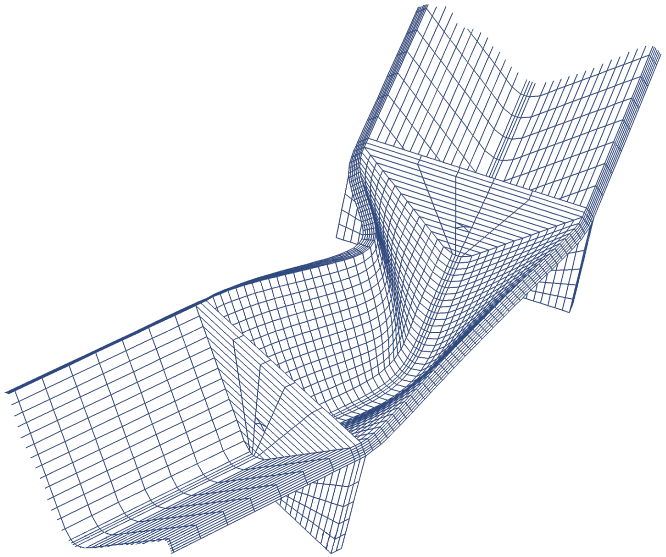
The Finite Element Method for the Analysis of Linear Systems



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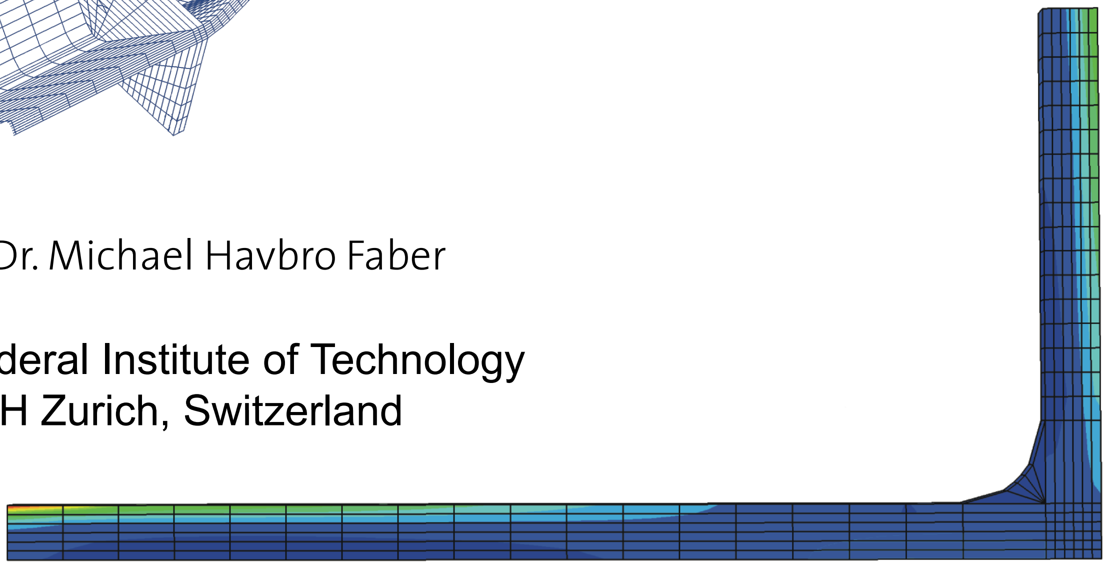
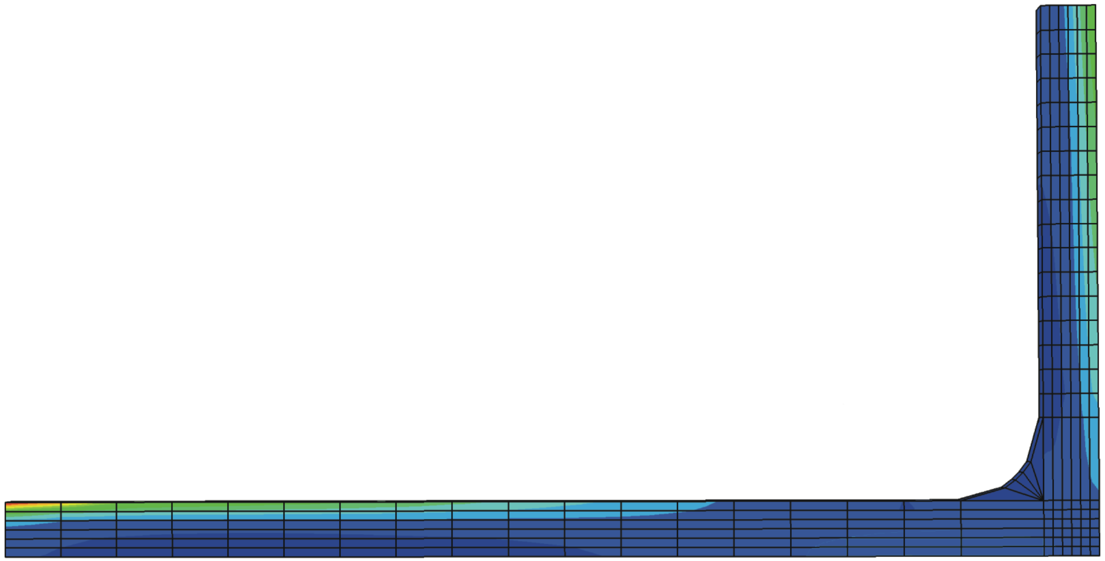


Contents of Today's Lecture

- **The Method of Finite Elements (principle of virtual displacements)**
- **Properties of FEM solutions**
- **On the choice of shape functions**
  - Lagrange
  - Hermitian
  - Serendipity
- **Natural coordinates**



The Method of Finite Elements

The principle of virtual displacements:

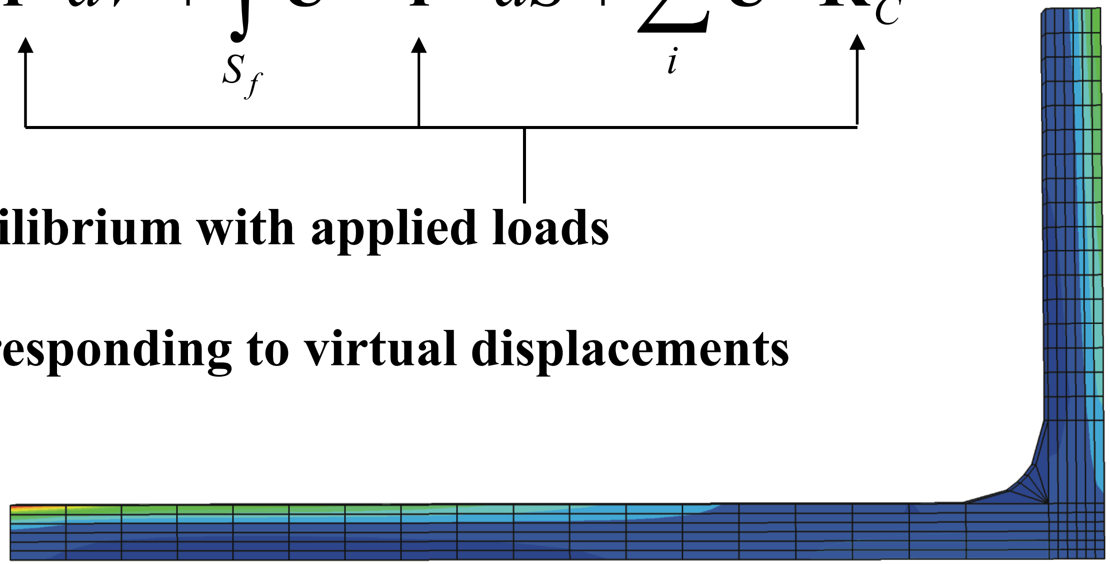
We may now express the principle of virtual displacements in a more general form:

Internal virtual work — **External virtual work**

$$\int_V \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\tau}\, dV = \int_V \bar{\mathbf{U}}^T \mathbf{f}^B\, dV + \int_{S_f} \bar{\mathbf{U}}^{S_f T} \mathbf{f}^{S_f}\, dS + \sum_i \bar{\mathbf{U}}^{iT} \mathbf{R}_C^i$$

Stresses in equilibrium with applied loads

Virtual strains corresponding to virtual displacements



The Method of Finite Elements

Finite Element Equations:

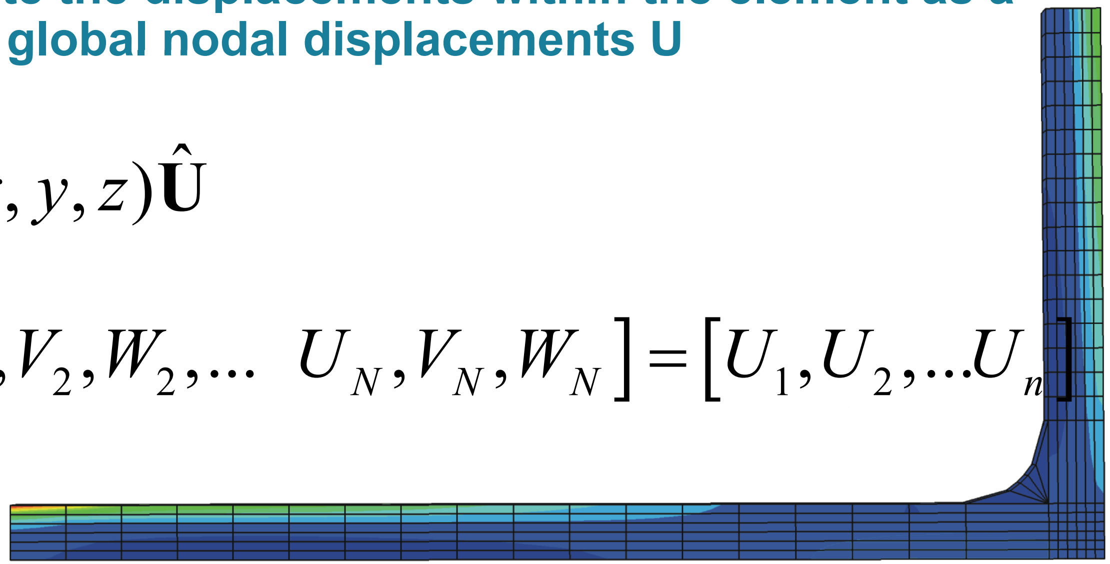
We now consider the volume modeled as an assemblage of *N* elements connected in the nodal points on the element boundaries

