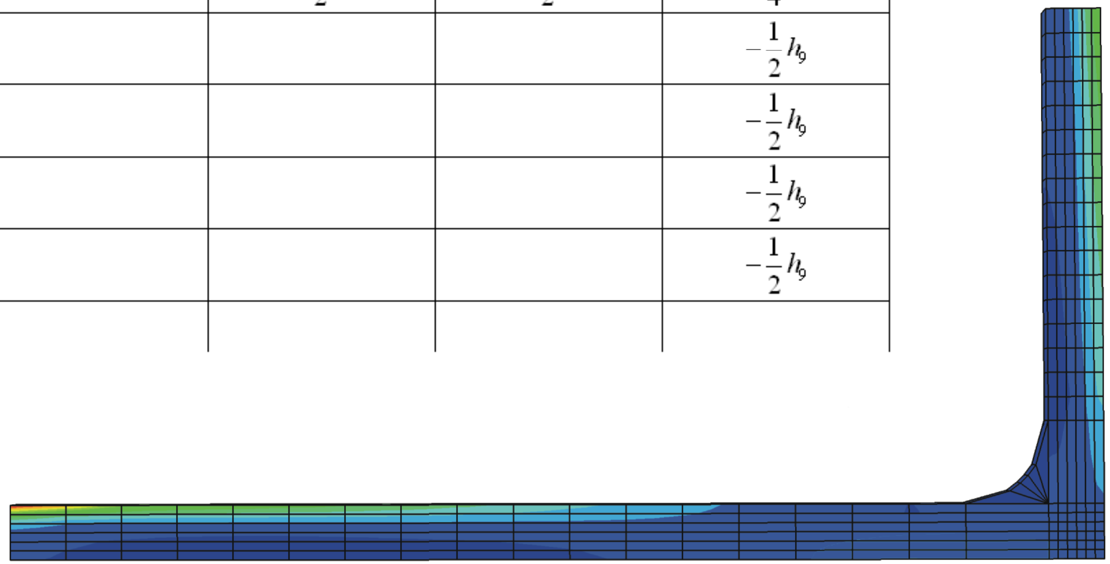
Iso-parametric Elements



Quadrilateral elements:

Include only if:

		$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$					$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1-r)(1-s^2)$					$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$					$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1+r)(1-s^2)$					$-\frac{1}{2}h_9$
$h_9 =$	$(1-r^2)(1-s^2)$					



Iso-parametric Elements

Quadrilateral elements:

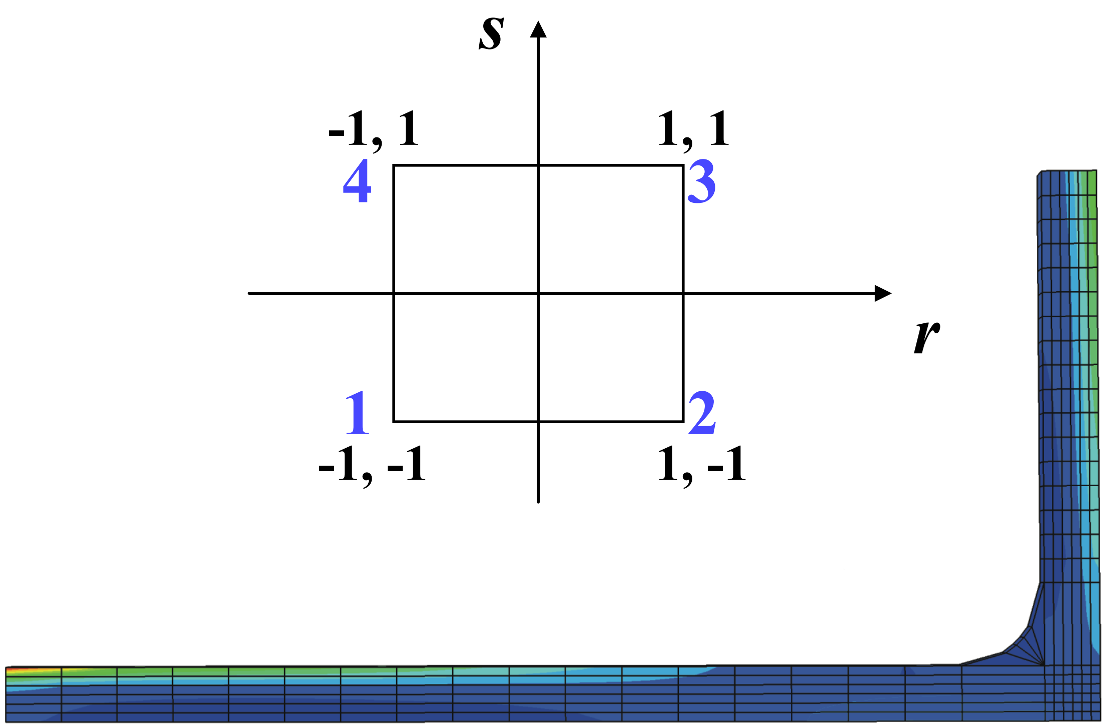
For the bi-linear four node element the shape functions in this coordinate system become:

$$h_1 = \frac{1}{2}(1-r)\frac{1}{2}(1-s)$$

$$h_2 = \frac{1}{2}(1+r)\frac{1}{2}(1-s)$$

$$h_3 = \frac{1}{2}(1+r)\frac{1}{2}(1+s)$$

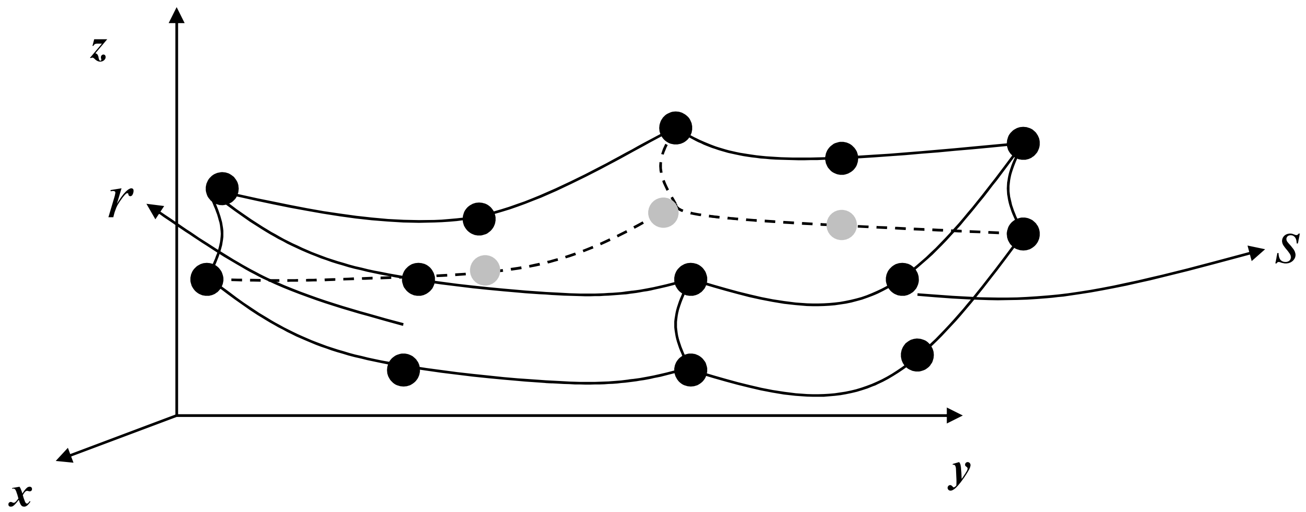
$$h_4 = \frac{1}{2}(1-r)\frac{1}{2}(1+s)$$

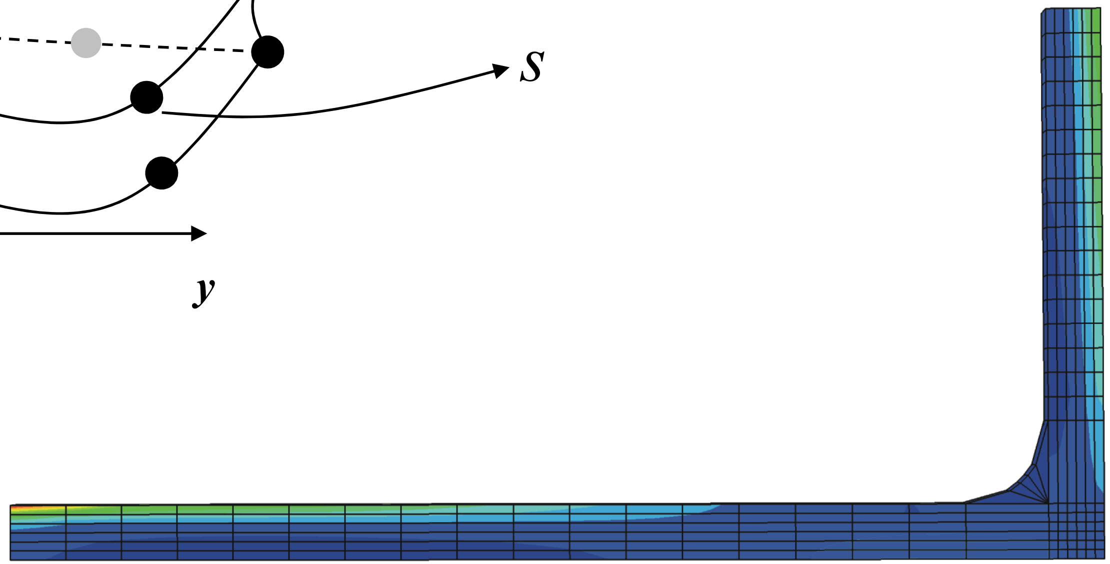


Iso-parametric Elements

Quadrilateral elements :

Following the same principle we may proceed and define iso-parametric shape functions for **three-dimensional quadrilateral** elements (see Bathe pp. 344-345.)





Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

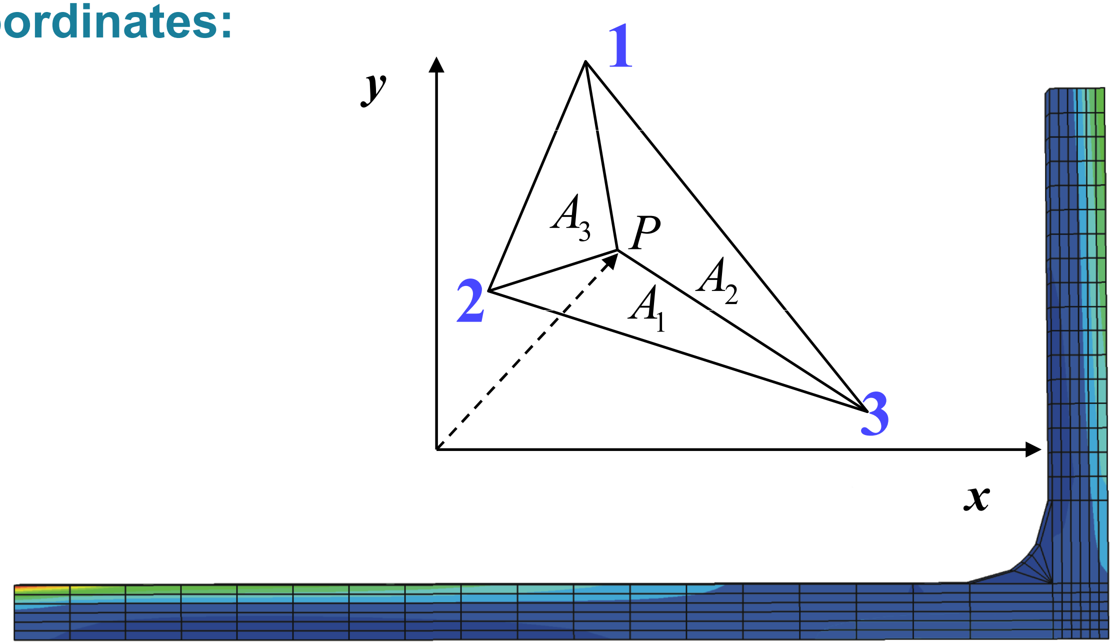
For the **triangular elements** a Cartesian coordinate system is not convenient – so we introduce the **area coordinates**:

