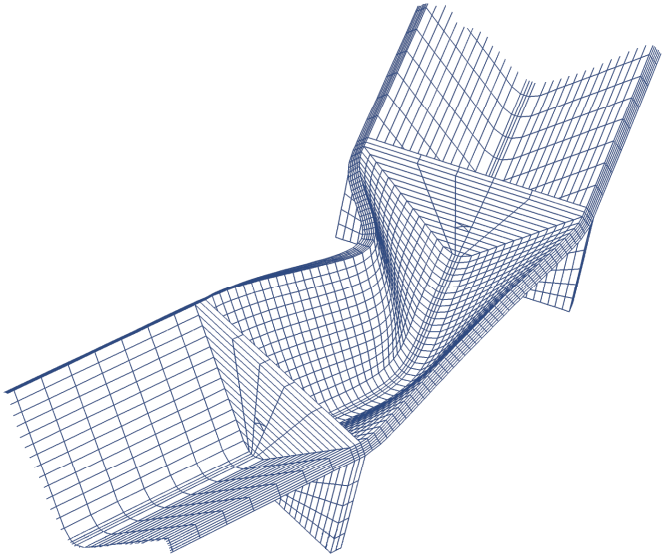
The Finite Element Method for the Analysis of Linear Systems



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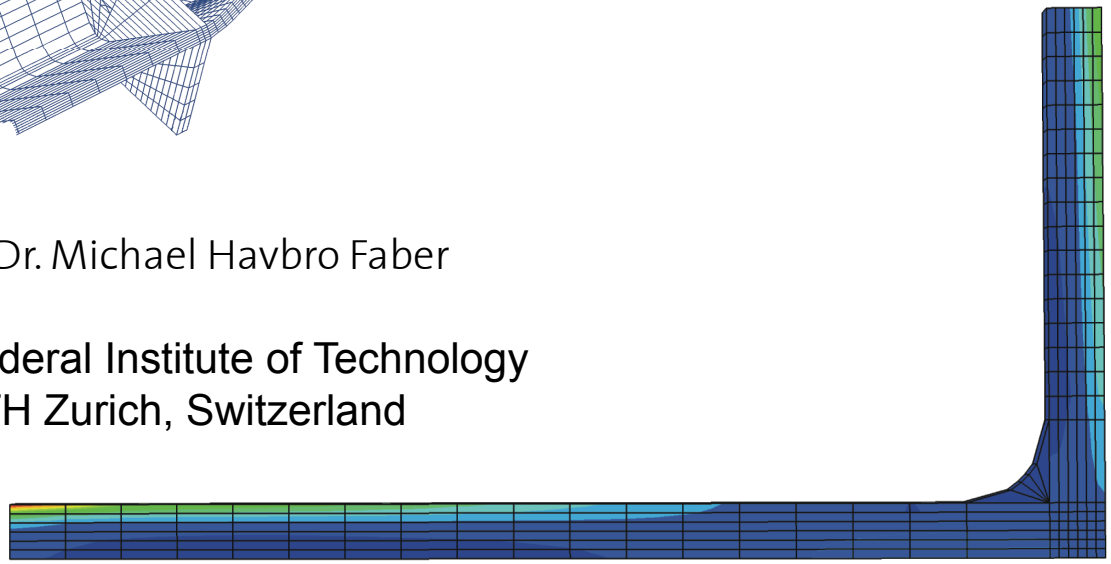

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  - The assumption about stress equilibrium
  - Local to global coordinate transformations
  - Imposition of displacement boundary conditions
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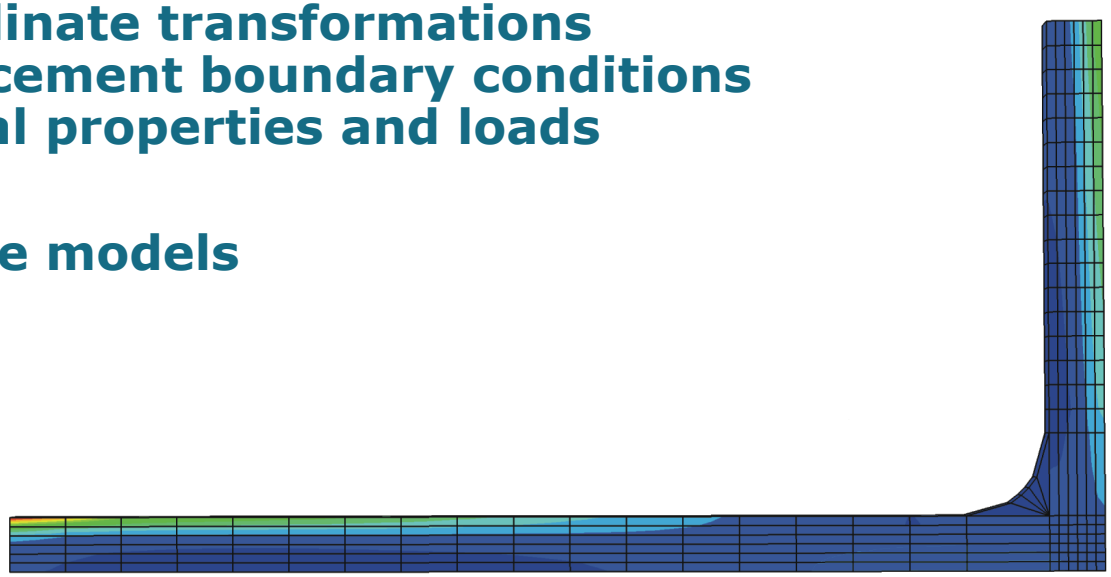


Introduction

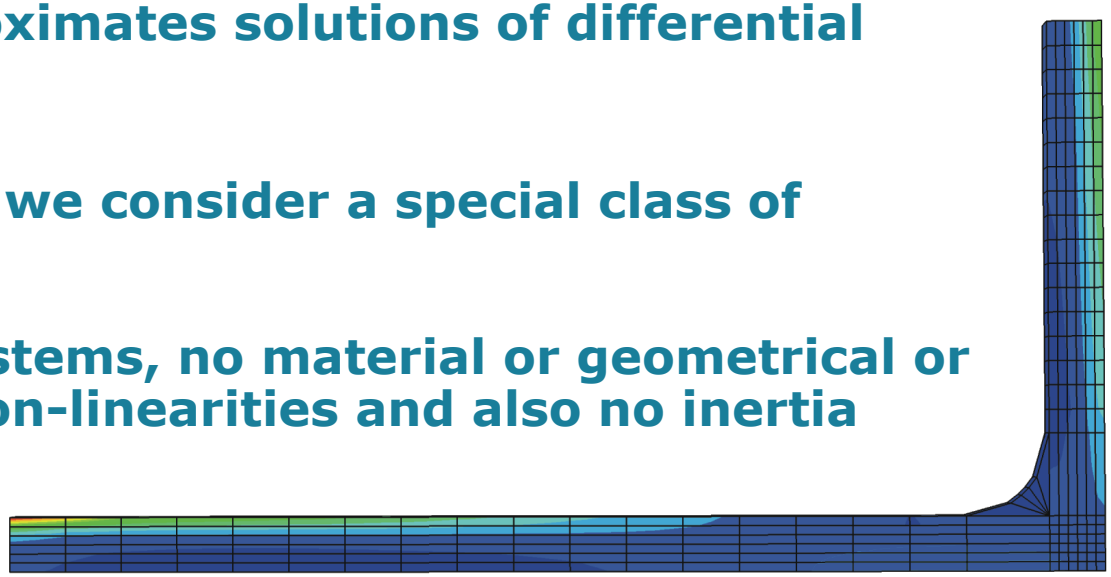
What we would like to establish is the response of a structure subject to "loading".

The Method of Finite Elements provides a framework for the analysis of such responses – however for very general problems.

The Method of Finite Elements provides a very general approach to the approximates solutions of differential equations.