The displacements within the individual elements are measured in a convenient local coordinate system *x,y,z*

For element *m* we now write the displacements within the element as a function of the total set of global nodal displacements U

$$\mathbf{u}^{(m)}(x,y,z)=\mathbf{H}^{(m)}(x,y,z)\hat{\mathbf{U}}$$

$$\hat{\mathbf{U}}^{T}=\left[U_1,V_1,W_1,\;\; U_2,V_2,W_2,\ldots\;\; U_N,V_N,W_N\right]=\left[U_1,U_2,\ldots U_n\right]$$



The Method of Finite Elements

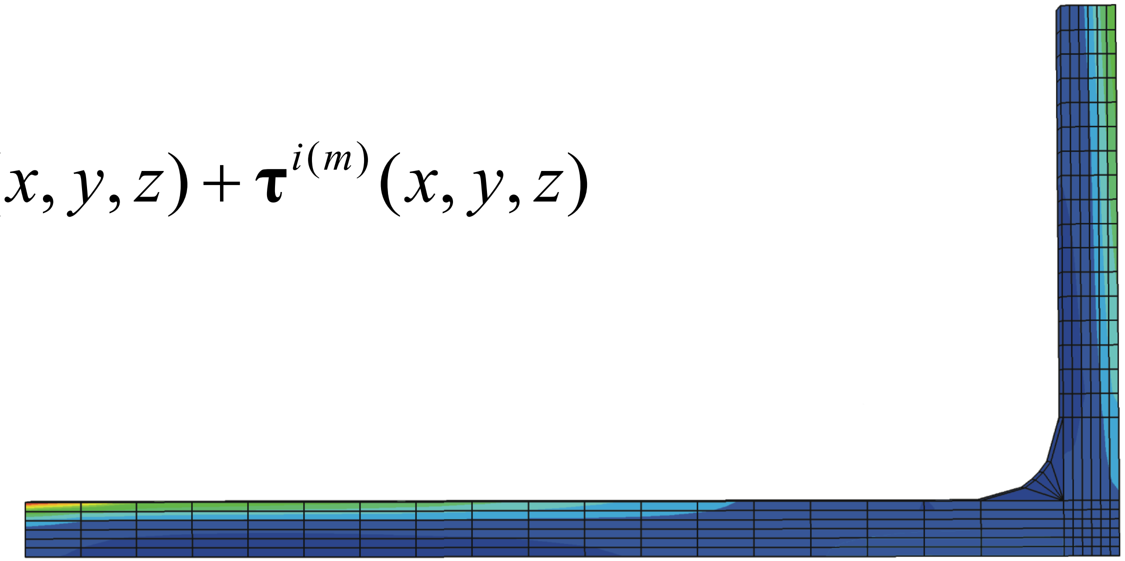
Finite Element Equations:

For element *m* we now write the strains within the element as a function of the total set of global nodal displacements U

$$\boldsymbol{\varepsilon}^{(m)}(x,y,z)=\mathbf{B}^{(m)}(x,y,z)\hat{\mathbf{U}}$$

The stresses are then:

$$\boldsymbol{\tau}^{(m)}(x,y,z)=\mathbf{C}\boldsymbol{\varepsilon}^{(m)}(x,y,z)+\boldsymbol{\tau}^{i(m)}(x,y,z)$$



The Method of Finite Elements

Finite Element Equations:

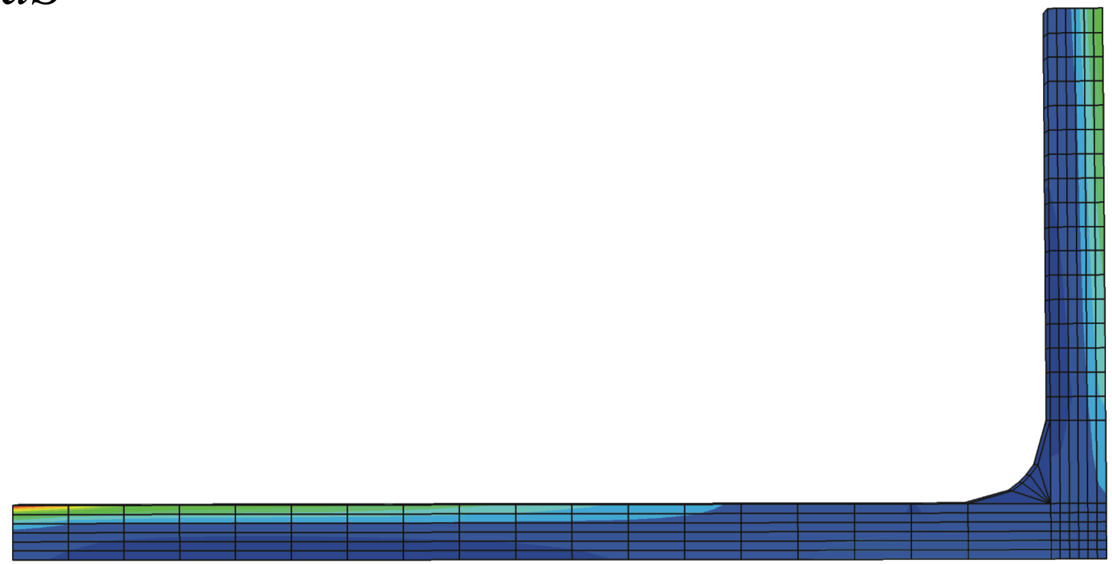
We can now write the equlibrium equations for the total volume by summing up over the *N* elements

$$\sum_{m=1}^{N} \int_{V^{(m)}} \overline{\boldsymbol{\varepsilon}}^{(m)T} \boldsymbol{\tau}^{(m)} dV^{(m)} = \sum_{m=1}^{N} \int_{V^{(m)}} \overline{\mathbf{U}}^{(m)T} \mathbf{f}^{B(m)} dV^{(m)} +$$

$$\sum_{m=1}^{N} \int_{S_{f1}^{(m)}, S_{f2}^{(m)}, \ldots} \overline{\mathbf{U}}^{S_f (m)T} \mathbf{f}^{S_f (m)} dS^{(m)} -$$

$$\sum_{m=1}^{N} \int_{V^{(m)}} \overline{\boldsymbol{\varepsilon}}^{(m)T} \boldsymbol{\tau}^{i(m)} dV^{(m)} +$$

$$\sum_{i} \overline{\mathbf{U}}^{iT} \mathbf{R}_C^i$$



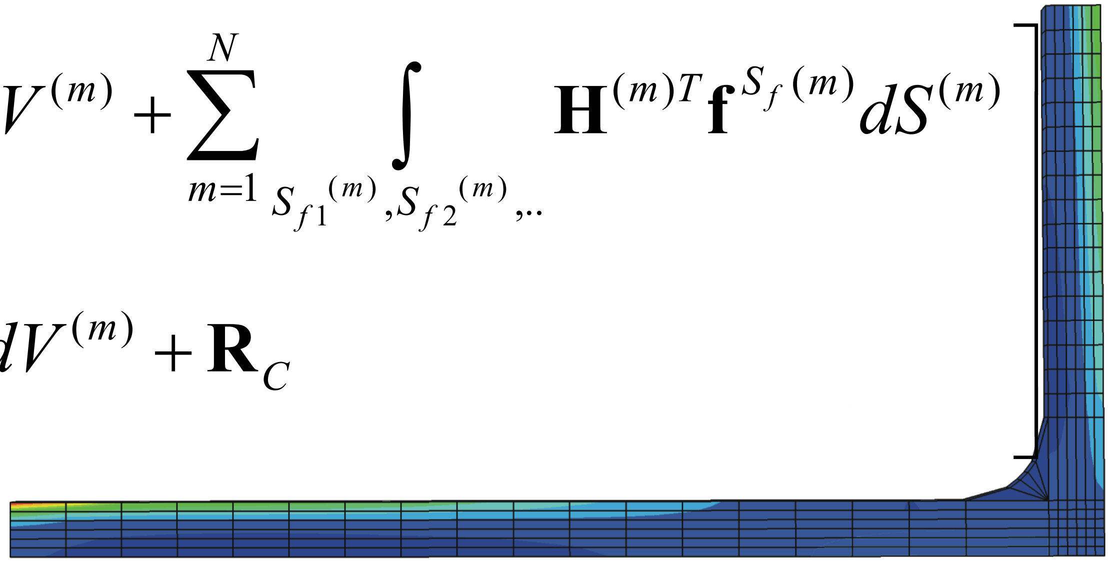
The Method of Finite Elements

Finite Element Equations:

As a next step we represent both the real unknown displacement fields as well as the virtual displacement fields through the interpolation functions (provides symmetrical stiffness matrixes ☺)

$$\hat{\bar{\mathbf{U}}}^{T}\left[\sum_{m=1}^{N}\int_{V^{(m)}}\mathbf{B}^{(m)T}\mathbf{C}^{(m)}\mathbf{B}^{(m)}dV^{(m)}\right]\bar{\mathbf{U}}=$$

$$\hat{\bar{\mathbf{U}}}^{T}\left[\sum_{m=1}^{N}\int_{V^{(m)}}\mathbf{H}^{(m)T}\mathbf{f}^{B(m)}dV^{(m)}+\sum_{m=1}^{N}\int_{S_{f1}^{(m)},S_{f2}^{(m)},..}\mathbf{H}^{(m)T}\mathbf{f}^{S_f(m)}dS^{(m)} - \sum_{m=1}^{N}\int_{V^{(m)}}\mathbf{B}^{(m)T}\boldsymbol{\tau}^{i(m)}dV^{(m)}+\mathbf{R}_{C}\right]$$



The Method of Finite Elements

Finite Element Equations:

Now we may finally simplify as

$$\mathbf{K}\mathbf{U} = \mathbf{R}$$

$$\mathbf{K} = \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T}\mathbf{C}^{(m)}\mathbf{B}^{(m)} dV^{(m)}$$

$$\mathbf{R} = \mathbf{R}_B + \mathbf{R}_S - \mathbf{R}_I + \mathbf{R}_C$$

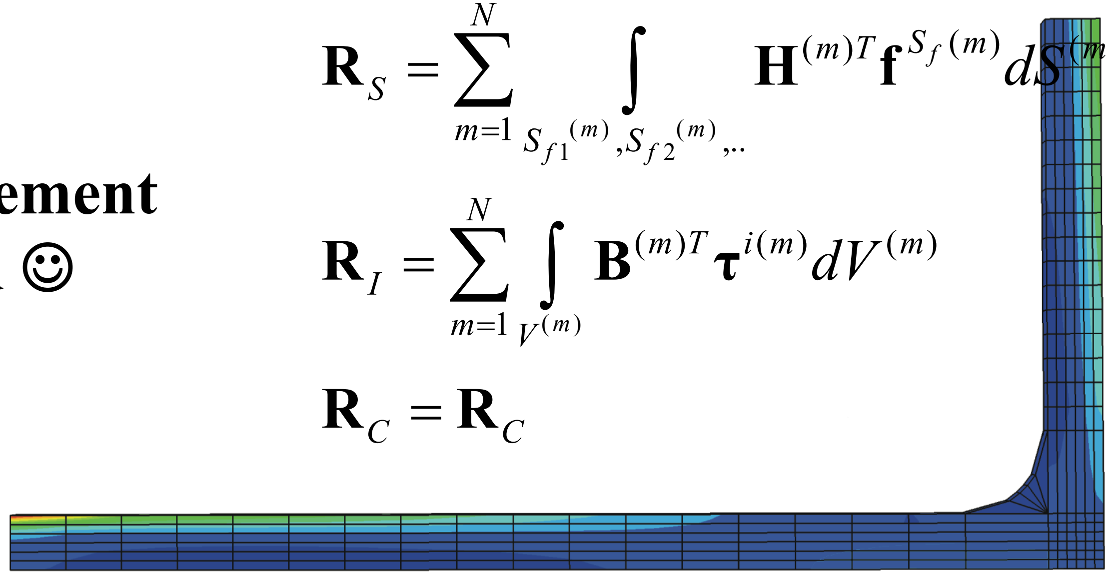
$$\mathbf{R}_B = \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{H}^{(m)T}\mathbf{f}^{B(m)} dV^{(m)}$$

$$\mathbf{R}_S = \sum_{m=1}^{N} \int_{S_{f1}^{(m)}, S_{f2}^{(m)}, ..} \mathbf{H}^{(m)T}\mathbf{f}^{S_f(m)} dS^{(m)}$$

$$\mathbf{R}_I = \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T}\boldsymbol{\tau}^{i(m)} dV^{(m)}$$

$$\mathbf{R}_C = \mathbf{R}_C$$

These are the finite element equations to be solved ☺



Generalized coordinate models

Generalized coordinate models:

The principle behind these models is:

Formulate displacement field in terms of polynomials

one-dimensional

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \cdots$$

two-dimensional

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 xy + \alpha_4 x^2 + \cdots$$

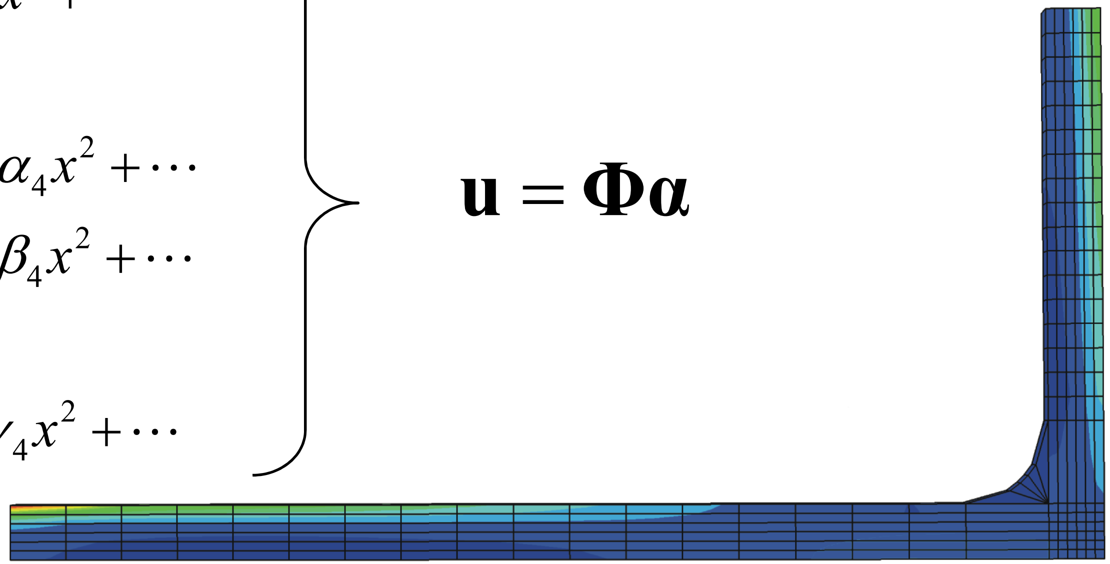
$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 xy + \beta_4 x^2 + \cdots$$

plate bending

$$w(x,y) = \gamma_1 + \gamma_2 x + \gamma_3 xy + \gamma_4 x^2 + \cdots$$

$$\mathbf{u} = \mathbf{\Phi} \boldsymbol{\alpha}$$

$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$: Genenralized coordinates



Generalized coordinate models

Generalized coordinate models:

Next step is to relate the generalized coordinates to the nodal displacements:

We insert the nodal coordinates into

$$\mathbf{u} = \mathbf{\Phi}\boldsymbol{\alpha}$$

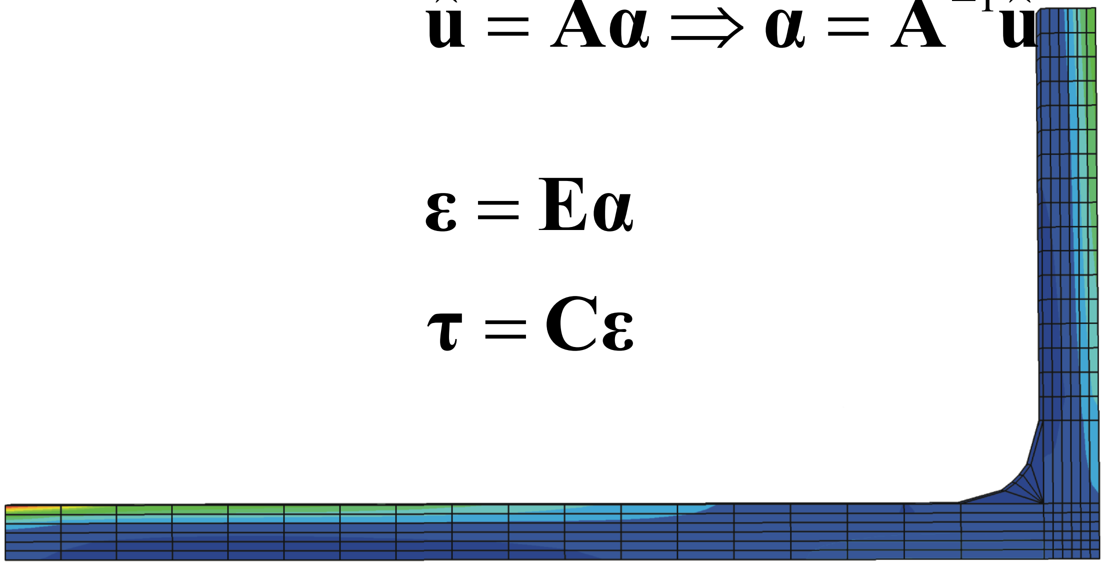
and get:

$$\hat{\mathbf{u}} = \mathbf{A}\boldsymbol{\alpha} \Rightarrow \boldsymbol{\alpha} = \mathbf{A}^{-1}\hat{\mathbf{u}}$$

Now we can obtain

$$\boldsymbol{\varepsilon} = \mathbf{E}\boldsymbol{\alpha}$$

$$\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\varepsilon}$$



On the choice of shape functions

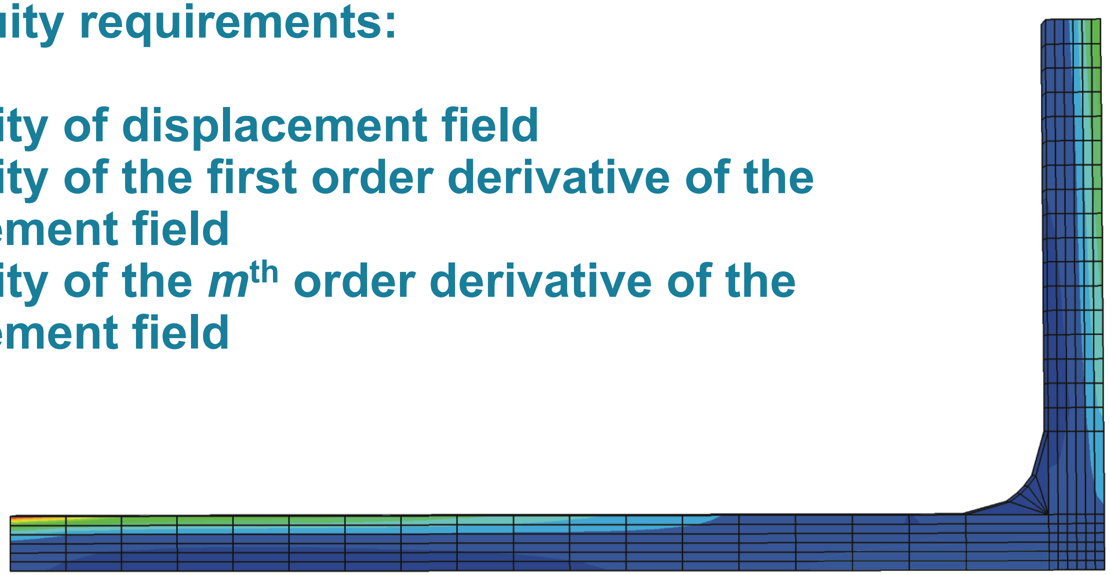
Requirements to shape functions:

First of all the functions which we want to represent (displacements and their derivatives) need to be able to represent the physics of the type of problem we are aiming to model within the individual elements.