Any point in the triangle is uniquely determined by the area coordinates:

$$L_1 = \frac{A_1}{A}$$

$$L_2 = \frac{A_2}{A}$$

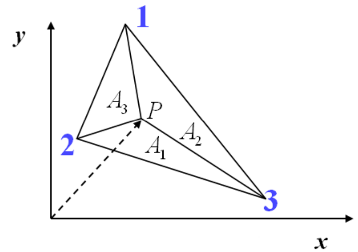
$$L_3 = \frac{A_3}{A} \qquad A = \sum_{i=1}^{3} A_i$$



Iso-parametric Elements

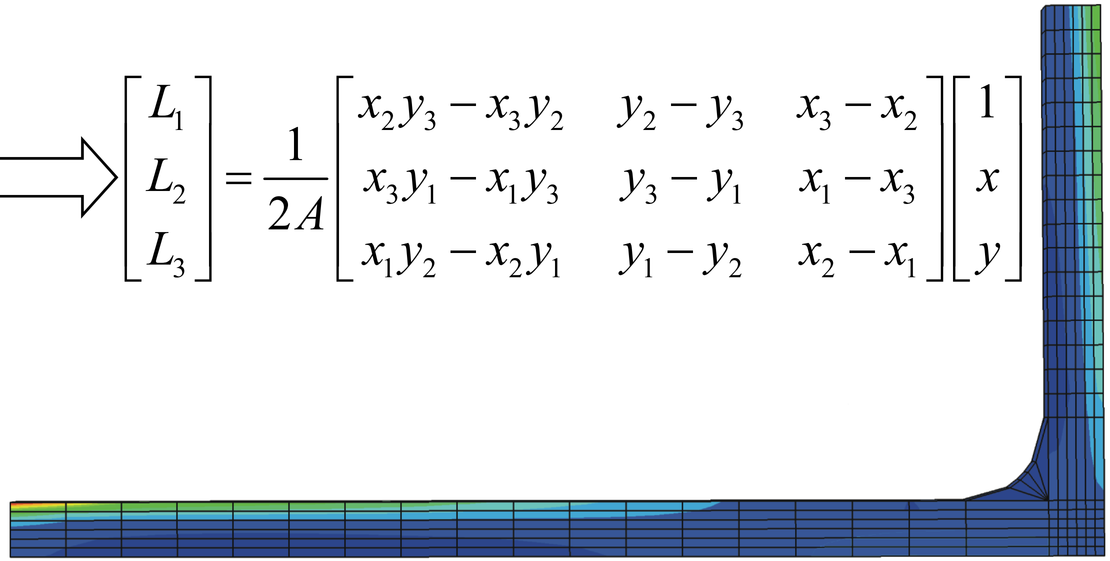
Shape functions – Natural coordinates – Triangular element:

In order to be able to establish the strains (derivatives of the displacements with respect to x and y), we need to relate the area coordinates to x and y :



$$A = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$L_1 = \frac{1}{A}\det\begin{bmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x & y \end{bmatrix}, \text{ etc.} \Rightarrow \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$



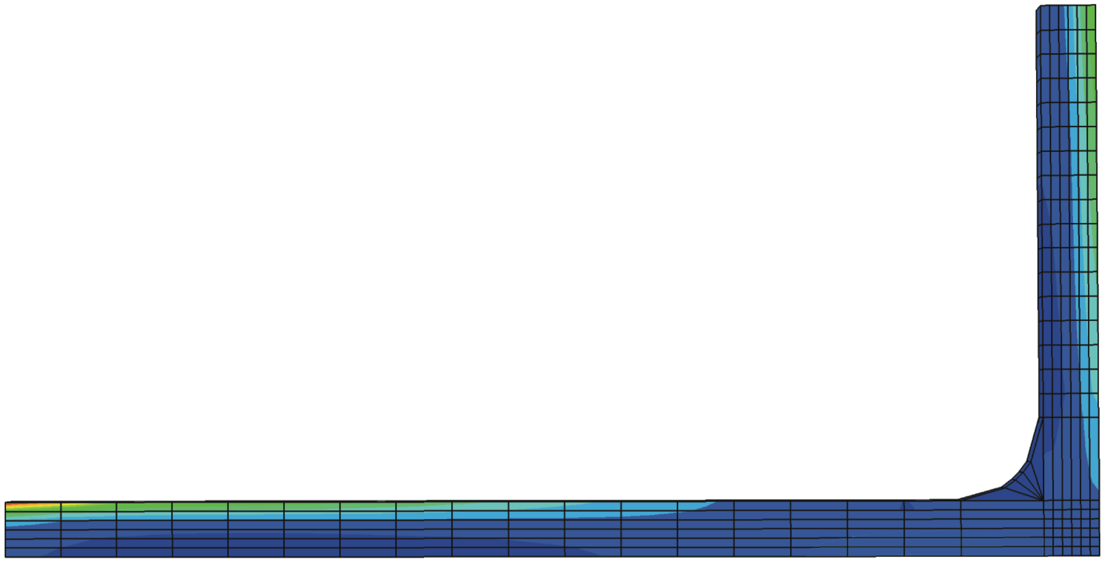
Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

We now realize that the area coordinates are identical to the shape functions for the constant strain triangular element we considered previously (**using generalized coordinates**) and we can write:

$$u = \sum_{i=1}^{3} h_i \hat{u}_i; \qquad x = \sum_{i=1}^{3} h_i \hat{x}_i$$

$$v = \sum_{i=1}^{3} h_i \hat{v}_i; \qquad y = \sum_{i=1}^{3} h_i \hat{y}_i$$



Iso-parametric Elements

Transformation from natural to global coordinates :

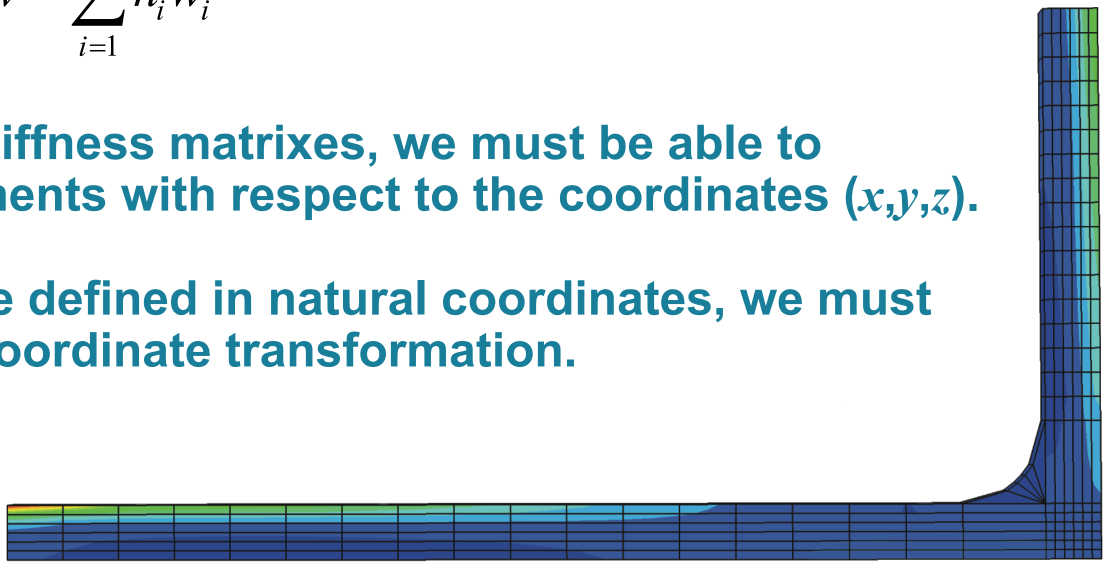
We return to the representation of the coordinates and the displacements

$$x = \sum_{i=1}^{n} h_i \hat{x}_i; \quad y = \sum_{i=1}^{n} h_i \hat{y}_i; \quad z = \sum_{i=1}^{n} h_i \hat{z}_i$$

$$u = \sum_{i=1}^{n} h_i \hat{u}_i; \quad v = \sum_{i=1}^{n} h_i \hat{v}_i; \quad w = \sum_{i=1}^{n} h_i \hat{w}_i$$

In order to establish the stiffness matrixes, we must be able to differentiate the displacements with respect to the coordinates (x,y,z).

As the shape functions are defined in natural coordinates, we must introduce the necessary coordinate transformation.



Iso-parametric Elements

Transformation from natural to global coordinates :

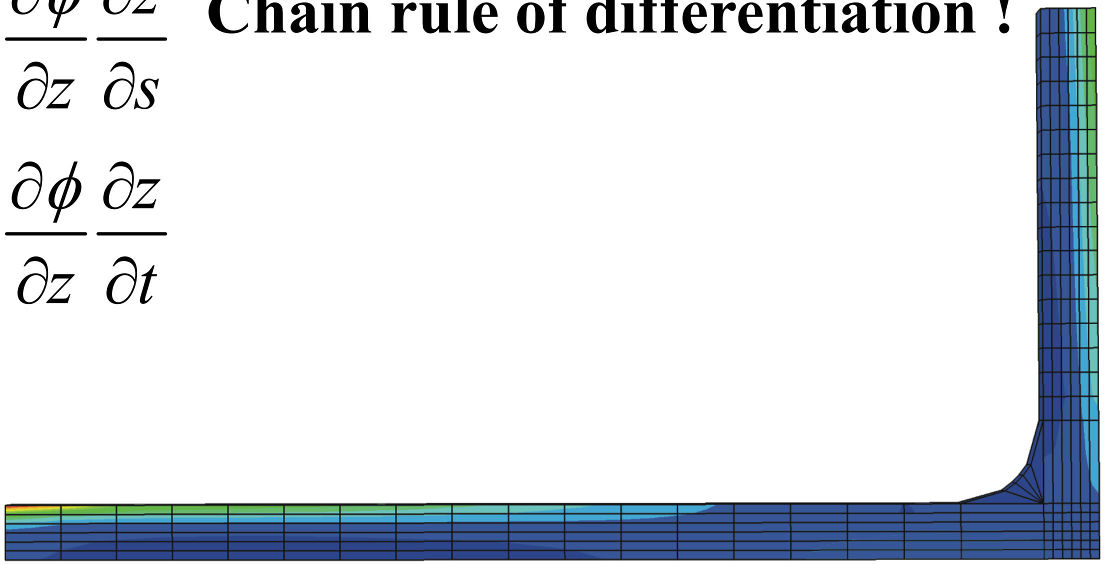
Considering the general three-dimensional case there is:

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial t}$$

Chain rule of differentiation !