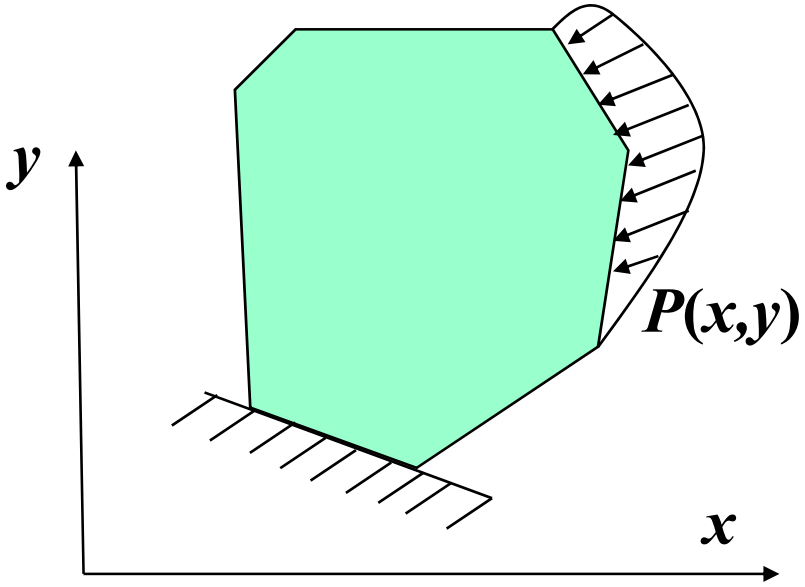
In the present course we consider a special class of problems, namely:

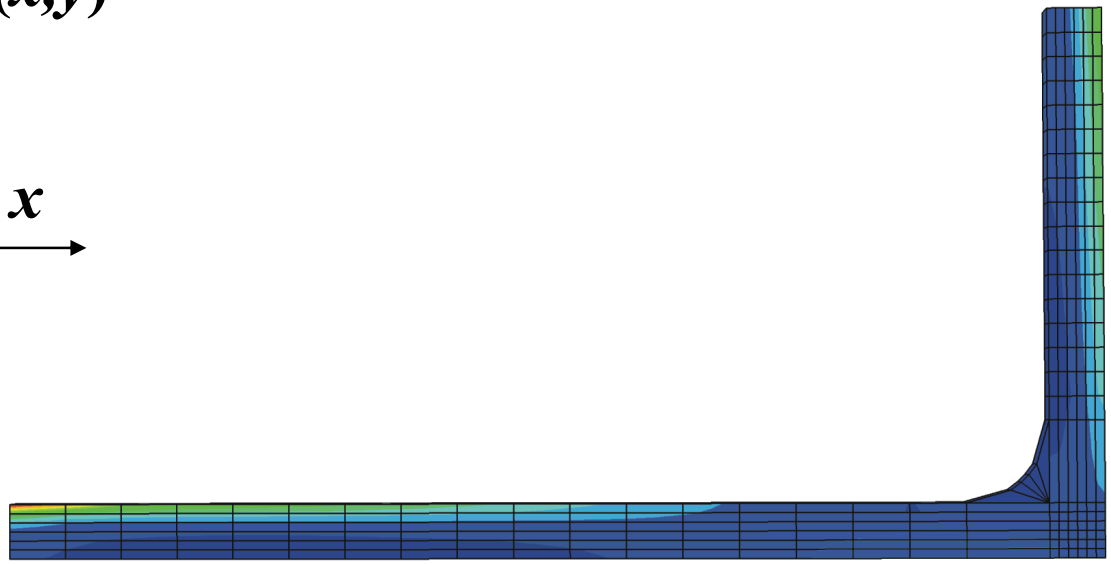
Linear quasi-static systems, no material or geometrical or boundary condition non-linearities and also no inertia effect!



Introduction

In principle the structures/systems we consider can be represented like show in the figure

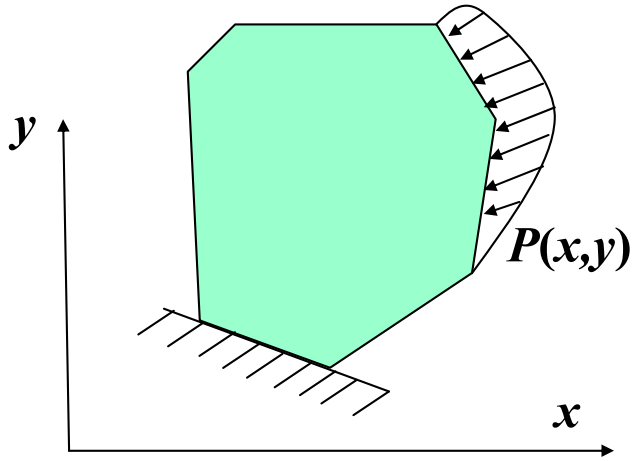




Introduction

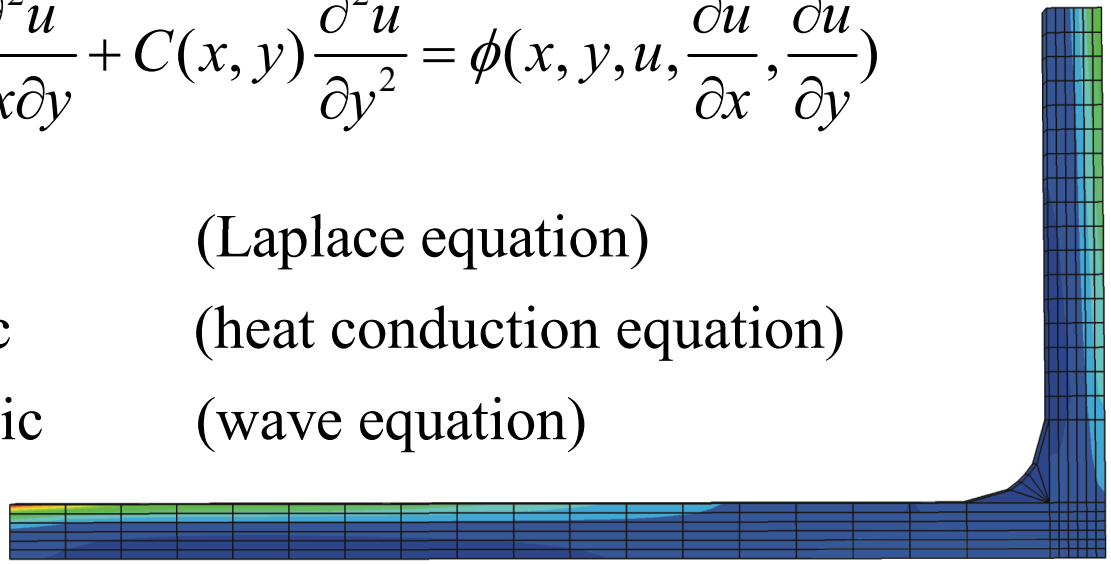
Continuous-system:

The governing differential equation we consider in general have the form (second order differential equations)



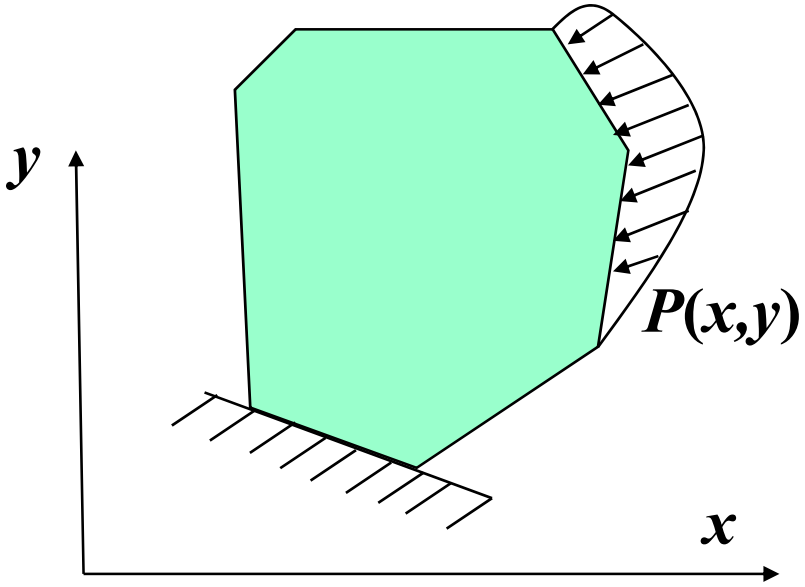
$$A(x,y)\frac{\partial^2 u}{\partial x^2}+2B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}=\phi(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$$

$$B^2-AC\begin{cases}<0 & \text{elliptic} \quad \text{(Laplace equation)}\\ =0 & \text{parabolic} \quad \text{(heat conduction equation)}\\ >0 & \text{hyperbolic} \quad \text{(wave equation)}\end{cases}$$