Secondly we also need to be concerned with continuity over the borders of the elements.

We here introduce continuity requirements:

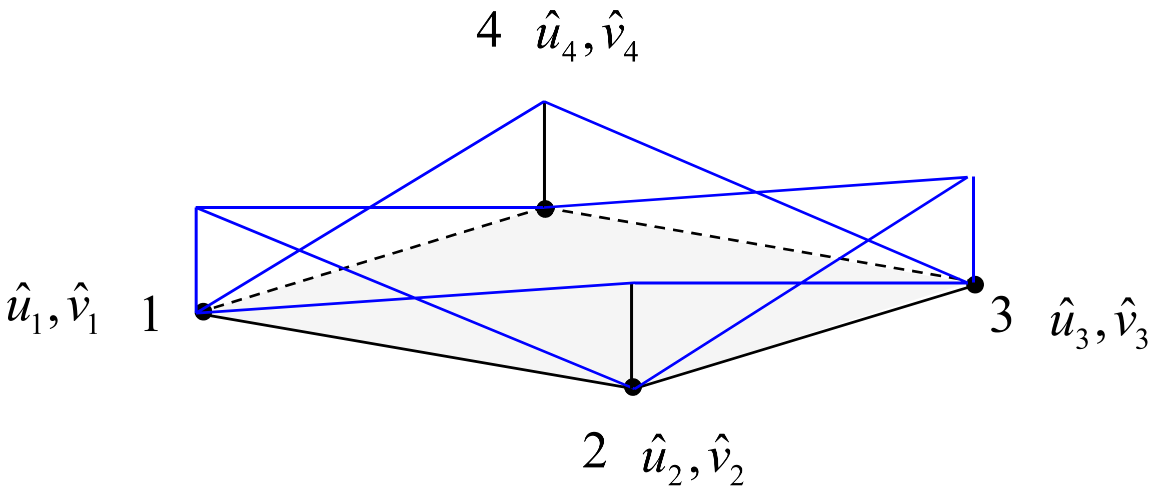
C^0 continuity	Continuity of displacement field
C^1 continuity	Continuity of the first order derivative of the displacement field
C^m continuity	Continuity of the *m*th order derivative of the displacement field



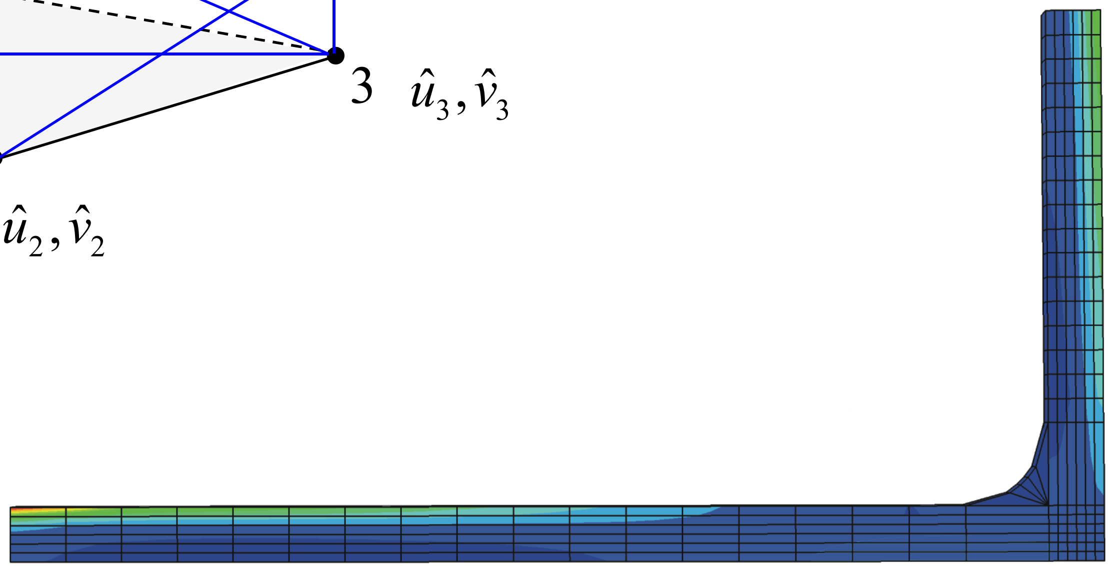
On the choice of shape functions

Shape functions:

The displacement e.g. *u* or *v* at any location within the element (*x,y*) can be represented as a function of the nodal displacements (*u* or *v*):



$$u(x,y) = f_1(\hat{\mathbf{u}})$$
$$v(x,y) = f_2(\hat{\mathbf{v}})$$



On the choice of shape functions

Shape functions:

In general we may write the approximate relation between the field representation and the nodal displacements as:

$$u(x,y) = \mathbf{H}^T(x,y)\hat{\mathbf{u}}$$

We consider an element with n nodes

Displacements in the u-direction

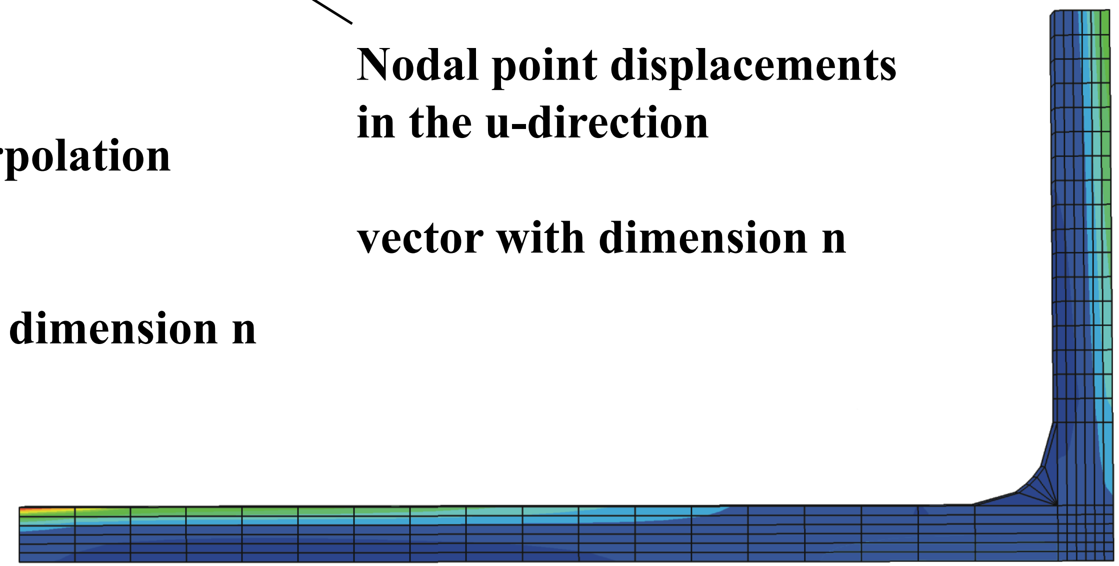
scalar

Shape/interpolation functions

vector with dimension n

Nodal point displacements in the u-direction

vector with dimension n



On the choice of shape functions

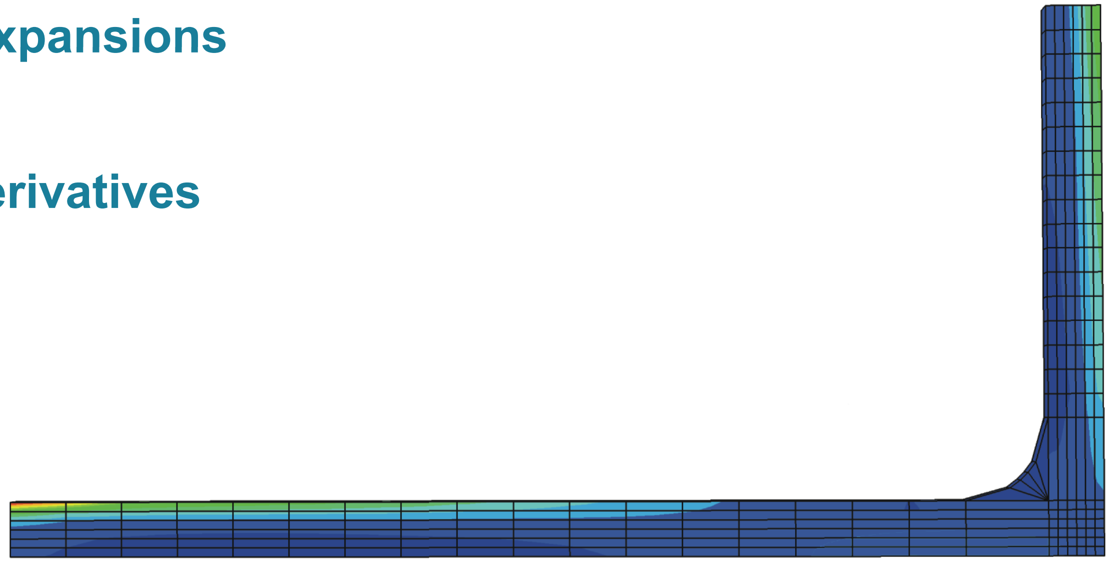
Shape functions:

Polynomials are usually applied for the development of shape functions (polynomials are easily differentiated analytically)