Iso-parametric Elements

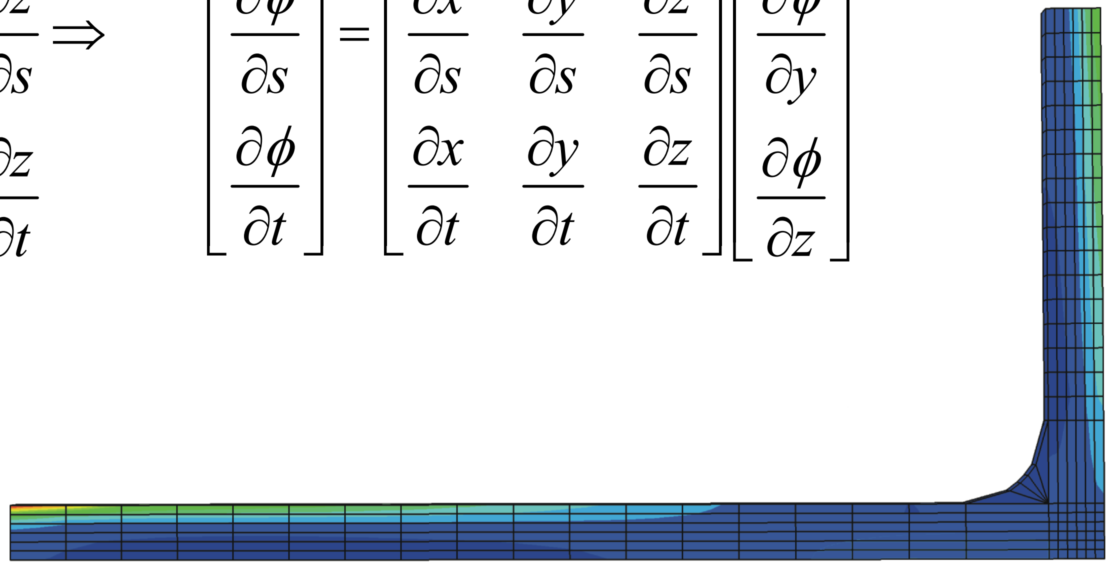
Transformation from natural to global coordinates :

Considering the general three-dimensional case there is:

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial z}{\partial s} \Rightarrow \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial s} \\ \frac{\partial \phi}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial t}$$



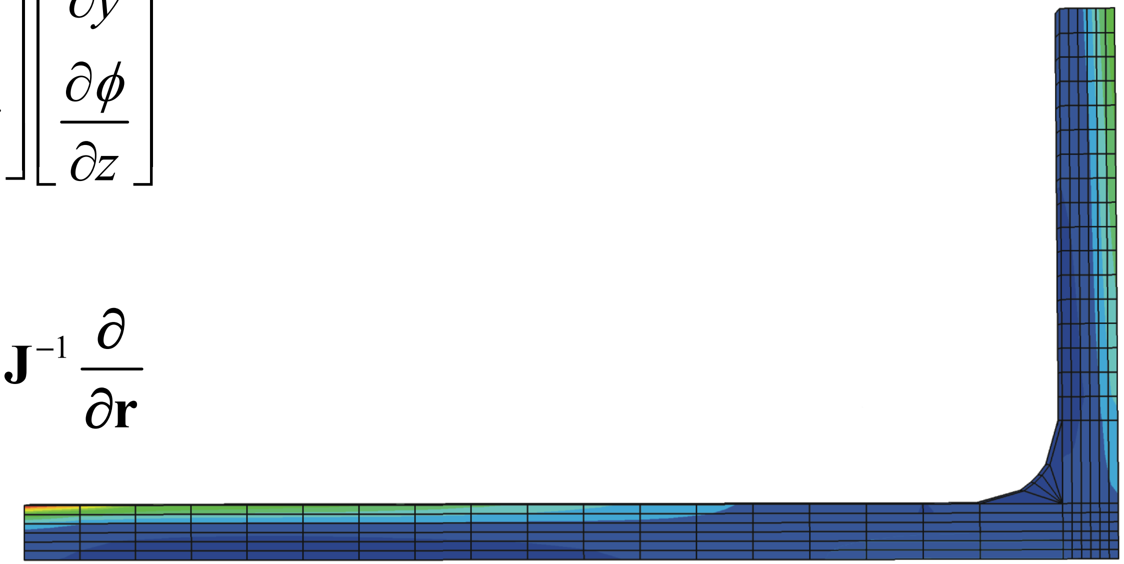
Iso-parametric Elements

Transformation from natural to global coordinates :

Considering the general three-dimensional case there is:

$$\begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial s} \\ \frac{\partial \phi}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix}$$

$$\frac{\partial}{\partial \mathbf{r}} = \mathbf{J} \frac{\partial}{\partial \mathbf{x}} \quad \Rightarrow \quad \frac{\partial}{\partial \mathbf{x}} = \mathbf{J}^{-1} \frac{\partial}{\partial \mathbf{r}}$$



Iso-parametric Elements

Transformation from natural to global coordinates :

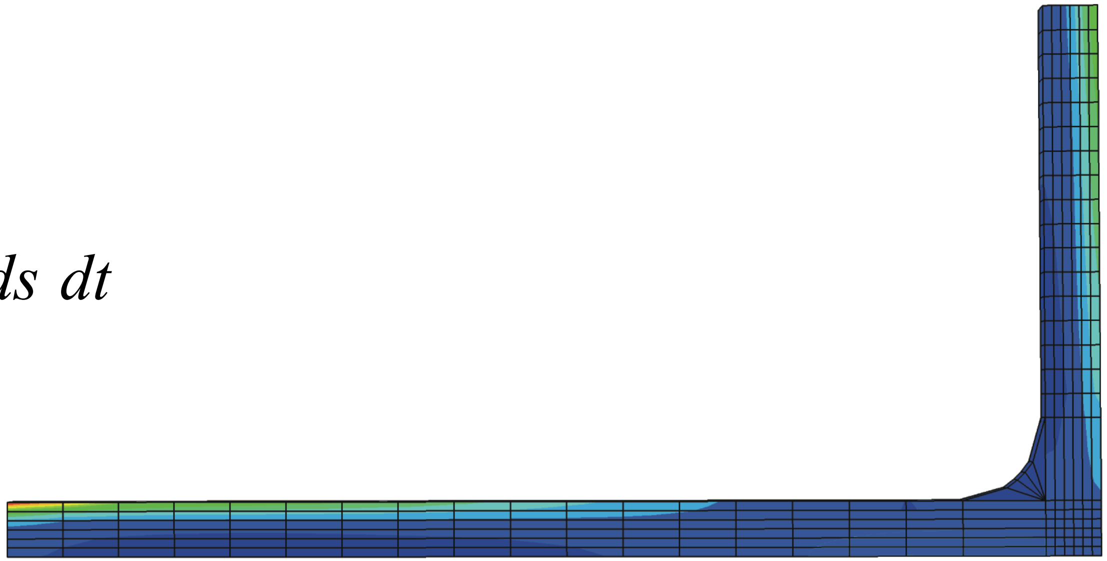
Having the transformation we can now write the strain-stress matrix as:

$$\boldsymbol{\varepsilon} = \mathbf{B}\hat{\mathbf{u}}$$

and we may write up the integrals for calculating the stiffness matrix:

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} dV$$

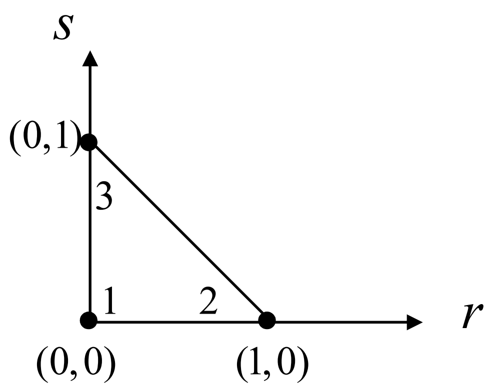
$$= \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} \det \mathbf{J} \; dr \; ds \; dt$$



Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

The natural coordinates for the triangular element may be represented as:

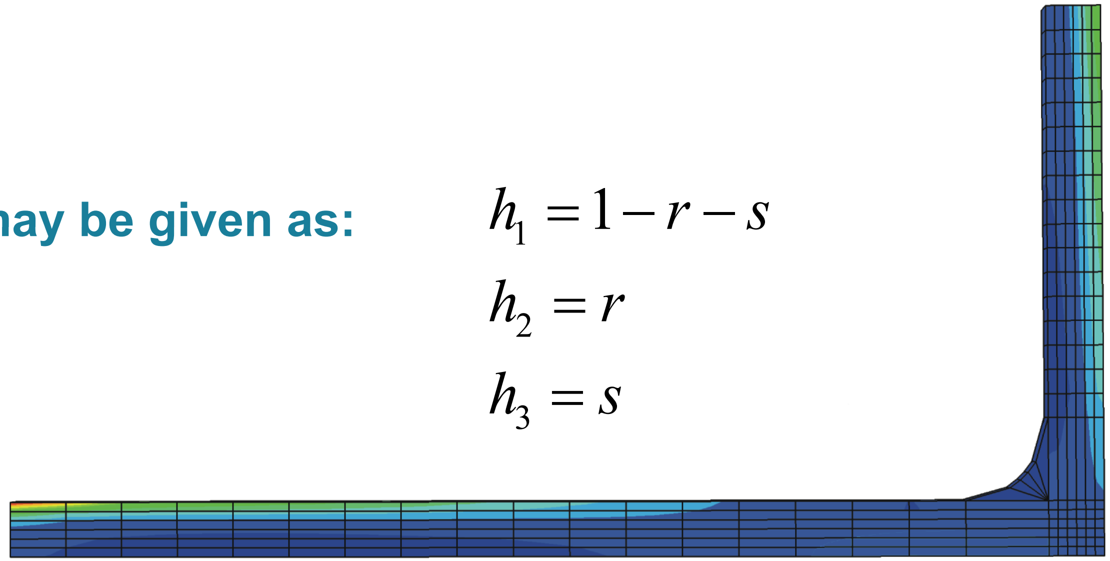


and the shape functions may be given as:

$$h_1 = 1 - r - s$$

$$h_2 = r$$

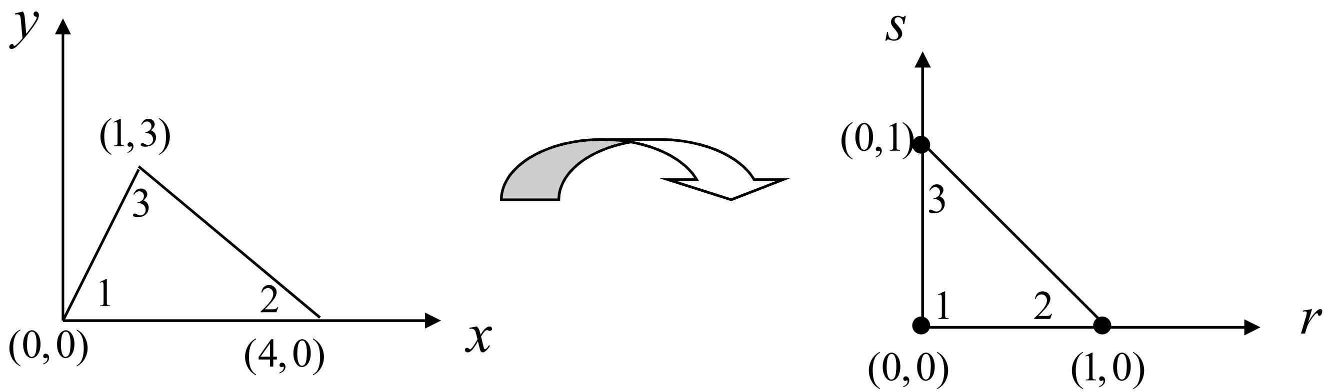
$$h_3 = s$$



Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

Let us try to establish the stiffness matrix for a triangular constant stress element:



$$h_1 = 1 - r - s$$

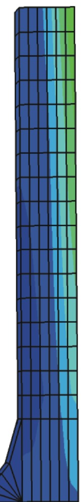
$$h_2 = r$$

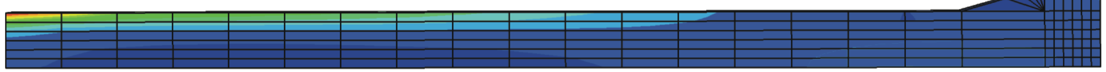
$$h_3 = s$$

$$x = \sum_{i=1}^{3} h_i \hat{x}_i, \quad y = \sum_{i=1}^{3} h_i \hat{y}_i$$

$$\Rightarrow \quad x = 4r + s$$

$$y = 3s$$





Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

In this case the Jacobi matrix becomes:

$$\begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial s} \\ \frac{\partial \phi}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix}$$

$$x = 4r + s$$

$$y = 3s$$

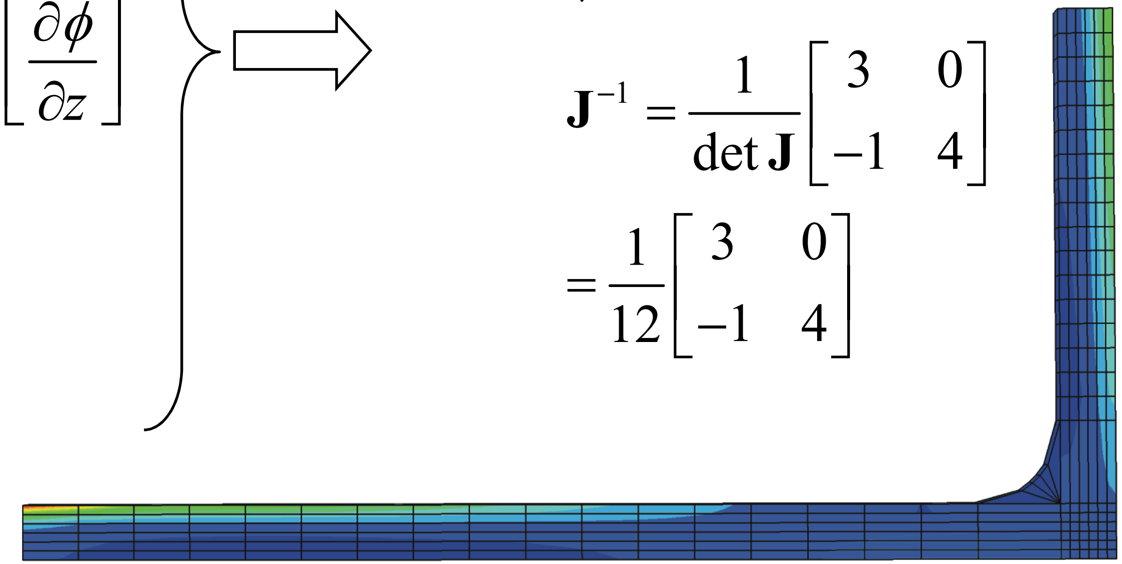
$$\Rightarrow$$

$$\mathbf{J} = \begin{bmatrix} 4 & 0 \\ 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}$$



Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

The we can finally write:

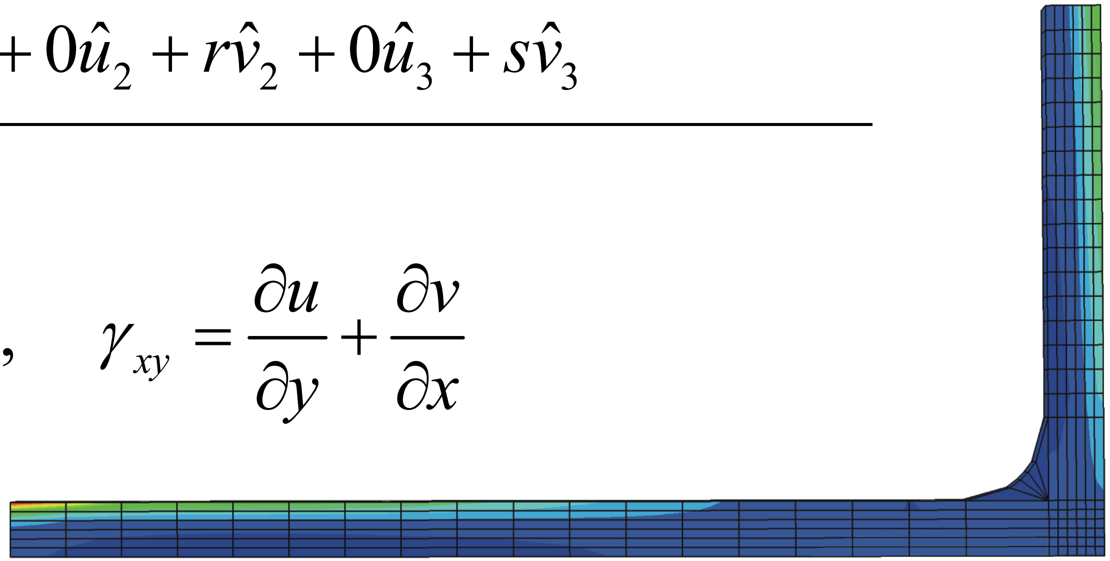
$$\mathbf{H} = \left[\begin{array}{cc:cc:cc} (1-r-s) & 0 & r & 0 & s & 0 \\ 0 & (1-r-s) & 0 & r & 0 & s \end{array}\right]$$

$$u = (1-r-s)\hat{u}_1 + 0\hat{v}_1 + r\hat{u}_2 + 0\hat{v}_2 + s\hat{u}_3 + 0\hat{v}_3$$

$$v = 0\hat{u}_1 + (1-r-s)\hat{v}_1 + 0\hat{u}_2 + r\hat{v}_2 + 0\hat{u}_3 + s\hat{v}_3$$

Plane stress

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



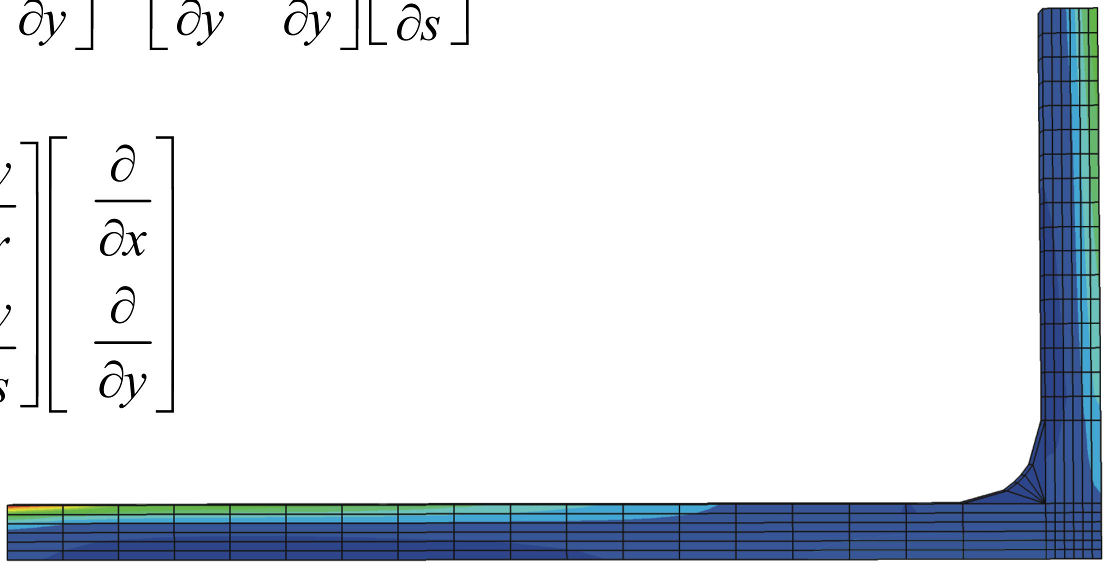
Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

Plane stress $\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial}{\partial s} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial s}{\partial y}\frac{\partial}{\partial s} \end{aligned}\right\} \Rightarrow \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$



Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

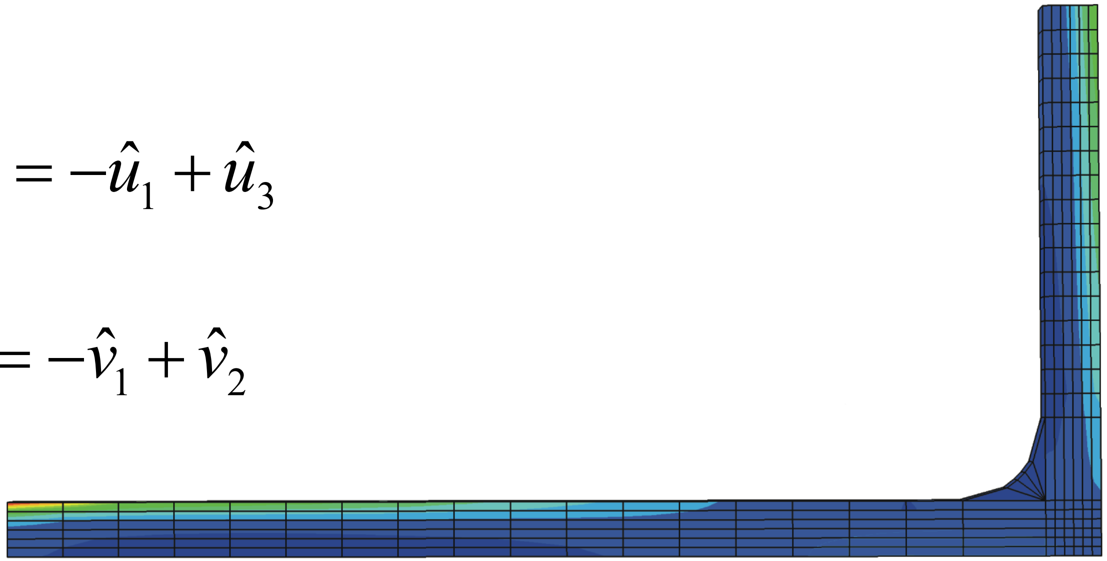
Plane stress $\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

$$u = (1 - r - s)\hat{u}_1 + 0\hat{v}_1 + r\hat{u}_2 + 0\hat{v}_2 + s\hat{u}_3 + 0\hat{v}_3$$

$$v = 0\hat{u}_1 + (1 - r - s)\hat{v}_1 + 0\hat{u}_2 + r\hat{v}_2 + 0\hat{u}_3 + s\hat{v}_3$$

$$\frac{\partial u}{\partial r} = -\hat{u}_1 + \hat{u}_2, \qquad \frac{\partial u}{\partial s} = -\hat{u}_1 + \hat{u}_3$$

$$\frac{\partial v}{\partial s} = -\hat{v}_1 + \hat{v}_3, \qquad \frac{\partial v}{\partial r} = -\hat{v}_1 + \hat{v}_2$$



Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

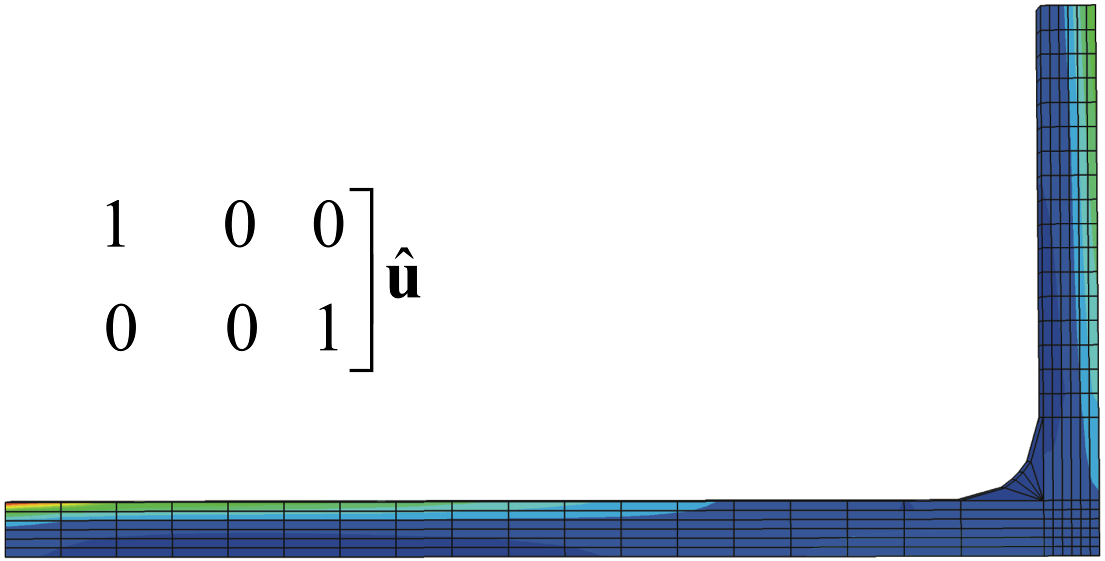
Plane stress

$$\frac{\partial u}{\partial r} = -\hat{u}_1 + \hat{u}_2, \qquad \frac{\partial u}{\partial s} = -\hat{u}_1 + \hat{u}_3$$

$$\frac{\partial v}{\partial s} = -\hat{v}_1 + \hat{v}_3, \qquad \frac{\partial v}{\partial r} = -\hat{v}_1 + \hat{v}_2$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \hat{\mathbf{u}}$$

$$\begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \hat{\mathbf{u}}$$



Iso-parametric Elements

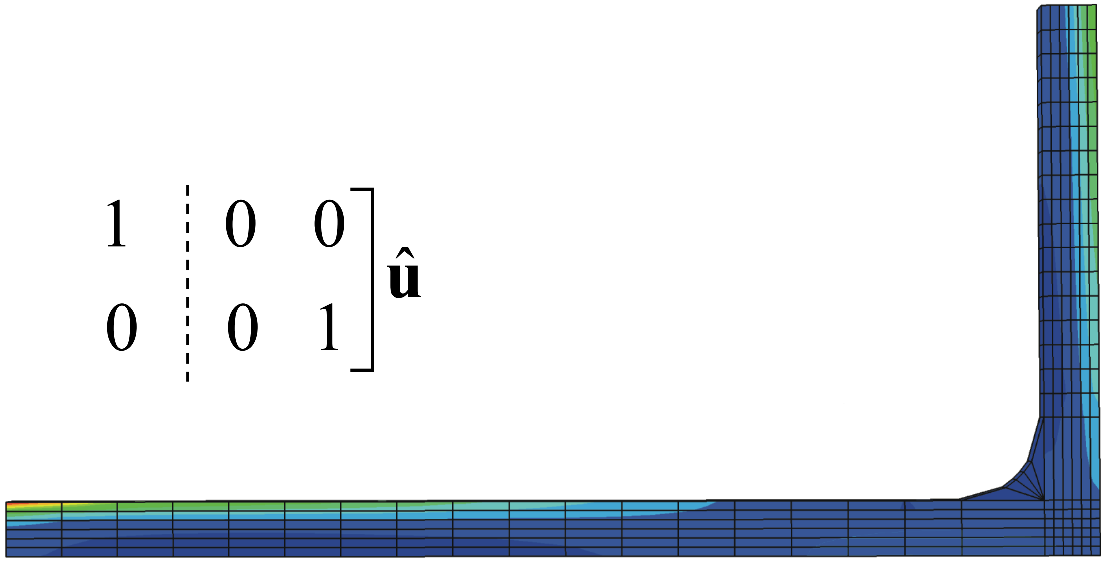
Shape functions – Natural coordinates – Triangular element:

Plane stress

$$\mathbf{J}^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \left[\begin{array}{cc:cc:cc} -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \hat{\mathbf{u}}$$

$$\begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \left[\begin{array}{cc:cc:cc} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \hat{\mathbf{u}}$$



Iso-parametric Elements

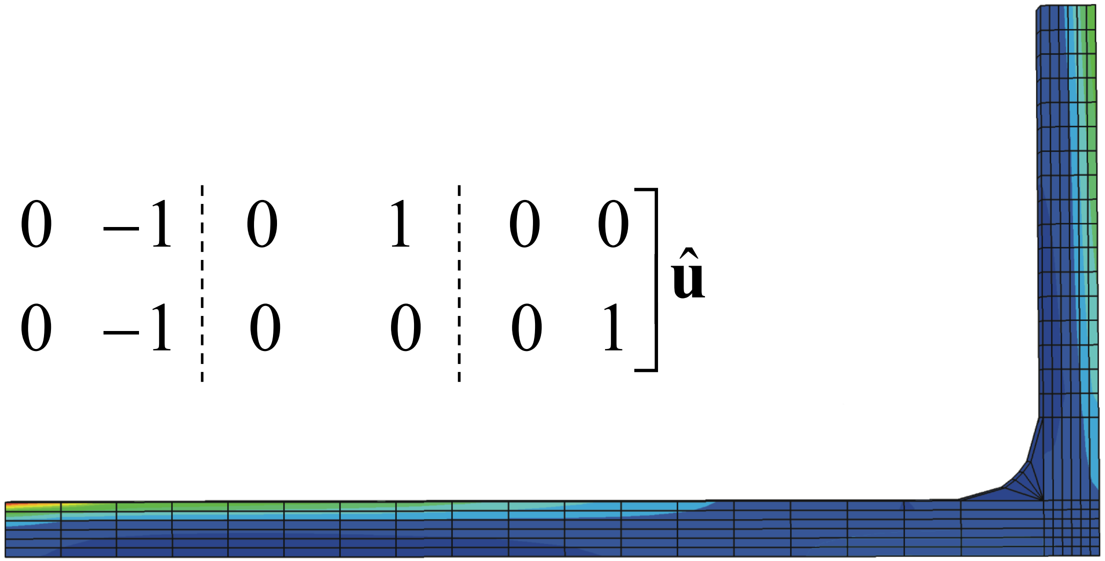
Shape functions – Natural coordinates – Triangular element:

Plane stress

$$\mathbf{J}^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}\left[\begin{array}{cc:cc:cc} -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]\hat{\mathbf{u}}$$

$$\begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}\left[\begin{array}{cc:cc:cc} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]\hat{\mathbf{u}}$$



Iso-parametric Elements

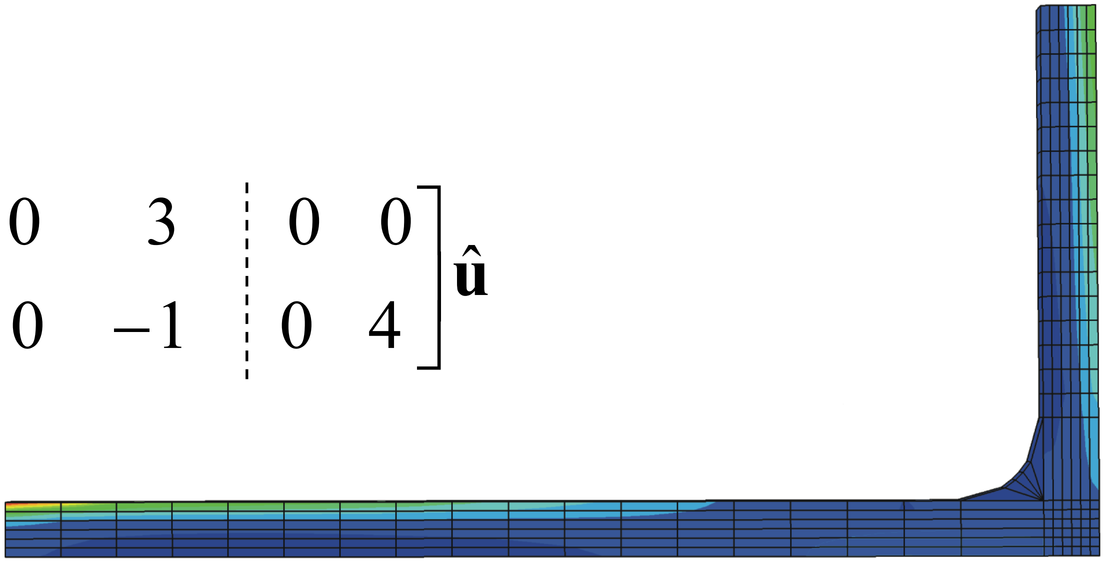
Shape functions – Natural coordinates – Triangular element:

Plane stress

$$\mathbf{J}^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \frac{1}{12}\left[\begin{array}{cc|cc|cc} -3 & 0 & 3 & 0 & 0 & 0 \\ -3 & 0 & -1 & 0 & 4 & 0 \end{array}\right]\hat{\mathbf{u}}$$

$$\begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{12}\left[\begin{array}{cc|cc|cc} 0 & -3 & 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -1 & 0 & 4 \end{array}\right]\hat{\mathbf{u}}$$



Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

Plane stress

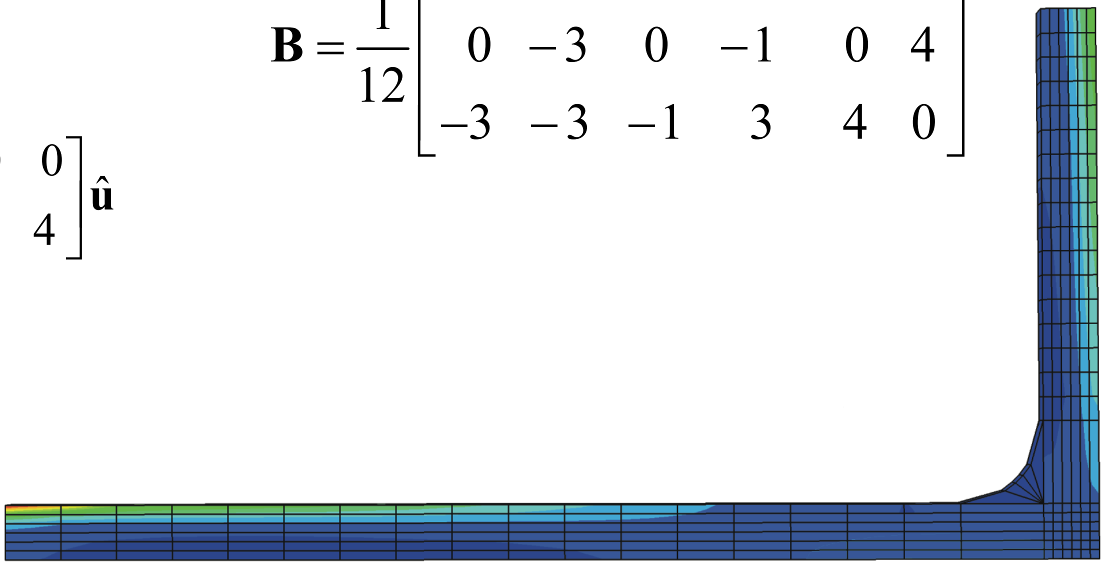
$$\mathbf{J}^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ -3 & 0 & -1 & 0 & 4 & 0 \end{bmatrix}\hat{\mathbf{u}}$$

$$\begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 0 & -3 & 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -1 & 0 & 4 \end{bmatrix}\hat{\mathbf{u}}$$

$$\mathbf{B} = \frac{1}{12}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -3 & 0 & -1 & 0 & 4 \\ -3 & -3 & -1 & 3 & 4 & 0 \end{bmatrix}$$