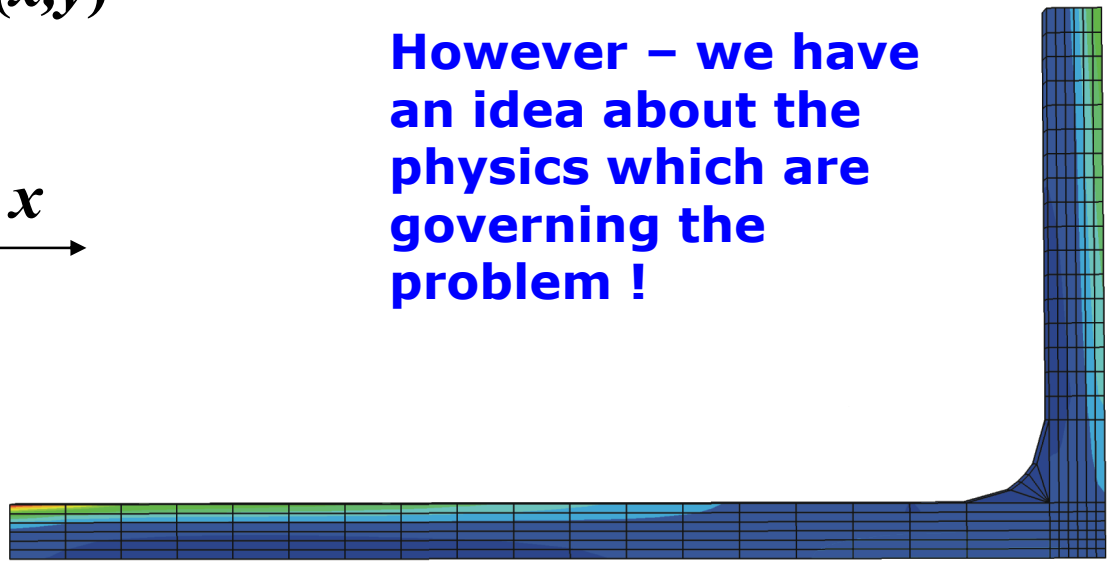
Introduction

We know that this type of problem can be analyzed taking basis in the governing differential equation



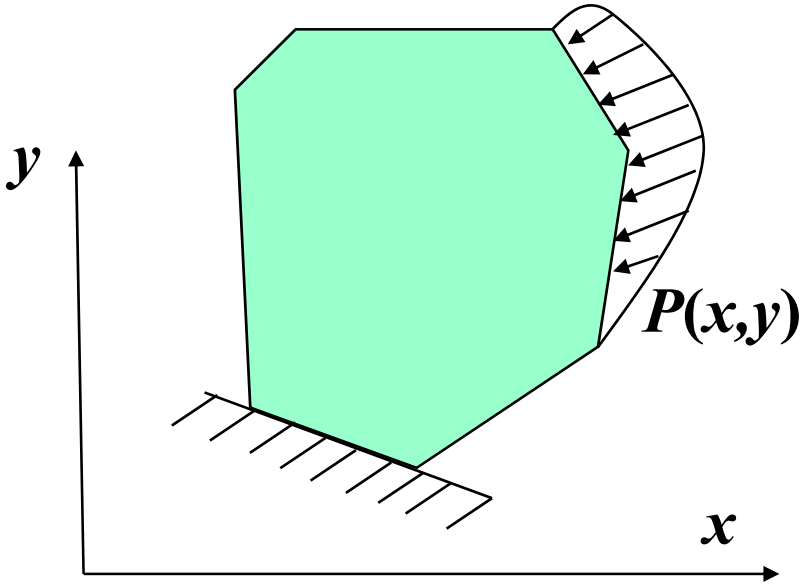
Only – the problem is that it is very difficult to find solutions for general cases

However – we have an idea about the physics which are governing the problem !

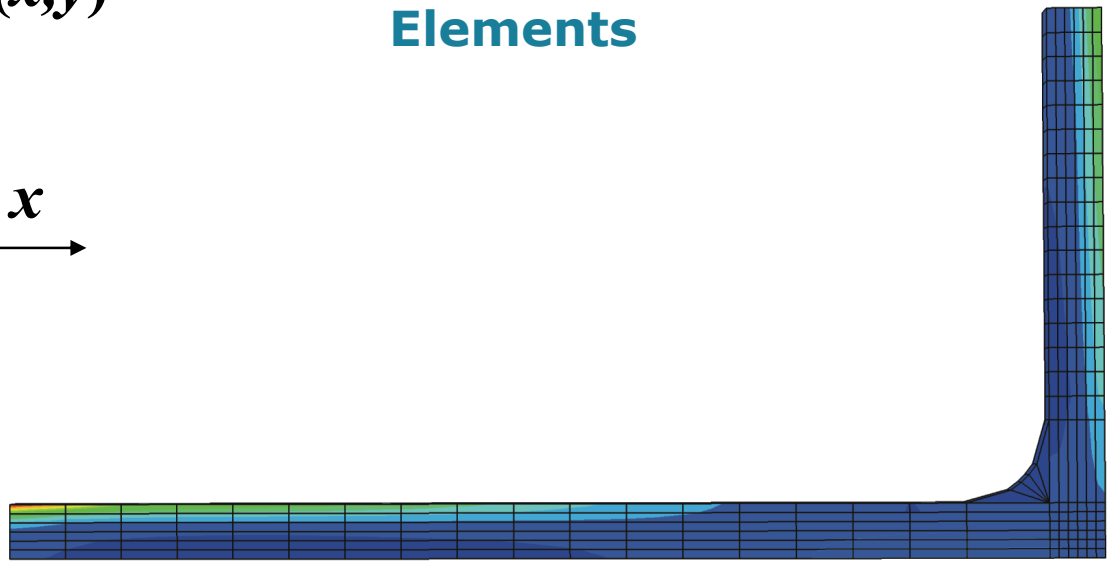


Introduction

General principles of mechanics on how to derive and solve the differential equations were developed by Ritz and Galerkin – taking basis in variational approaches