- **Langrange polynomials**
  complete polynomial expansions

- **Serendipity polynomials**
  incomplete polynomial expansions

- **Hermitian polynomials**
  polynomials including derivatives



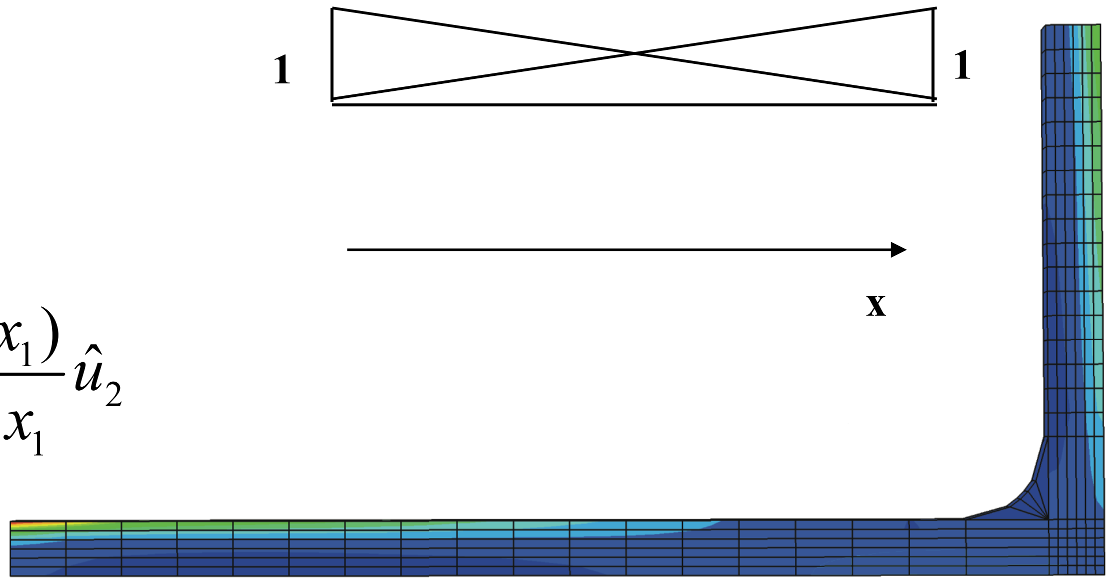
On the choice of shape functions

Shape functions:

Lagrange polynomials (one dimensional element):

$$H_i(x) = L_i(y) \Rightarrow u(x) = \sum_{i=1}^{n} L_i(x)\hat{u}_i$$

$$L_i(x) = \prod_{\substack{j=1,\\ j\neq i}}^{n+1} \frac{x - x_j}{x_i - x_j}$$



$$u(x) = L_1(x)\hat{u}_1 + L_2(x)\hat{u}_2$$

$$= \frac{(x_2 - x)}{x_2 - x_1}\hat{u}_1 + \frac{(x - x_1)}{x_2 - x_1}\hat{u}_2$$

On the choice of shape functions

Shape functions:

Lagrange polynomials (general):

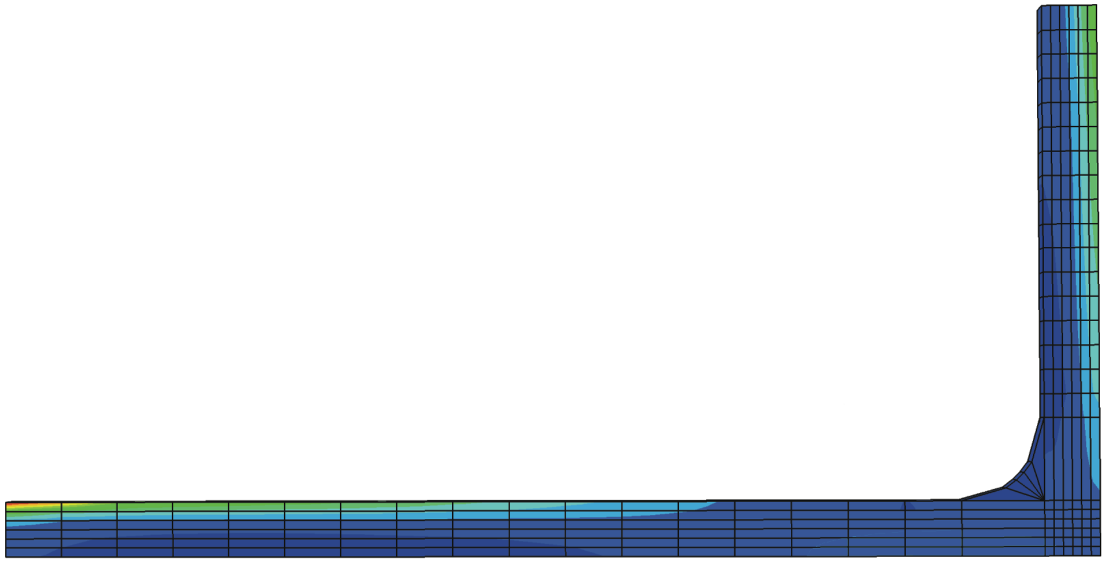
$$u(x,y,z) = \sum_{i=1}^{n} L_i(x,y,z)\hat{u}_i$$

$$u(x,y,z) = \sum_{i=1}^{n} \quad \hat{u}_i$$

$$L_i(x_j,y_j,z_j) = 1, \quad i = j$$

$$L_i(x_j,y_j,z_j) = 0, \quad i \neq j$$

$$\sum_{i=1}^{n} L_i(x,y,z) = 1$$



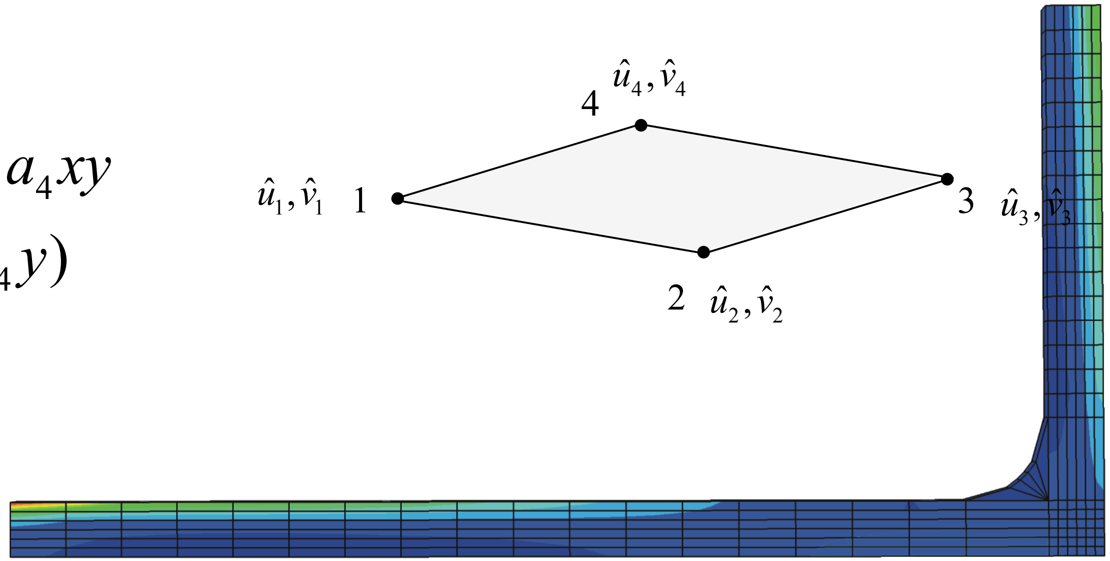
On the choice of shape functions

Shape functions:

Lagrange polynomials (four node rectangular element):

Products of two one dimensional first order Lagrange polynomials result in the bi-linear four-node element

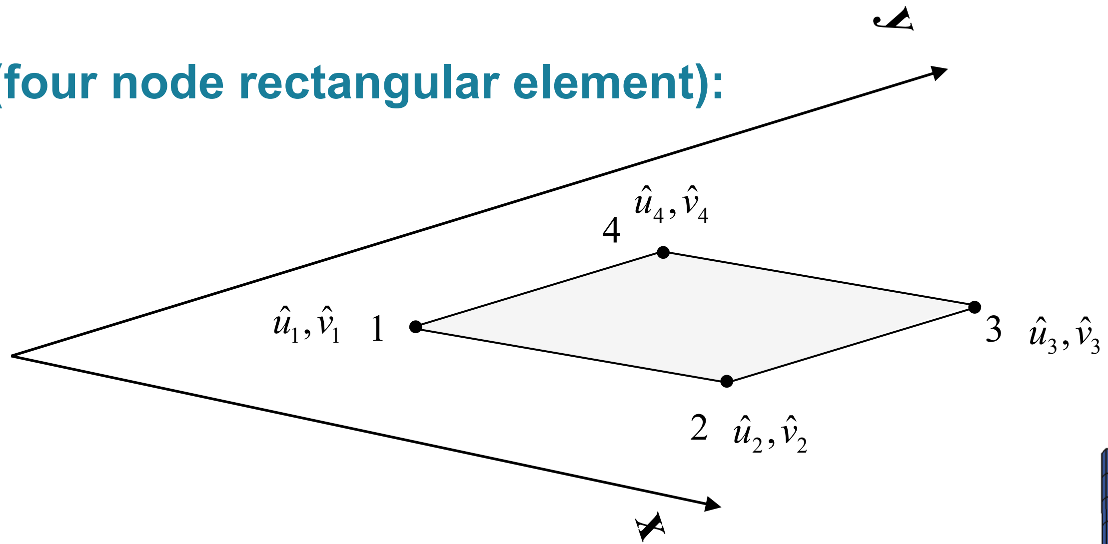
$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 xy$$
$$= (b_1 + b_2 x)(b_3 + b_4 y)$$