Iso-parametric Elements

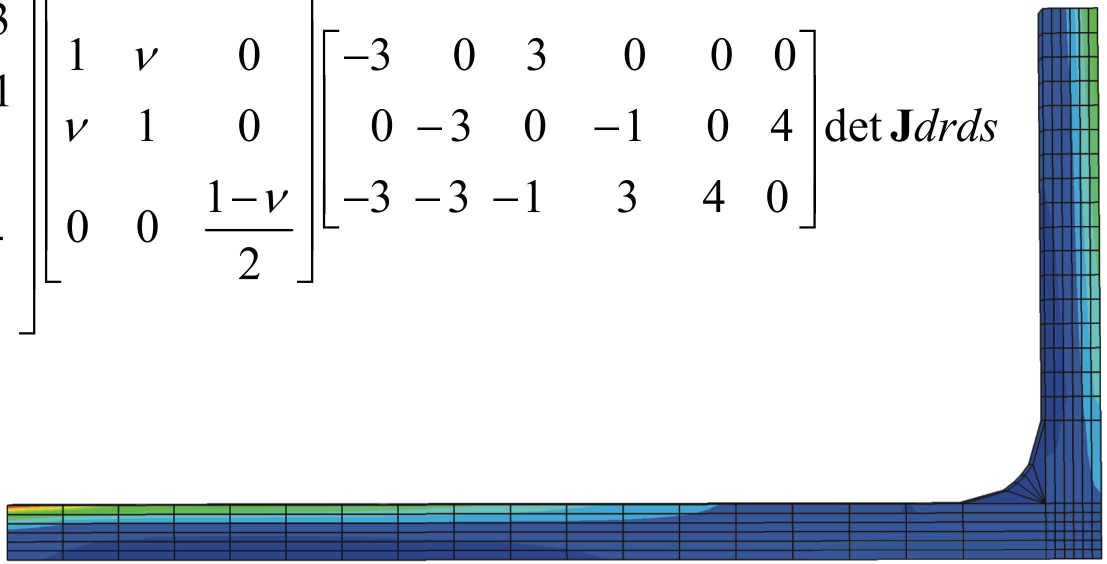
Shape functions – Natural coordinates – Triangular element:

In order to calculate the stiffness matrix we now insert into:

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} dV = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} \det \mathbf{J}\ dr\ ds\ dt$$

$$\mathbf{B} = \frac{1}{12}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -3 & 0 & -1 & 0 & 4 \\ -3 & -3 & -1 & 3 & 4 & 0 \end{bmatrix}$$

$$\mathbf{K} = \frac{E \cdot t}{144(1-\nu^2)} \int_V \begin{bmatrix} -3 & 0 & -3 \\ 0 & -3 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -3 & 0 & -1 & 0 & 4 \\ -3 & -3 & -1 & 3 & 4 & 0 \end{bmatrix} \det \mathbf{J} dr ds$$



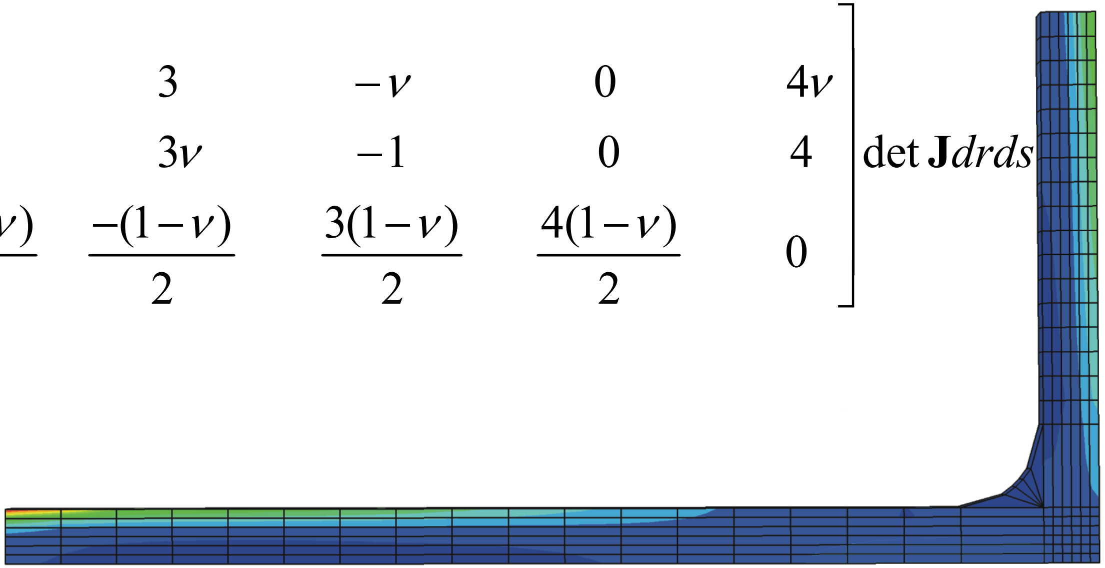
Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

In order to calculate the stiffness matrix we now insert into:

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} dV = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} \det \mathbf{J} \; dr \; ds \; dt$$

$$\mathbf{K} = \frac{E \cdot t}{144(1-\nu^2)} \int_V \mathbf{B}^T \begin{bmatrix} -3 & -3\nu & 3 & -\nu & 0 & 4\nu \\ -3\nu & -3 & 3\nu & -1 & 0 & 4 \\ \frac{-3(1-\nu)}{2} & \frac{-3(1-\nu)}{2} & \frac{-(1-\nu)}{2} & \frac{3(1-\nu)}{2} & \frac{4(1-\nu)}{2} & 0 \end{bmatrix} \det \mathbf{J} dr ds$$

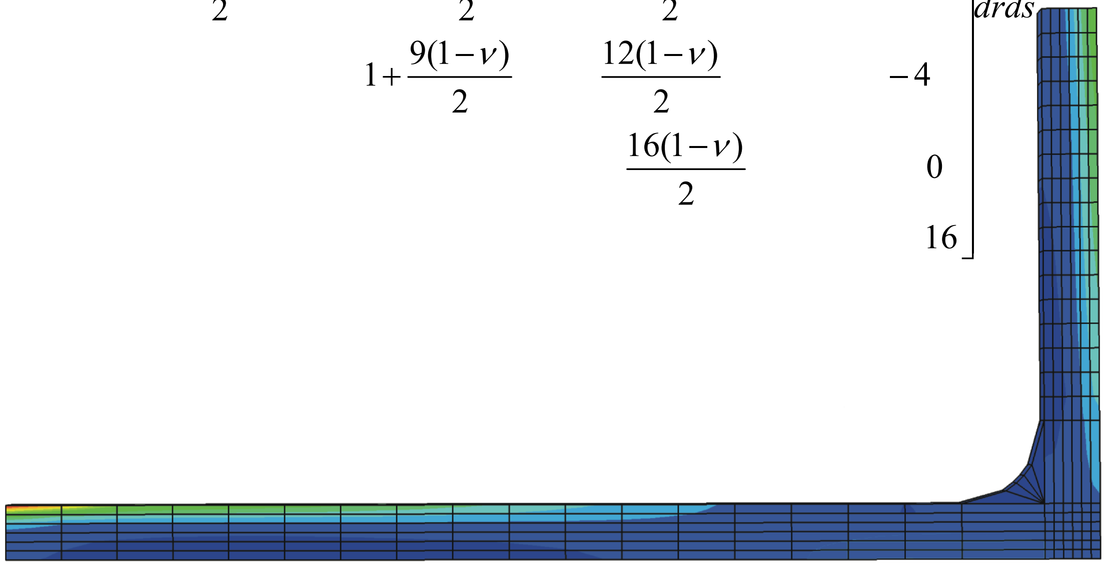


Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

In order to calculate the stiffness matrix we now insert into:

$$\mathbf{K} = \frac{E \cdot t}{12(1-\nu^2)} \int_V \begin{bmatrix} 9+\frac{9(1-\nu)}{2} & 9\nu+\frac{9(1-\nu)}{2} & -9+\frac{3(1-\nu)}{2} & 3\nu-\frac{9(1-\nu)}{2} & \frac{-12(1-\nu)}{2} & -12\nu \\ & 9+\frac{9(1-\nu)}{2} & -9\nu+\frac{3(1-\nu)}{2} & 3-\frac{9(1-\nu)}{2} & \frac{-12(1-\nu)}{2} & -12 \\ & & 9+\frac{(1-\nu)}{2} & -3\nu-\frac{3(1-\nu)}{2} & \frac{-4(1-\nu)}{2} & 12\nu \\ & \text{Symmetrical} & & 1+\frac{9(1-\nu)}{2} & \frac{12(1-\nu)}{2} & -4 \\ & & & & \frac{16(1-\nu)}{2} & 0 \\ & & & & & 16 \end{bmatrix} drds$$

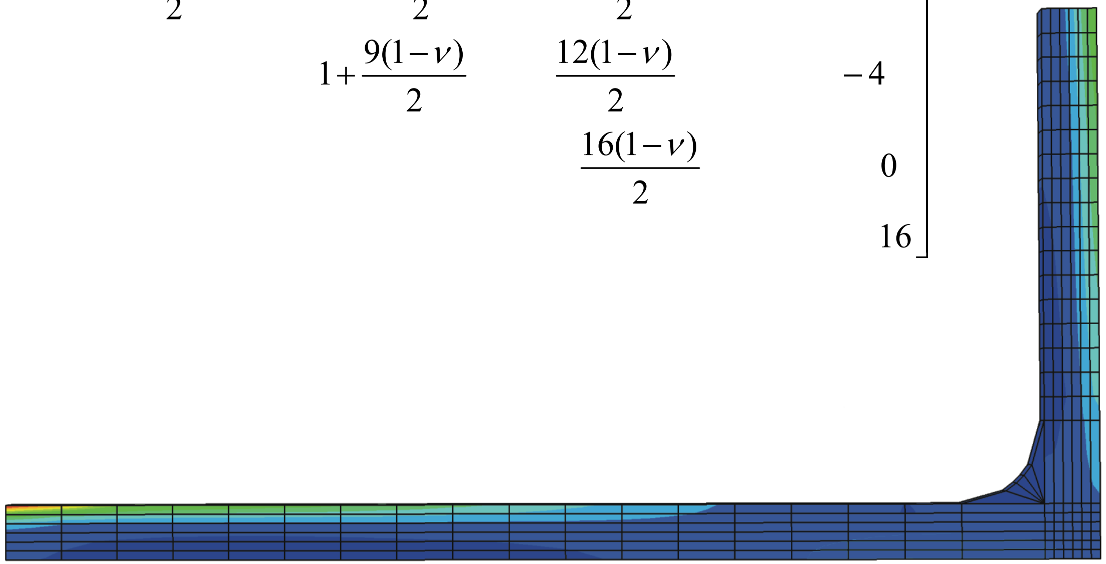


Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

In order to calculate the stiffness matrix we now insert into:

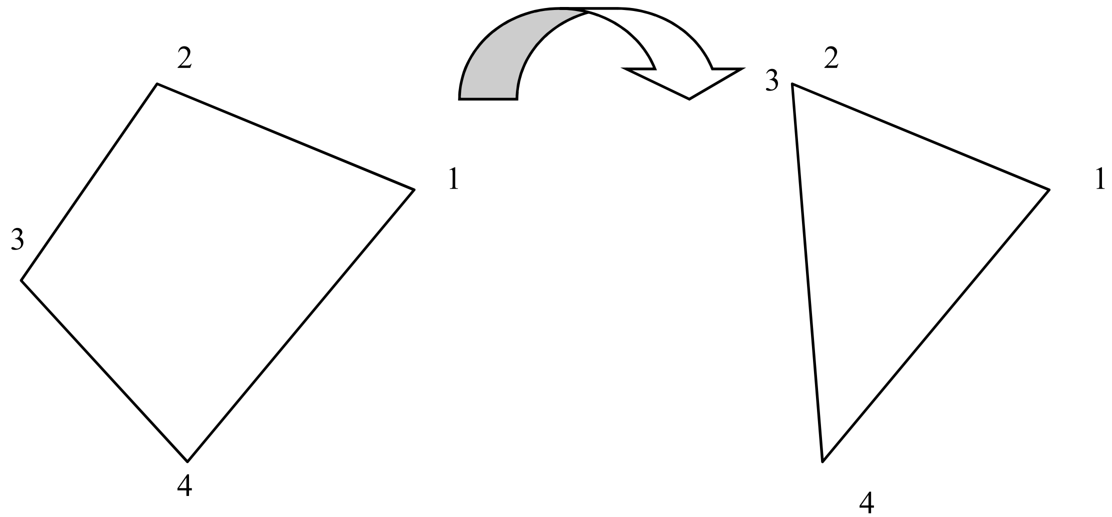
$$\mathbf{K} = \frac{E \cdot t}{24(1-\nu^2)} \begin{bmatrix} 9+\frac{9(1-\nu)}{2} & 9\nu+\frac{9(1-\nu)}{2} & -9+\frac{3(1-\nu)}{2} & 3\nu-\frac{9(1-\nu)}{2} & \frac{-12(1-\nu)}{2} & -12\nu \\ & 9+\frac{9(1-\nu)}{2} & -9\nu+\frac{3(1-\nu)}{2} & 3-\frac{9(1-\nu)}{2} & \frac{-12(1-\nu)}{2} & -12 \\ & & 9+\frac{(1-\nu)}{2} & -3\nu-\frac{3(1-\nu)}{2} & \frac{-4(1-\nu)}{2} & 12\nu \\ & \text{Symmetrical} & & 1+\frac{9(1-\nu)}{2} & \frac{12(1-\nu)}{2} & -4 \\ & & & & \frac{16(1-\nu)}{2} & 0 \\ & & & & & 16 \end{bmatrix}$$

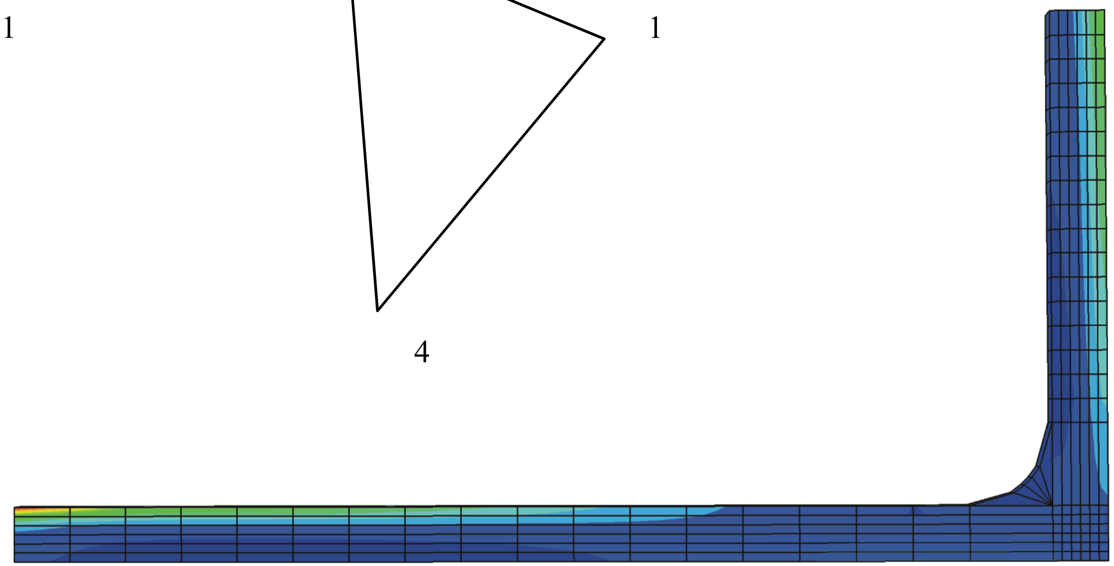


Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

Finally we can also construct the triangular element directly from the quadrilateral element – by so-called collapsing:





Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

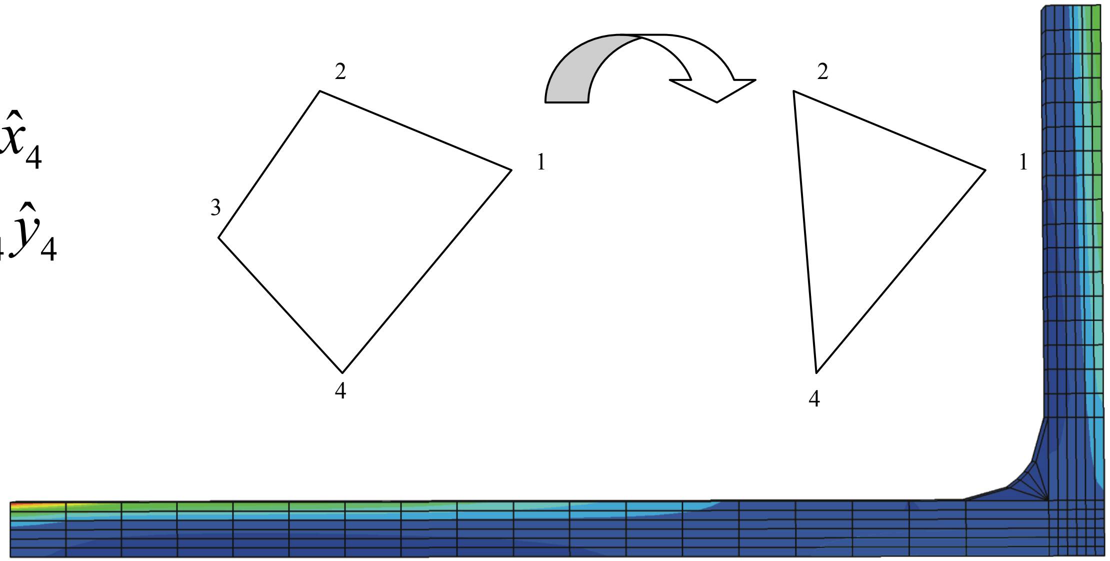
Finally we can also construct the triangular element directly from the quadrilateral element – by so-called **collapsing**:

$$x = h_1\hat{x}_1 + h_2\hat{x}_2 + h_3\hat{x}_3 + h_4\hat{x}_4 \qquad \hat{x}_3 = \hat{x}_2$$

$$y = h_1\hat{y}_1 + h_2\hat{y}_2 + h_3\hat{y}_3 + h_4\hat{y}_4 \qquad \hat{y}_3 = \hat{y}_2$$

$$x = h_1\hat{x}_1 + (h_2 + h_3)\hat{x}_2 + h_4\hat{x}_4$$

$$y = h_1\hat{y}_1 + (h_2 + h_3)\hat{y}_2 + h_4\hat{y}_4$$



Iso-parametric Elements

Shape functions – Natural coordinates – Triangular element:

Finally we can also construct the triangular element directly from the quadrilateral element – by so-called collapsing:

$$x = h_1 \hat{x}_1 + (h_2 + h_3)\hat{x}_2 + h_4 \hat{x}_4$$

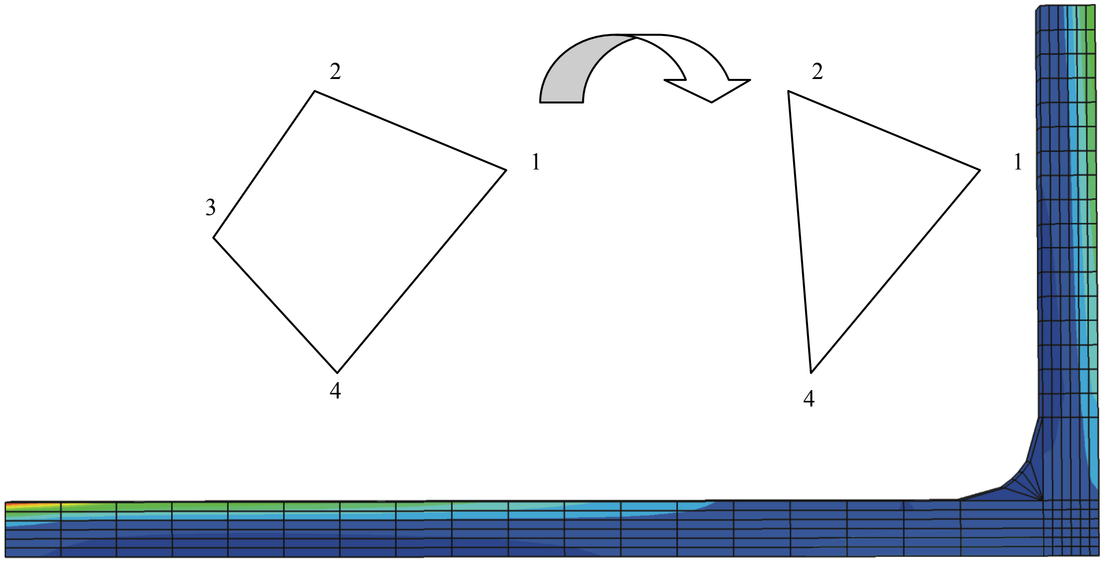
$$y = h_1 \hat{y}_1 + (h_2 + h_3)\hat{y}_2 + h_4 \hat{y}_4$$

$$h_1 = \frac{1}{2}(1-r)\frac{1}{2}(1-s)$$

$$h_2 = \frac{1}{2}(1+r)\frac{1}{2}(1-s)$$

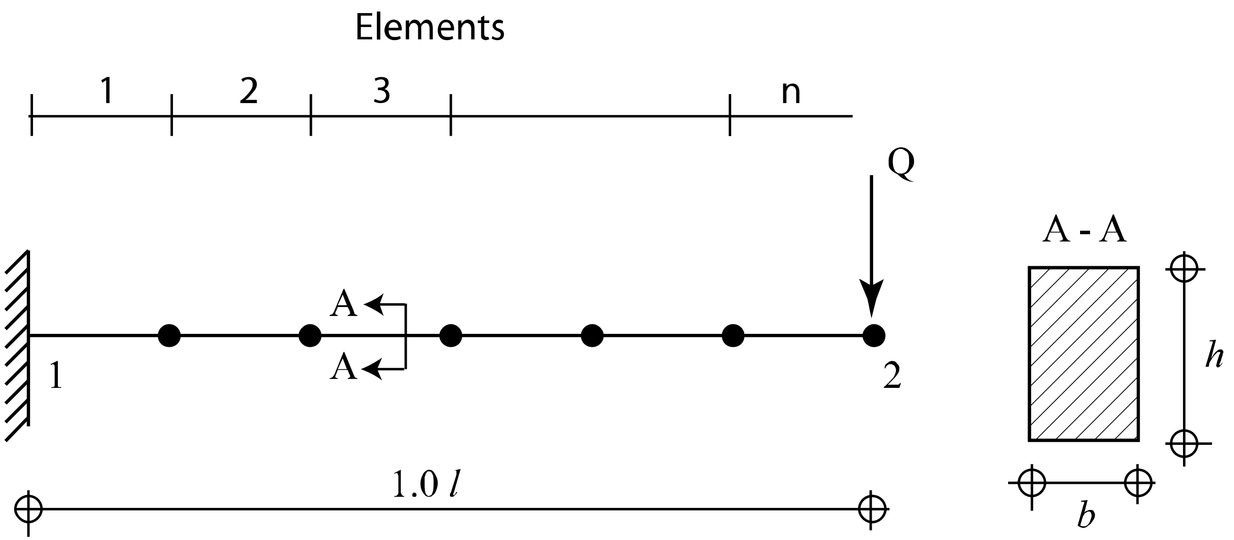
$$h_3 = \frac{1}{2}(1+r)\frac{1}{2}(1+s)$$