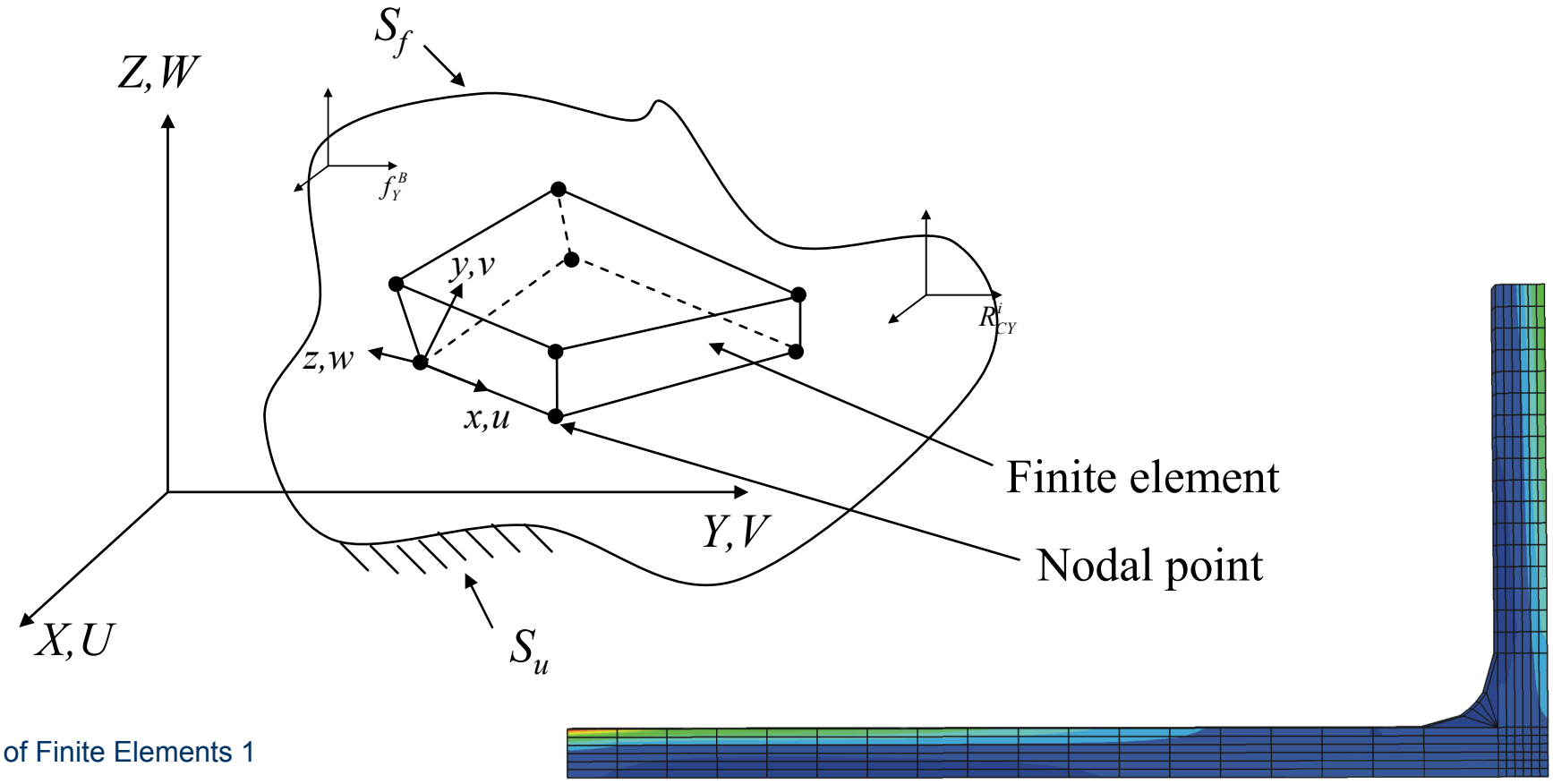


These developments led to the principle of virtual work - which essentially forms the basis for the Method of Finite Elements



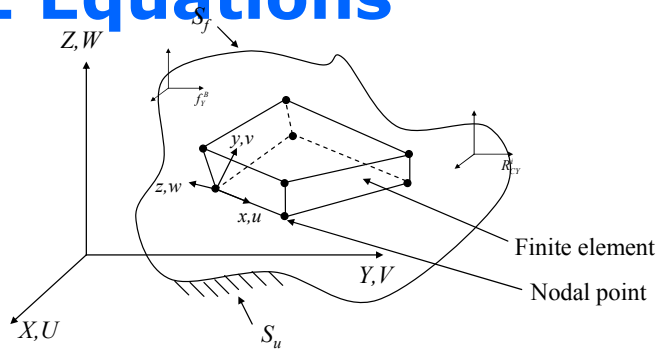
General Derivation of the FEE Equations

Based on the principle of virtual work we can derive the element and system equations for the analysis of the response



General Derivation of the FEE Equations

The principle of virtual displacements:



The loads are given as

Body loads

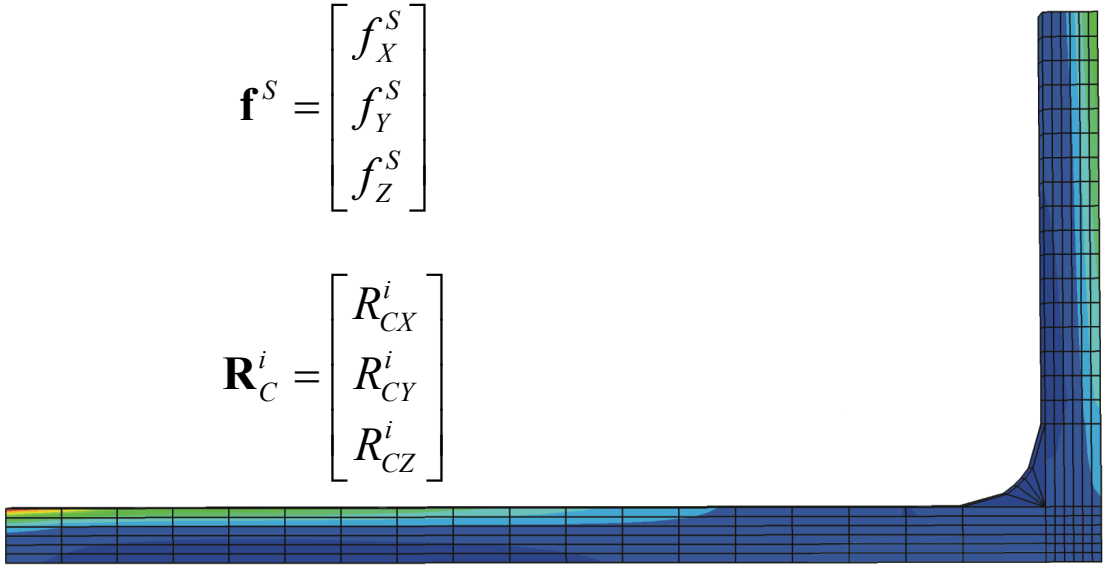
$$\mathbf{f}^B = \begin{bmatrix} f_X^B \\ f_Y^B \\ f_Z^B \end{bmatrix}$$

Surface loads

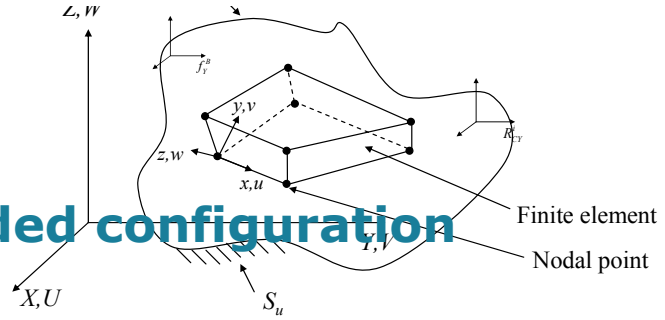
$$\mathbf{f}^S = \begin{bmatrix} f_X^S \\ f_Y^S \\ f_Z^S \end{bmatrix}$$

Concentrated loads

$$\mathbf{R}_C^i = \begin{bmatrix} R_{CX}^i \\ R_{CY}^i \\ R_{CZ}^i \end{bmatrix}$$



General Derivation of the FEE Equations



The principle of virtual displacements:

Displacements are measured in unloaded configuration

$$\mathbf{U}(X,Y,Z)=\begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

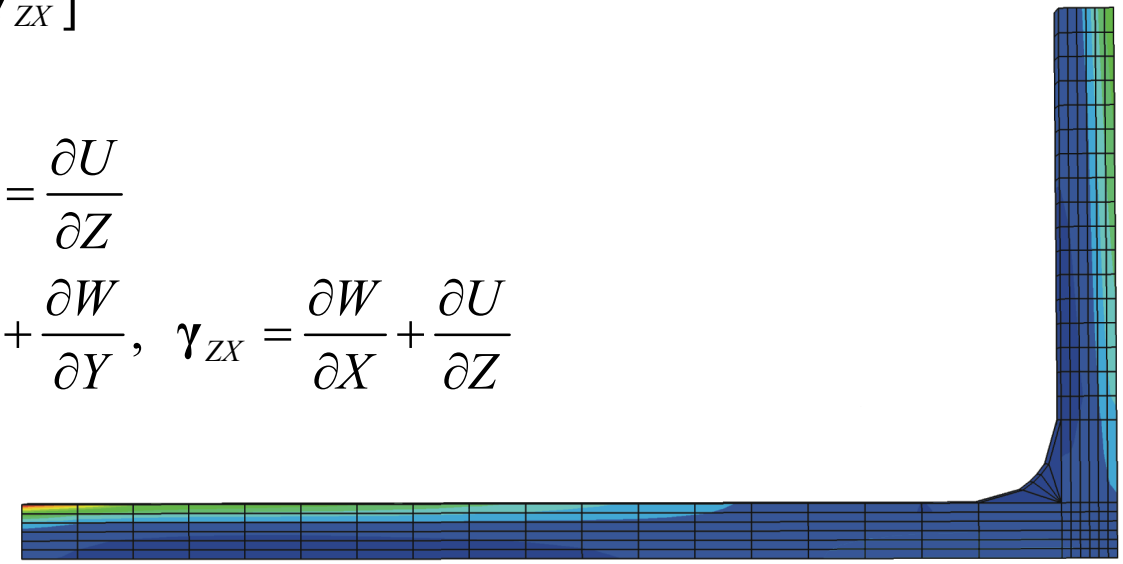
The strains corresponding to U are:

$$\boldsymbol{\varepsilon}^T=\left[\boldsymbol{\varepsilon}_{XX},\boldsymbol{\varepsilon}_{YY},\boldsymbol{\varepsilon}_{ZZ},\boldsymbol{\gamma}_{XY},\boldsymbol{\gamma}_{YZ},\boldsymbol{\gamma}_{ZX}\right]$$

with

$$\boldsymbol{\varepsilon}_{XX}=\frac{\partial U}{\partial X},\quad \boldsymbol{\varepsilon}_{YY}=\frac{\partial U}{\partial Y},\quad \boldsymbol{\varepsilon}_{ZZ}=\frac{\partial U}{\partial Z}$$