On the choice of shape functions

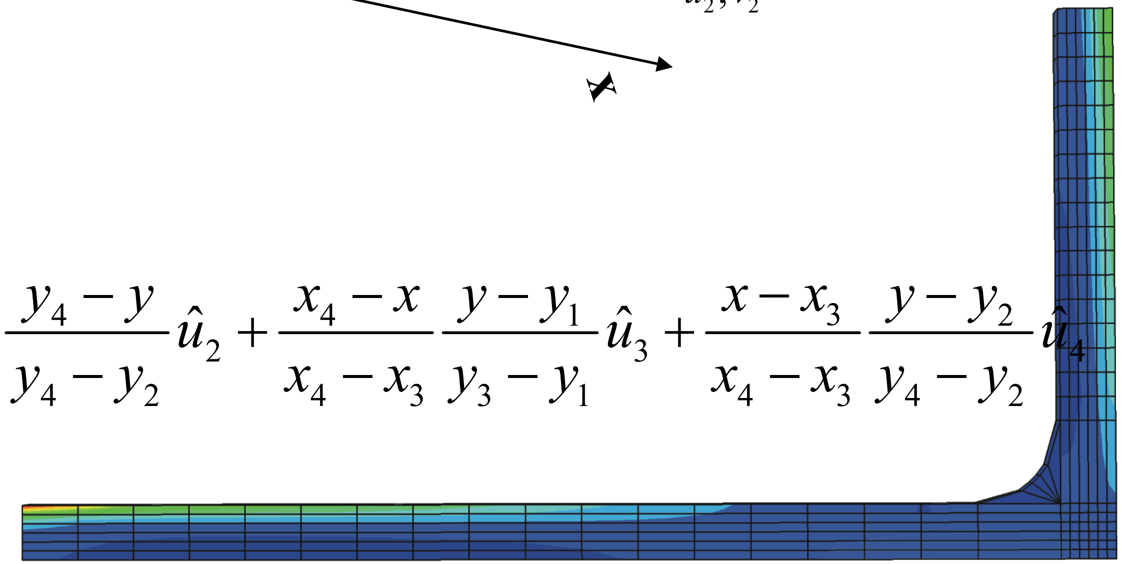
Shape functions:

Lagrange polynomials (four node rectangular element):



$$u(x,y) =$$

$$\frac{x_2 - x}{x_2 - x_1}\frac{y_3 - y}{y_3 - y_1}\hat{u}_1 + \frac{x - x_1}{x_2 - x_1}\frac{y_4 - y}{y_4 - y_2}\hat{u}_2 + \frac{x_4 - x}{x_4 - x_3}\frac{y - y_1}{y_3 - y_1}\hat{u}_3 + \frac{x - x_3}{x_4 - x_3}\frac{y - y_2}{y_4 - y_2}\hat{u}_4$$

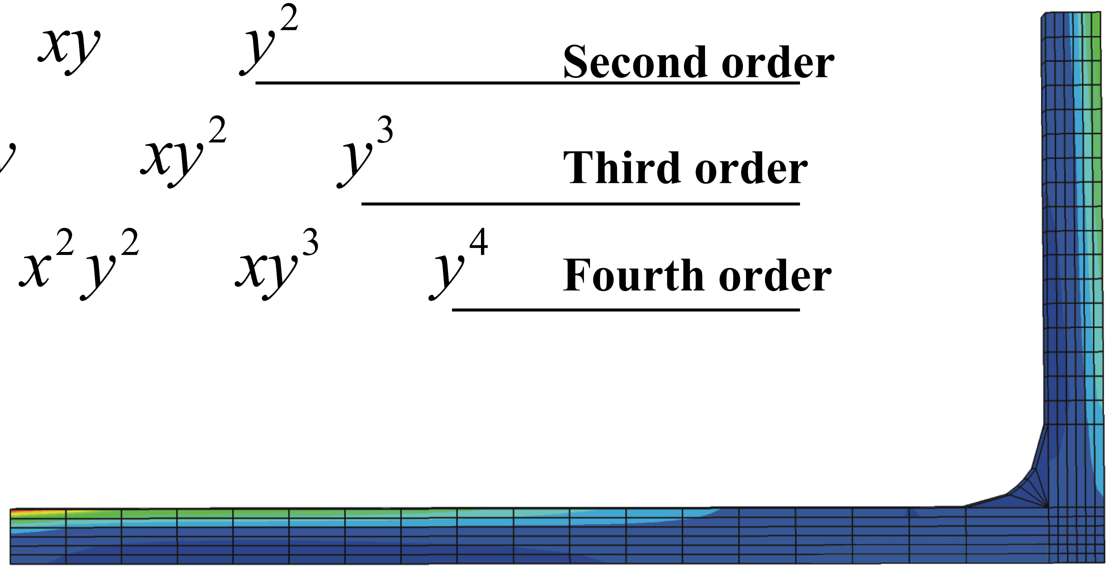


On the choice of shape functions

Shape functions:

From Pascal's triangle we can see how many nodes are required for the representation of displacement fields of any order and completeness:

1	**Zero order**
$x \quad y$	**First order**
$x^2 \quad xy \quad y^2$	**Second order**
$x^3 \quad x^2y \quad xy^2 \quad y^3$	**Third order**
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	**Fourth order**



On the choice of shape functions

Shape functions:

Products of Lagrange polynomials (bi-linear four node rectangular)

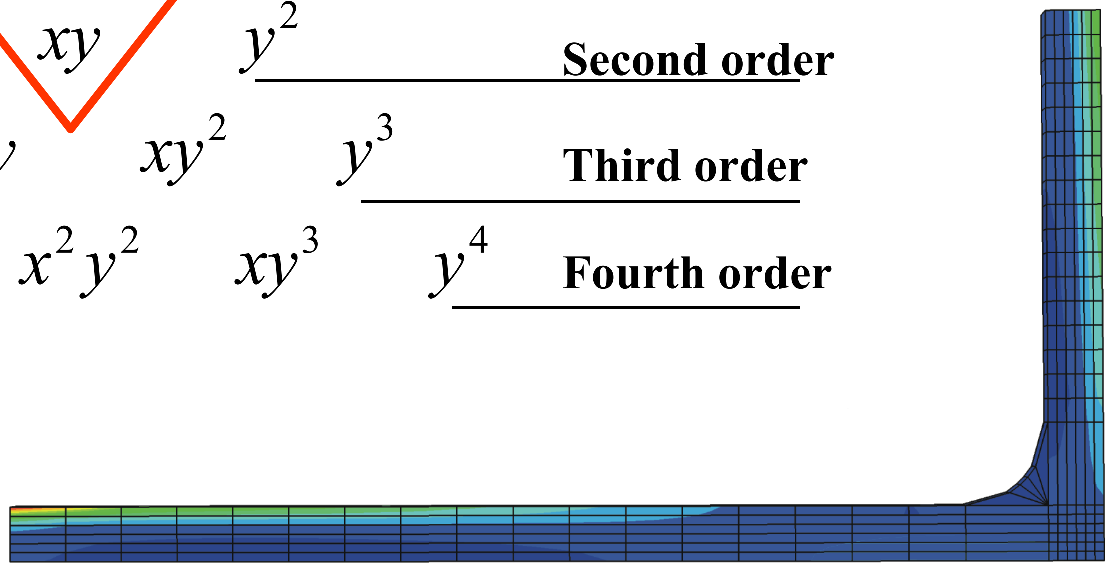
1 — **Zero order**

$x \quad y$ — **First order**

$x^2 \quad xy \quad y^2$ — **Second order**

$x^3 \quad x^2y \quad xy^2 \quad y^3$ — **Third order**

$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$ — **Fourth order**



On the choice of shape functions

Shape functions:

Products of Lagrange polynomials (quadratic nine-node rectangular)

This requires an inner node
a difficulty !