$$h_4 = \frac{1}{2}(1-r)\frac{1}{2}(1+s)$$

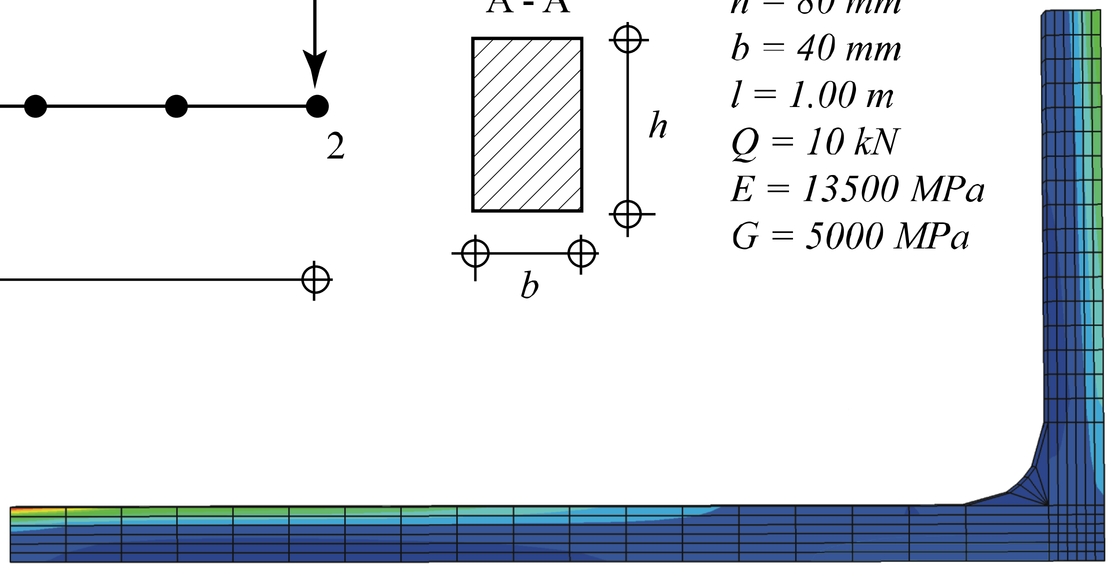


Assignment 3

Create a finite element code to calculate the deformation of a shear flexible cantilever beam (Timoshenko beam) using Matlab.



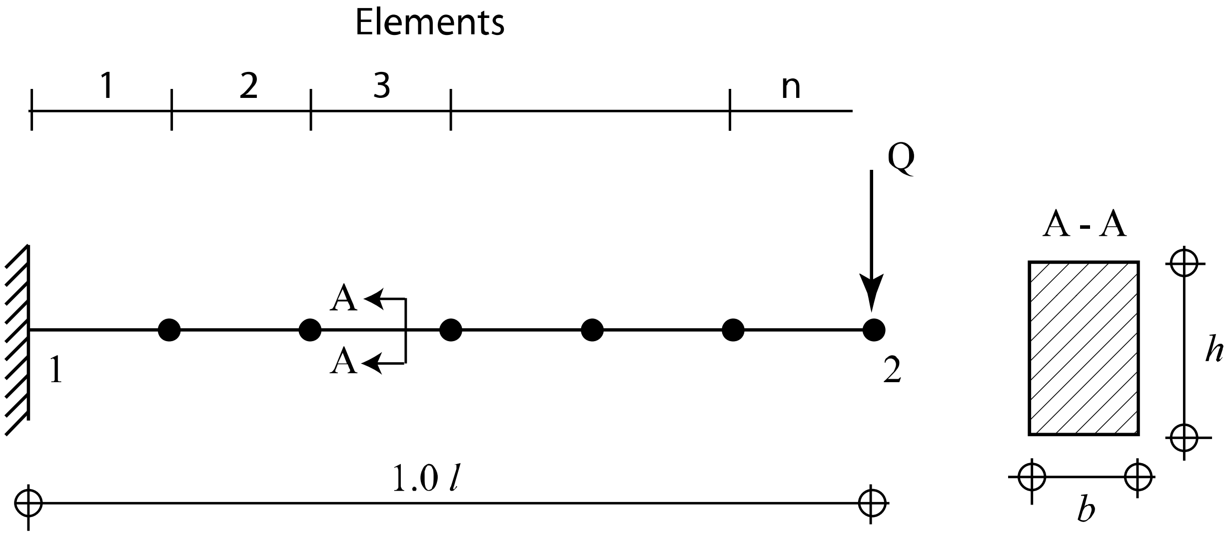
$h = 80\ mm$
$b = 40\ mm$
$l = 1.00\ m$
$Q = 10\ kN$
$E = 13500\ MPa$
$G = 5000\ MPa$



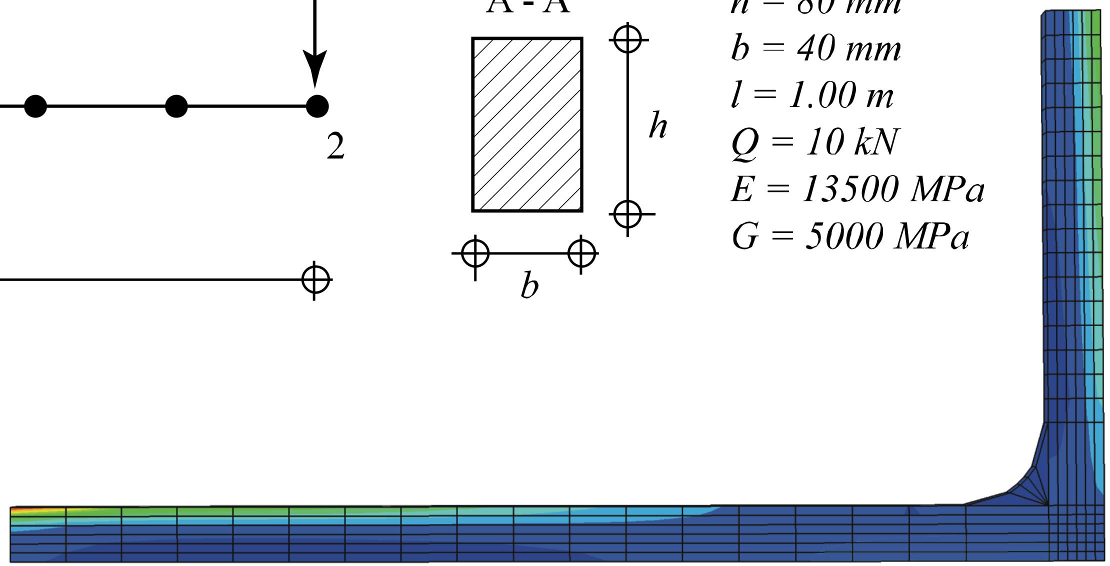
Assignment 3

Calculate the displacement at the endpoint and plot the calculated displacement over different numbers of elements.

Compare the results with the exact solution given in Equation and check the convergence of the results by increasing the number of elements.



$h = 80\ mm$
$b = 40\ mm$
$l = 1.00\ m$
$Q = 10\ kN$
$E = 13500\ MPa$
$G = 5000\ MPa$



Assignment 3

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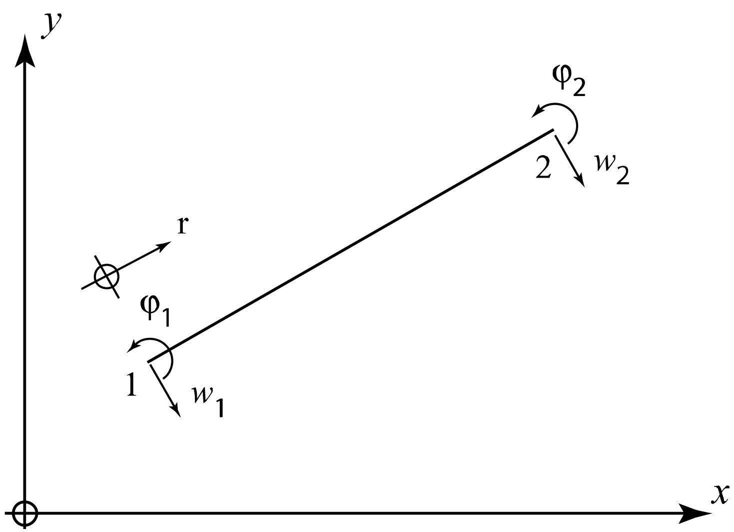
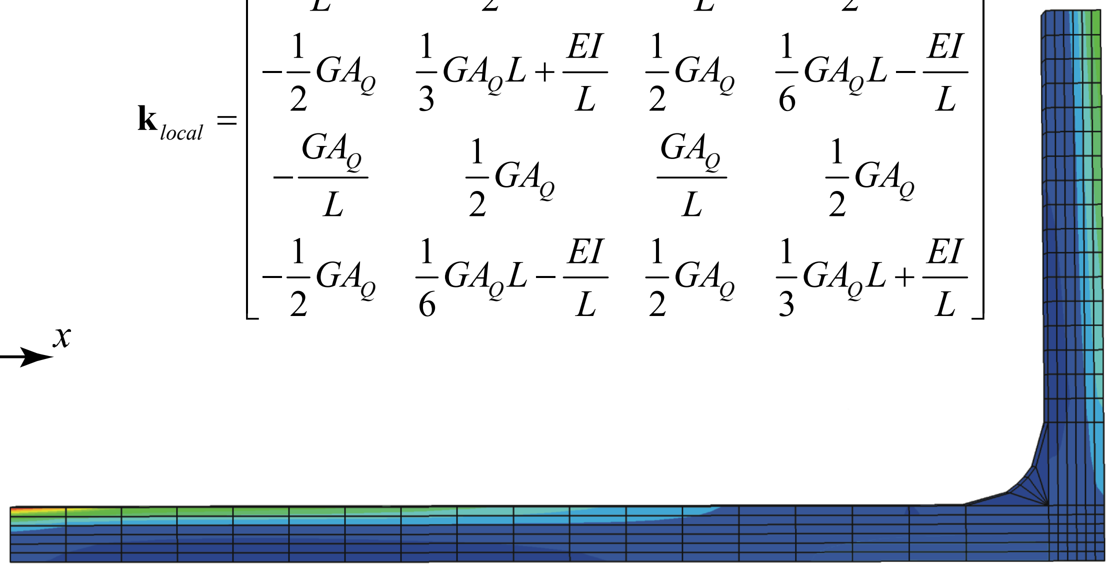
Compare the results with the exact solution given in Equation and check the convergence of the results by increasing the number of elements.



$$\mathbf{k}_{local} = \begin{bmatrix} \frac{GA_Q}{L} & -\frac{1}{2}GA_Q & -\frac{GA_Q}{L} & -\frac{1}{2}GA_Q \\ -\frac{1}{2}GA_Q & \frac{1}{3}GA_QL+\frac{EI}{L} & \frac{1}{2}GA_Q & \frac{1}{6}GA_QL-\frac{EI}{L} \\ -\frac{GA_Q}{L} & \frac{1}{2}GA_Q & \frac{GA_Q}{L} & \frac{1}{2}GA_Q \\ -\frac{1}{2}GA_Q & \frac{1}{6}GA_QL-\frac{EI}{L} & \frac{1}{2}GA_Q & \frac{1}{3}GA_QL+\frac{EI}{L} \end{bmatrix}$$



Assignment 3

Calculate the displacement at the endpoint and plot the calculated displacement over different numbers of elements.

Compare the results with the exact solution given in Equation and check the convergence of the results by increasing the number of elements.