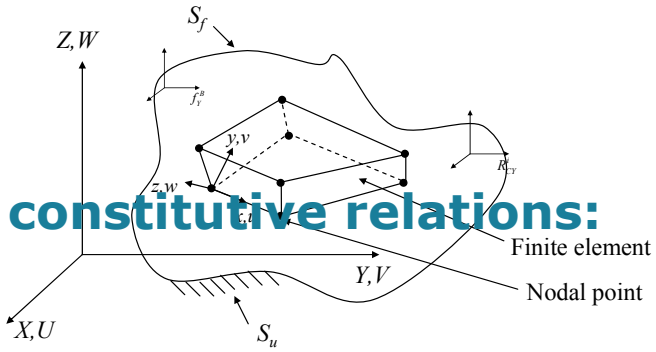
$$\boldsymbol{\gamma}_{XY}=\frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X},\quad \boldsymbol{\gamma}_{YZ}=\frac{\partial V}{\partial Z}+\frac{\partial W}{\partial Y},\quad \boldsymbol{\gamma}_{ZX}=\frac{\partial W}{\partial X}+\frac{\partial U}{\partial Z}$$



General Derivation of the FEE Equations

The principle of virtual displacements:

The stresses are obtained through the constitutive relations:



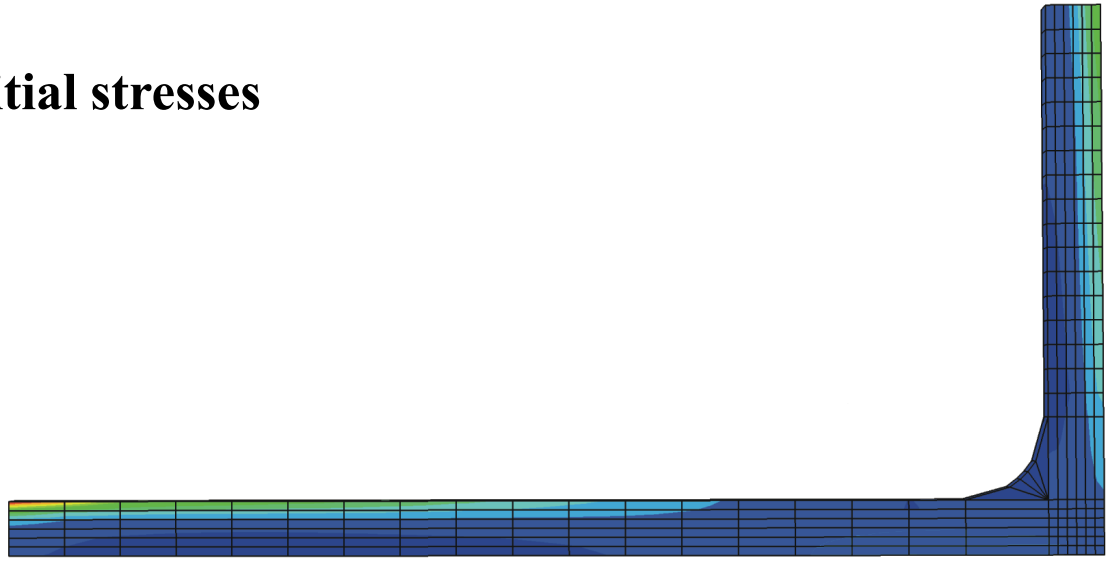
$$\boldsymbol{\tau}^T = \left[\boldsymbol{\tau}_{XX}, \boldsymbol{\tau}_{YY}, \boldsymbol{\tau}_{ZZ}, \boldsymbol{\tau}_{XY}, \boldsymbol{\tau}_{YZ}, \boldsymbol{\tau}_{ZX}\right]$$

with

$$\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\varepsilon} + \boldsymbol{\tau}'$$

Initial stresses

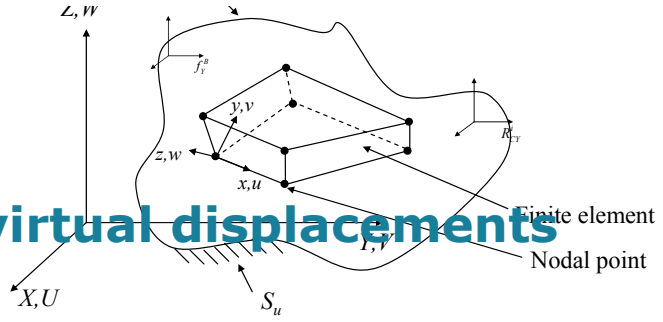
Stress-strain matrix



General Derivation of the FEE Equations

The principle of virtual displacements:

We may now express the principle of virtual displacements in a more general form:

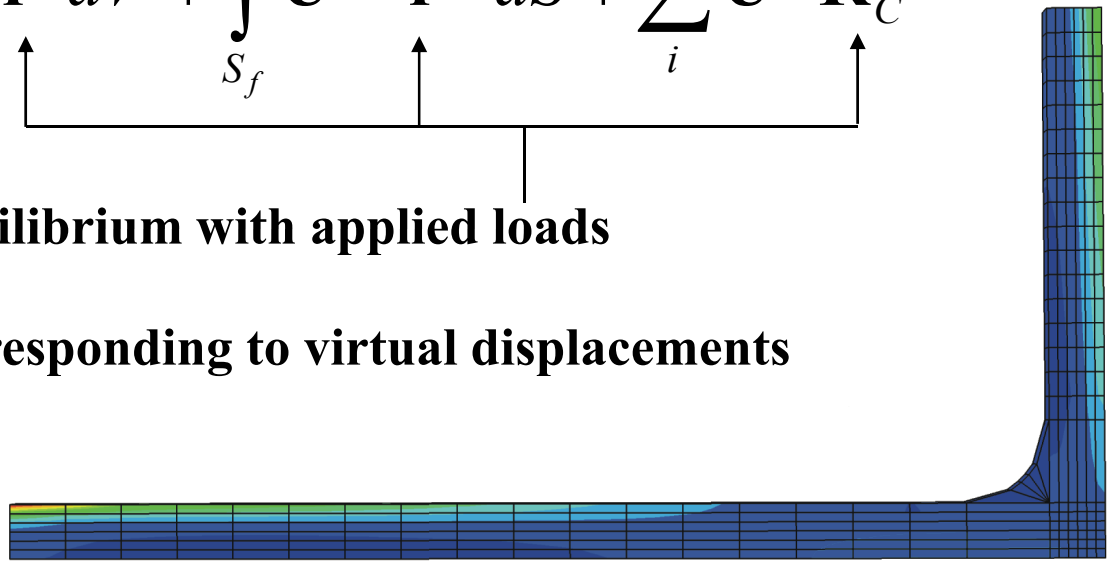


Internal virtual work **External virtual work**

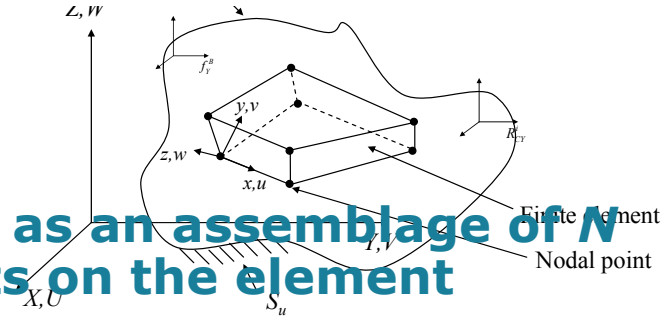
$$\int_V \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\tau} dV = \int_V \bar{\mathbf{U}}^T \mathbf{f}^B dV + \int_{S_f} \bar{\mathbf{U}}^{S_f T} \mathbf{f}^{S_f} dS + \sum_i \bar{\mathbf{U}}^{iT} \mathbf{R}_C^i$$