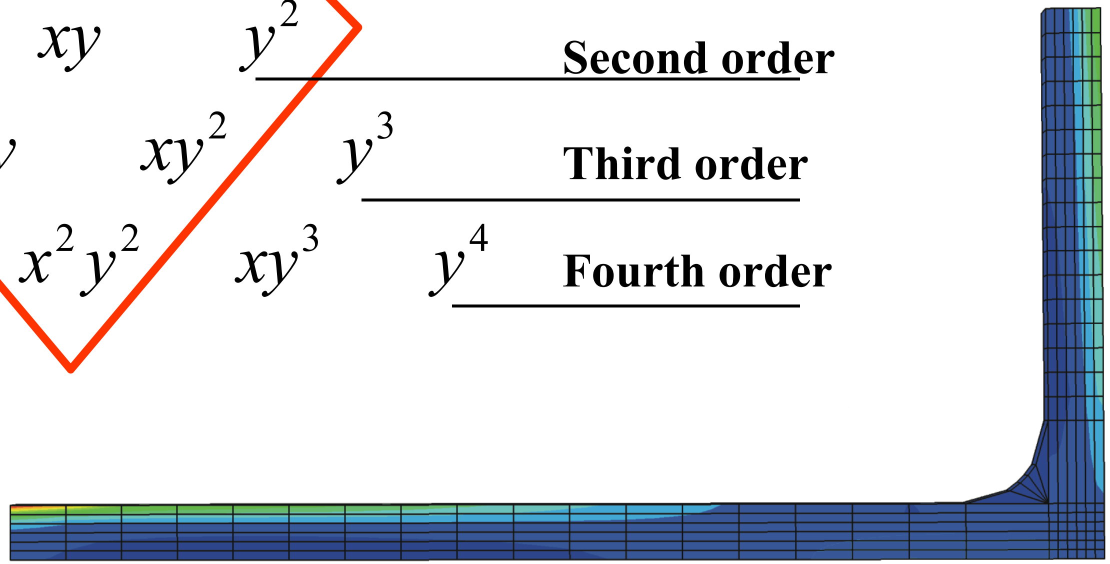
1 — **Zero order**

$x \quad y$ — **First order**

$x^2 \quad xy \quad y^2$ — **Second order**

$x^3 \quad x^2y \quad xy^2 \quad y^3$ — **Third order**

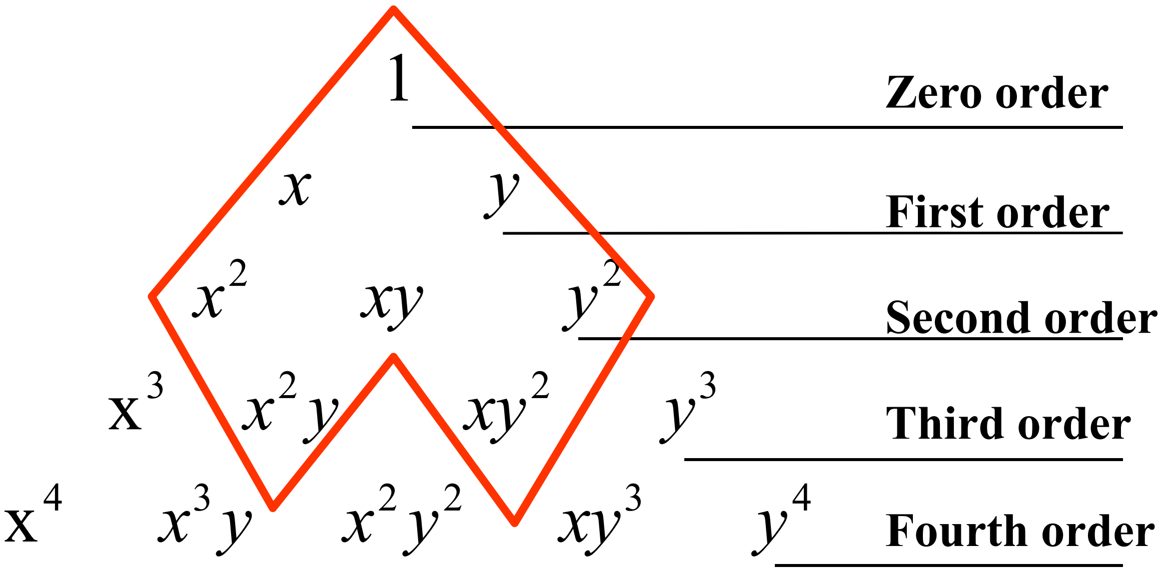
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$ — **Fourth order**

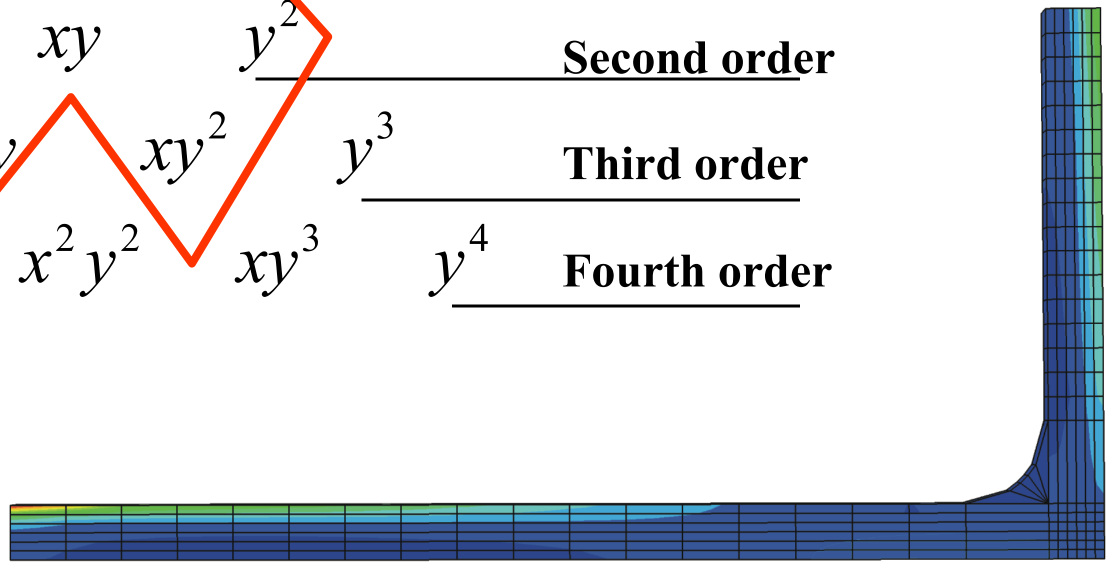


On the choice of shape functions

Shape functions:

Serendipity shape functions are constructed by incomplete polynomials – avoiding inner nodes



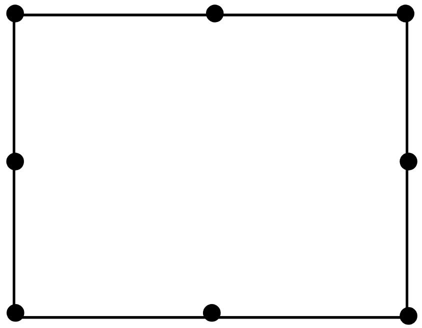


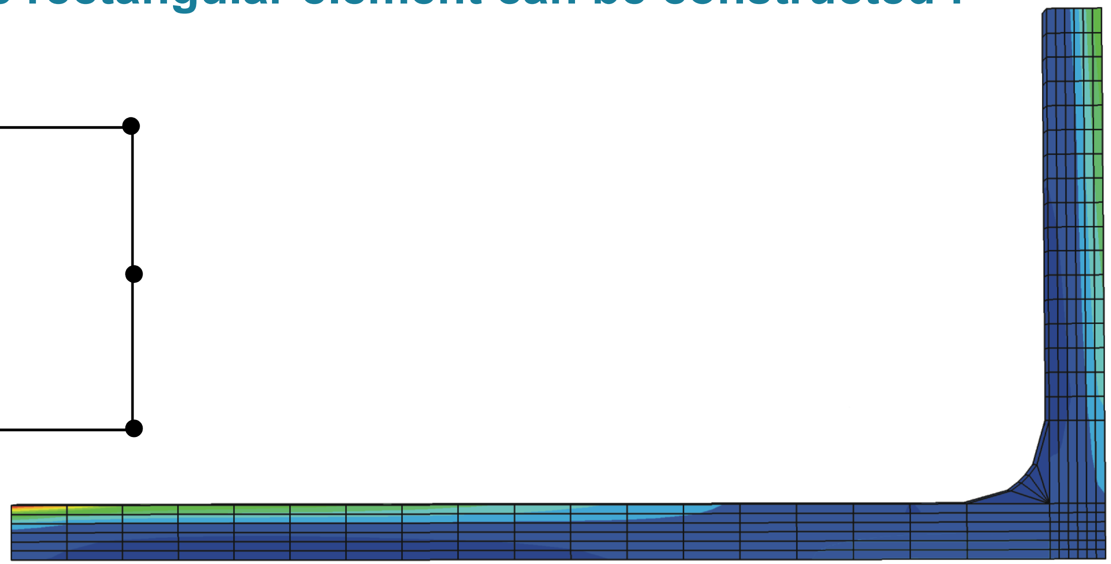
On the choice of shape functions

Shape functions:

Whereas difficulties may arise (inner nodes) when aiming to develop quadratic shape functions for rectangular elements using Lagrange polynomials the shape functions developed by incomplete polynomials (serendipity shape functions) – less terms necessitates less nodes !

An bi-quadratic eight node rectangular element can be constructed !

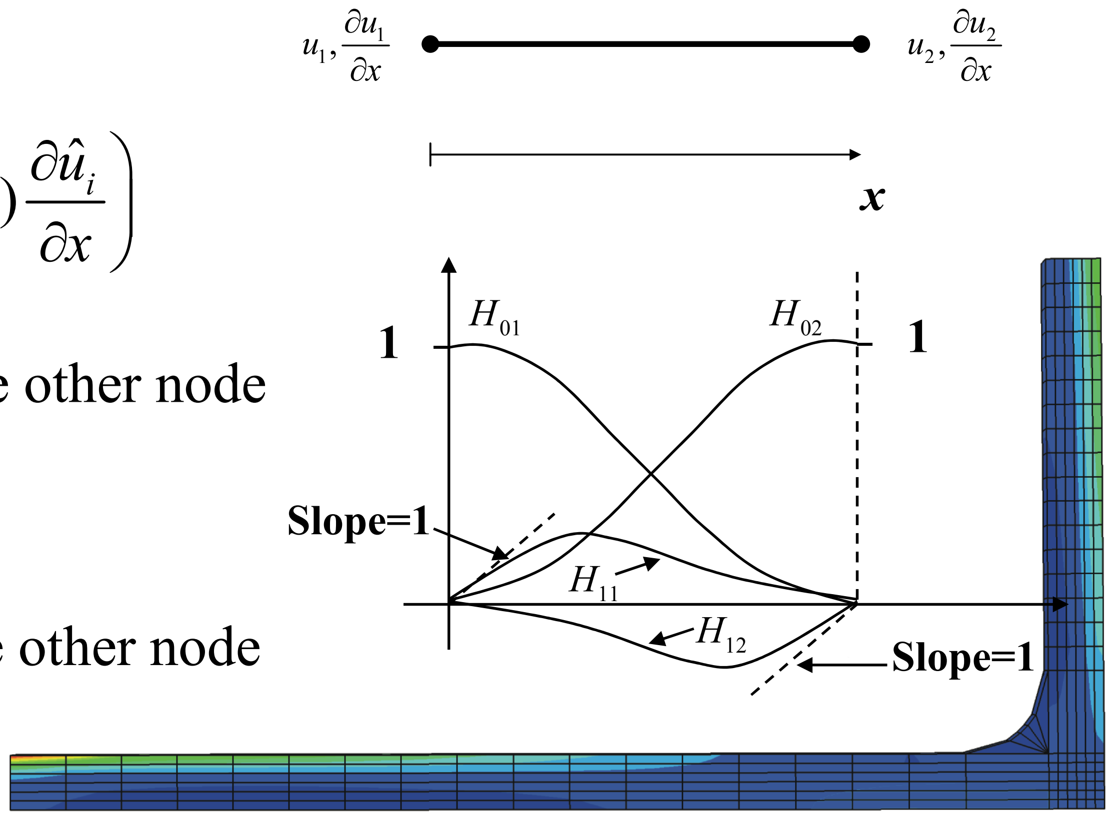




On the choice of shape functions

Shape functions:

Hermitian shape functions relate not only the displacements at nodes to displacements within the elements but also the first order derivatives



$$u(x) = \sum_{i=1}^{2} \left(H_{0i}(x)\hat{u}_i + H_{1i}(x)\frac{\partial \hat{u}_i}{\partial x} \right)$$