Stresses in equilibrium with applied loads

Virtual strains corresponding to virtual displacements



General Derivation of the FEE Equations



Finite Element Equations:

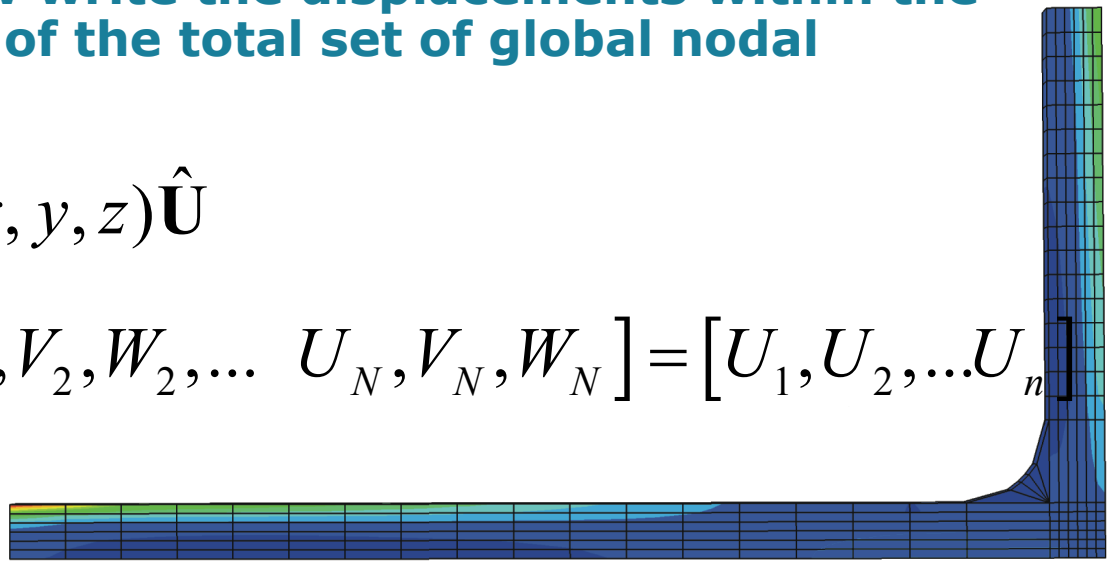
We now consider the volume modeled as an assemblage of *N* elements connected in the nodal points on the element boundaries

The displacements within the individual elements are measured in a convenient local coordinate system *x,y,z*

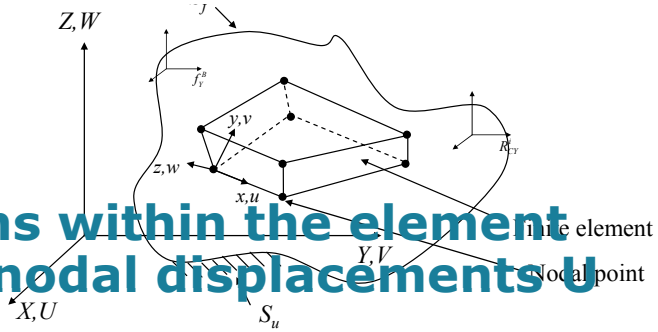
For element *m* we now write the displacements within the element as a function of the total set of global nodal displacements U

$$\mathbf{u}^{(m)}(x,y,z) = \mathbf{H}^{(m)}(x,y,z)\hat{\mathbf{U}}$$

$$\hat{\mathbf{U}}^T = \left[U_1, V_1, W_1, \;\; U_2, V_2, W_2, \ldots \;\; U_N, V_N, W_N\right] = \left[U_1, U_2, \ldots U_n\right]$$



General Derivation of the FEE Equations



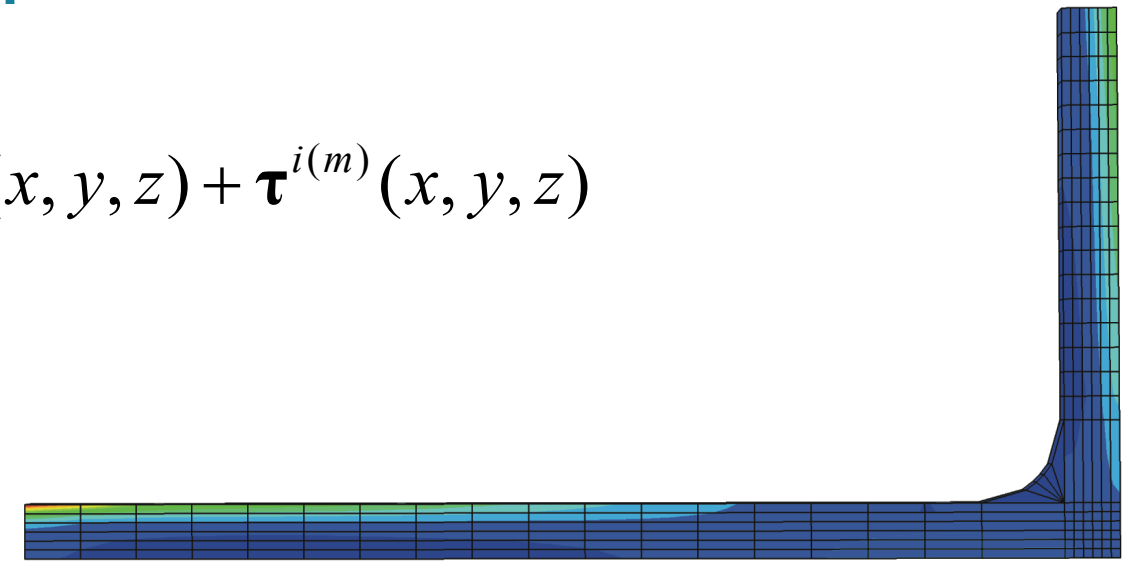
Finite Element Equations:

For element *m* we now write the strains within the element as a function of the total set of global nodal displacements U

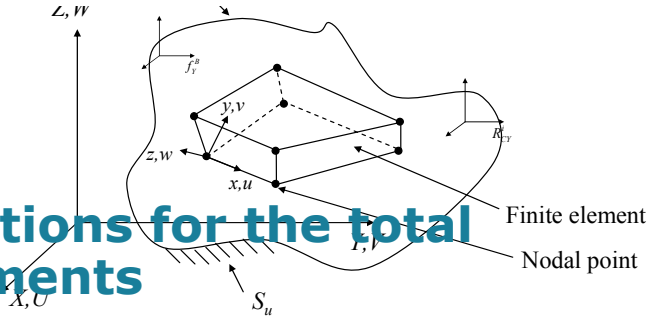
$$\boldsymbol{\varepsilon}^{(m)}(x,y,z) = \mathbf{B}^{(m)}(x,y,z)\hat{\mathbf{U}}$$

The stresses are then:

$$\boldsymbol{\tau}^{(m)}(x,y,z) = \mathbf{C}\boldsymbol{\varepsilon}^{(m)}(x,y,z) + \boldsymbol{\tau}^{i(m)}(x,y,z)$$



General Derivation of the FEE Equations



Finite Element Equations:

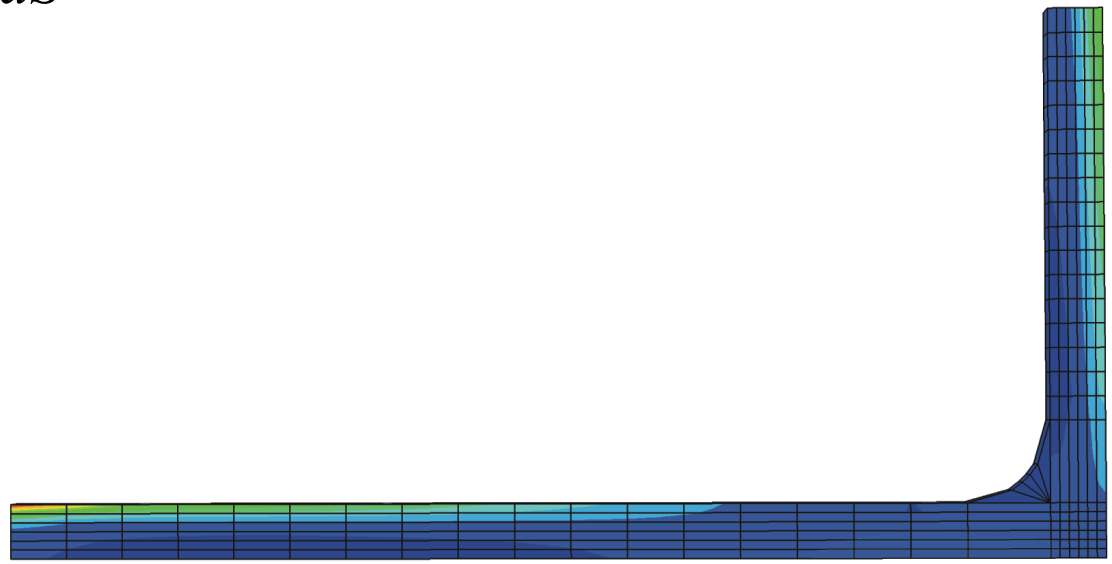
We can now write the equlibrium equations for the total volume by summing up over the *N* elements

$$\sum_{m=1}^{N} \int_{V^{(m)}} \bar{\boldsymbol{\varepsilon}}^{(m)T} \boldsymbol{\tau}^{(m)} dV^{(m)} = \sum_{m=1}^{N} \int_{V^{(m)}} \bar{\mathbf{U}}^{(m)T} \mathbf{f}^{B(m)} dV^{(m)} +$$

$$\sum_{m=1}^{N} \int_{S_{f1}^{(m)}, S_{f1}^{(m)}, \ldots} \bar{\mathbf{U}}^{S_f(m)T} \mathbf{f}^{S_f(m)} dS^{(m)} -$$

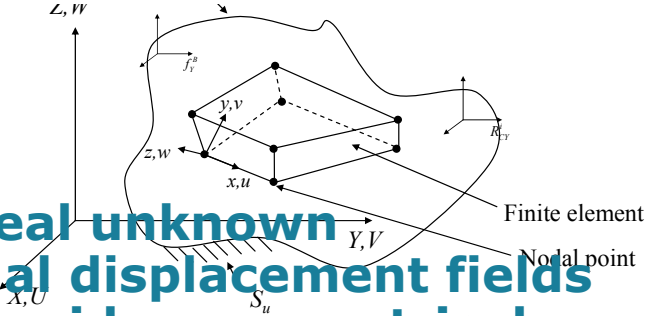
$$\sum_{m=1}^{N} \int_{V^{(m)}} \bar{\boldsymbol{\varepsilon}}^{(m)T} \boldsymbol{\tau}^{i(m)} dV^{(m)} +$$

$$\sum_{i} \bar{\mathbf{U}}^{iT} \mathbf{R}_C^i$$



General Derivation of the FEE Equations

Finite Element Equations:

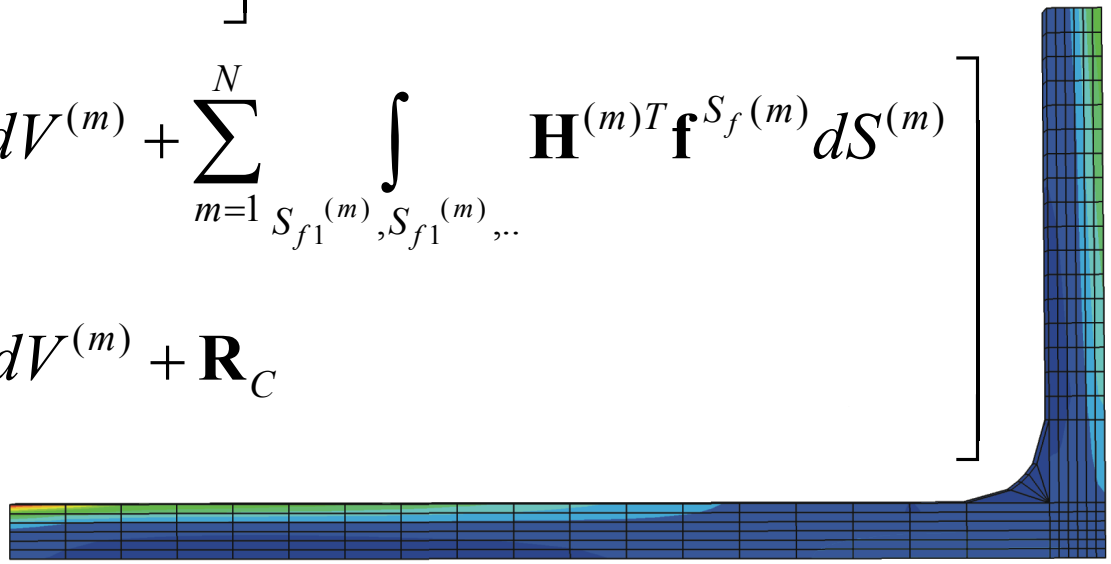


As a next step we represent both the real unknown displacement fields as well as the virtual displacement fields through the interpolation functions (provides symmetrical stiffness matrixes ☺)

$$\overline{\hat{\mathbf{U}}}^T \left[\sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T} \mathbf{C}^{(m)} \mathbf{B}^{(m)} dV^{(m)} \right] \overline{\mathbf{U}} =$$

$$\overline{\hat{\mathbf{U}}}^T \left[\sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{H}^{(m)T} \mathbf{f}^{B(m)} dV^{(m)} + \sum_{m=1}^{N} \int_{S_{f1}{}^{(m)}, S_{f1}{}^{(m)}, ..} \mathbf{H}^{(m)T} \mathbf{f}^{S_f(m)} dS^{(m)} \right.$$

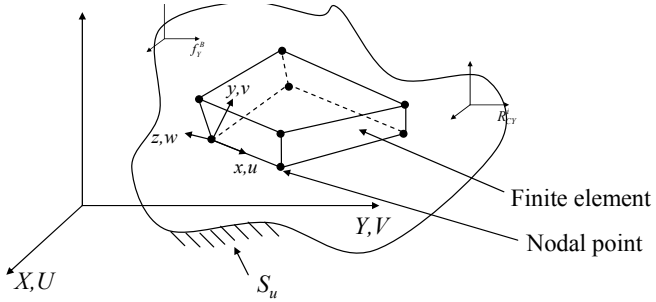
$$\left. - \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T} \boldsymbol{\tau}^{(m)} dV^{(m)} + \mathbf{R}_C \right]$$



General Derivation of the FEE Equations

Finite Element Equations:

Now we may finally simplify as



$$\mathbf{KU} = \mathbf{R}$$

$$\mathbf{K} = \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T} \mathbf{C}^{(m)} \mathbf{B}^{(m)} dV^{(m)}$$