$H_{0i}(x) = 1,$ and zero at the other node

$H'_{0i}(x) = 0$ at both nodes

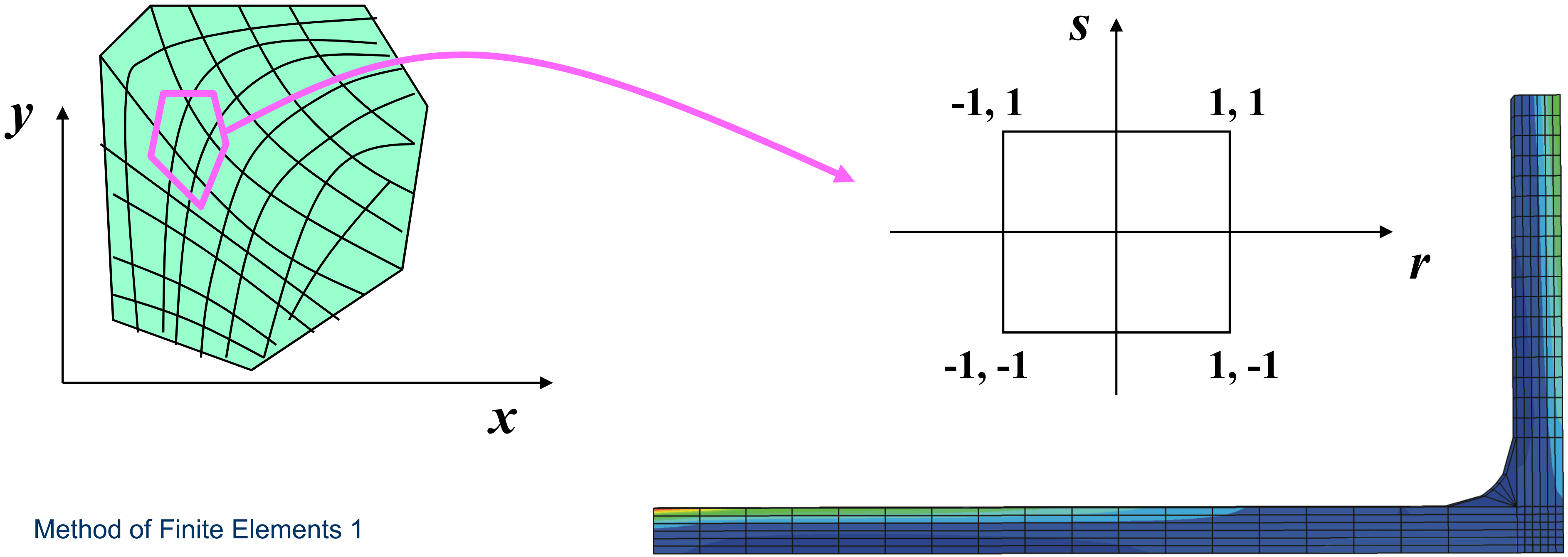
$H_{1i}(x) = 0,$ at both nodes

$H'_{1i}(x) = 1,$ and zero at the other node

Natural coordinates

Shape functions – Natural coordinates:

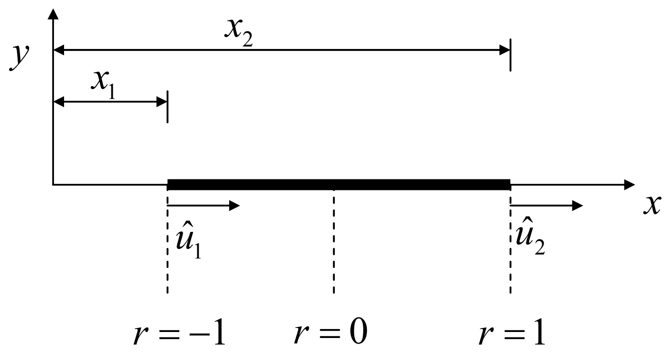
As we have seen we are able to establish shape functions in global or local coordinate systems as we please. However, for the purpose of standardizing the process of developing the element matrixes it is convenient to introduce the so-called natural coordinate system.



Natural coordinates

Shape functions – Natural coordinates:

Let us consider the simple bar element



The relation between the x-coordinate and the r-coordinate is given as:

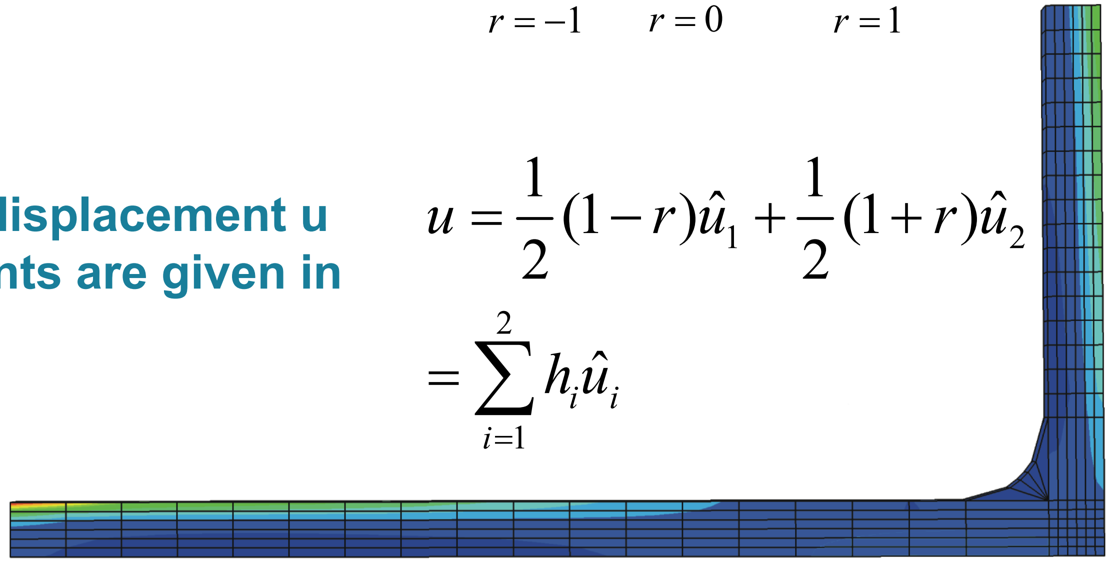
$$x = \frac{1}{2}(1-r)\hat{x}_1 + \frac{1}{2}(1+r)\hat{x}_2$$

$$= \sum_{i=1}^{2} h_i \hat{x}_i$$

The relation between the displacement u and the nodal displacements are given in the same way:

$$u = \frac{1}{2}(1-r)\hat{u}_1 + \frac{1}{2}(1+r)\hat{u}_2$$

$$= \sum_{i=1}^{2} h_i \hat{u}_i$$

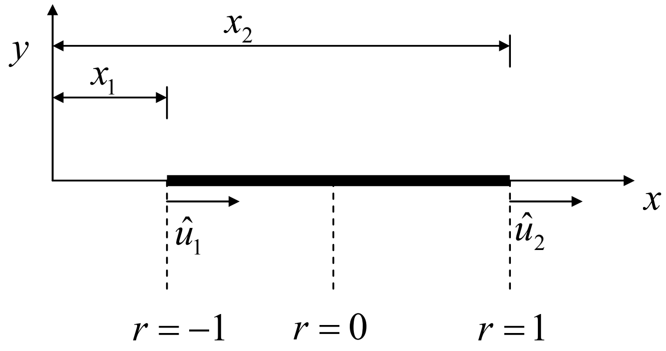


Natural coordinates

Shape functions – Natural coordinates:

Let us consider the simple bar element

We need to be able to establish the strains – meaning we need to be able to take the derivatives of the displacement filed in regard to the x-coordinate



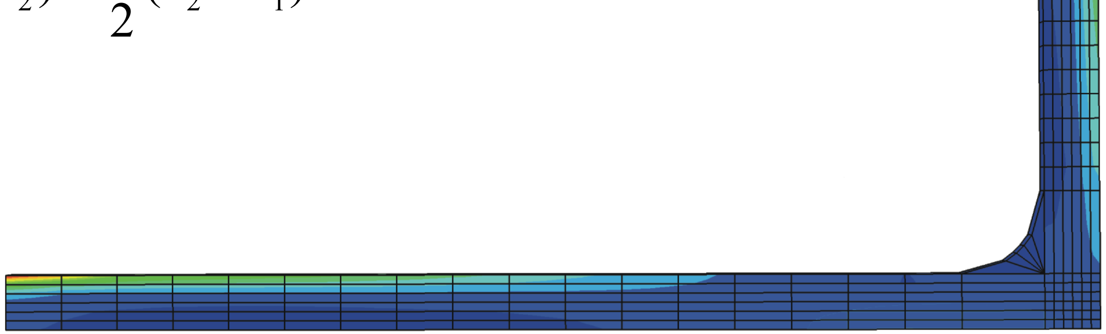
$$\varepsilon = \frac{du}{dx} = \frac{du}{dr}\frac{dr}{dx}$$

$$\frac{du}{dr} = \frac{d}{dr}\left(\frac{1}{2}(1-r)\hat{u}_1 + \frac{1}{2}(1+r)\hat{u}_2\right) = \frac{1}{2}(\hat{u}_2 - \hat{u}_1)$$

$$\frac{dx}{dr} = \frac{d}{dr}\left(\frac{1}{2}(1-r)x_1 + \frac{1}{2}(1+r)x_2\right) = \frac{1}{2}(x_2 - x_1)$$

$$\Downarrow$$

$$\frac{du}{dx} = \frac{(\hat{u}_2 - \hat{u}_1)}{(x_2 - x_1)} = \frac{(\hat{u}_2 - \hat{u}_1)}{L}$$



Natural coordinates

Shape functions – Natural coordinates:

Let us consider the simple bar element

The strain-displacement matrix then becomes:

$$\mathbf{B} = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

and the stiffness matrix is calculated as:

$$\mathbf{K} = \frac{AE}{L^2}\int_{-1}^{1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix}\mathbf{J}dr, \qquad \mathbf{J} = \frac{dx}{dr} = \frac{L}{2}$$

$$\Downarrow$$

$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$