These are the finite element equations to be solved ☺

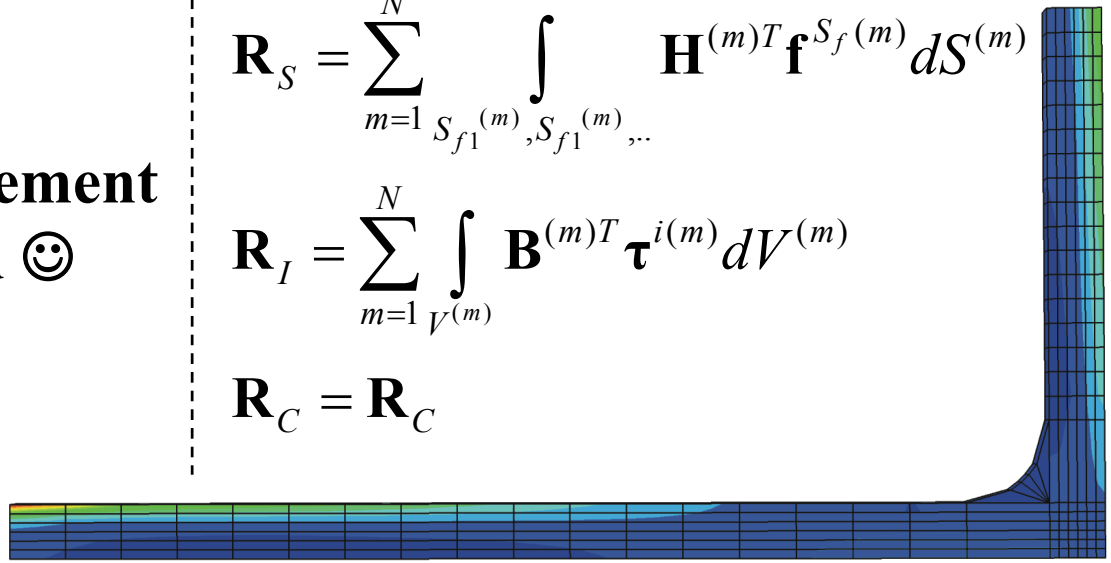
$$\mathbf{R} = \mathbf{R}_B + \mathbf{R}_S - \mathbf{R}_I + \mathbf{R}_C$$

$$\mathbf{R}_B = \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{H}^{(m)T} \mathbf{f}^{B(m)} dV^{(m)}$$

$$\mathbf{R}_S = \sum_{m=1}^{N} \int_{S_{f1}^{(m)}, S_{f1}^{(m)}, ..} \mathbf{H}^{(m)T} \mathbf{f}^{S_f(m)} dS^{(m)}$$

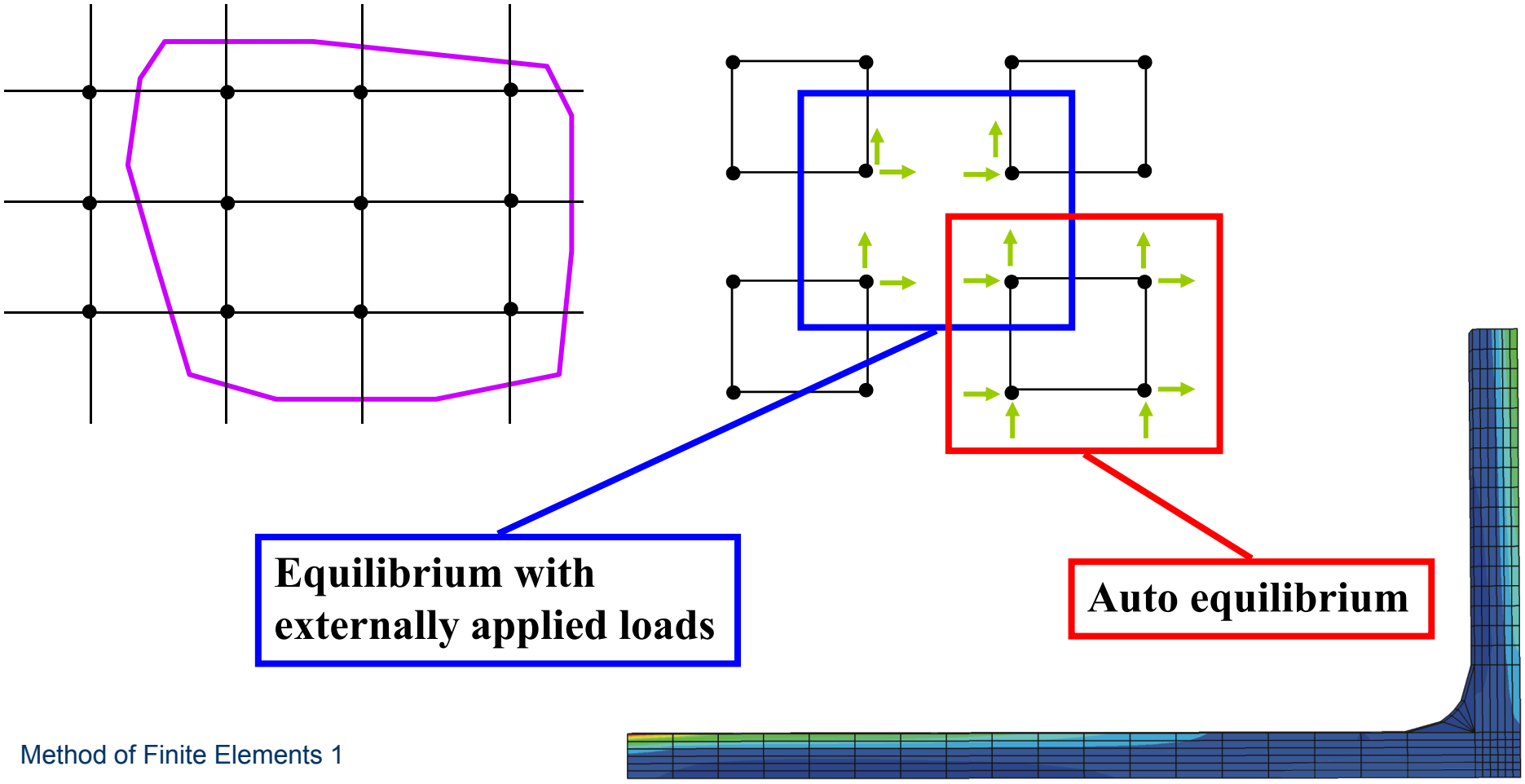
$$\mathbf{R}_I = \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T} \boldsymbol{\tau}^{i(m)} dV^{(m)}$$

$$\mathbf{R}_C = \mathbf{R}_C$$



General Derivation of the FEE Equations

The assumption about stress equilibrium:



General Derivation of the FEE Equations

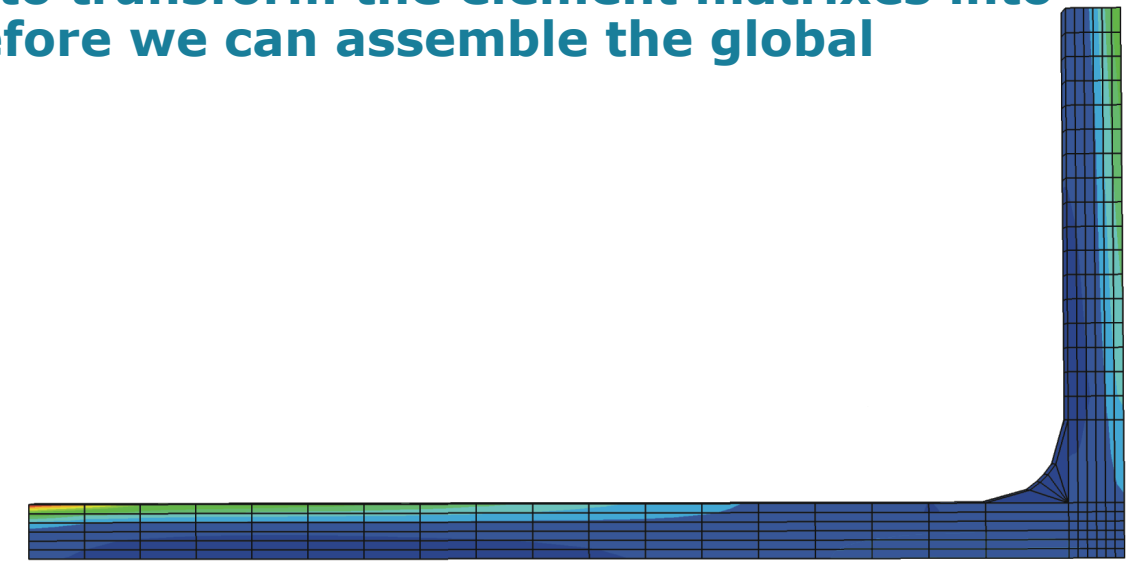
Local to global coordinate transformations:

It is often more convenient to define the element stiffness relations and to calculate their contributions to the load vector in a local coordinate system – this is often specific for each individual element.

$$\text{local} \quad \tilde{\mathbf{u}}$$

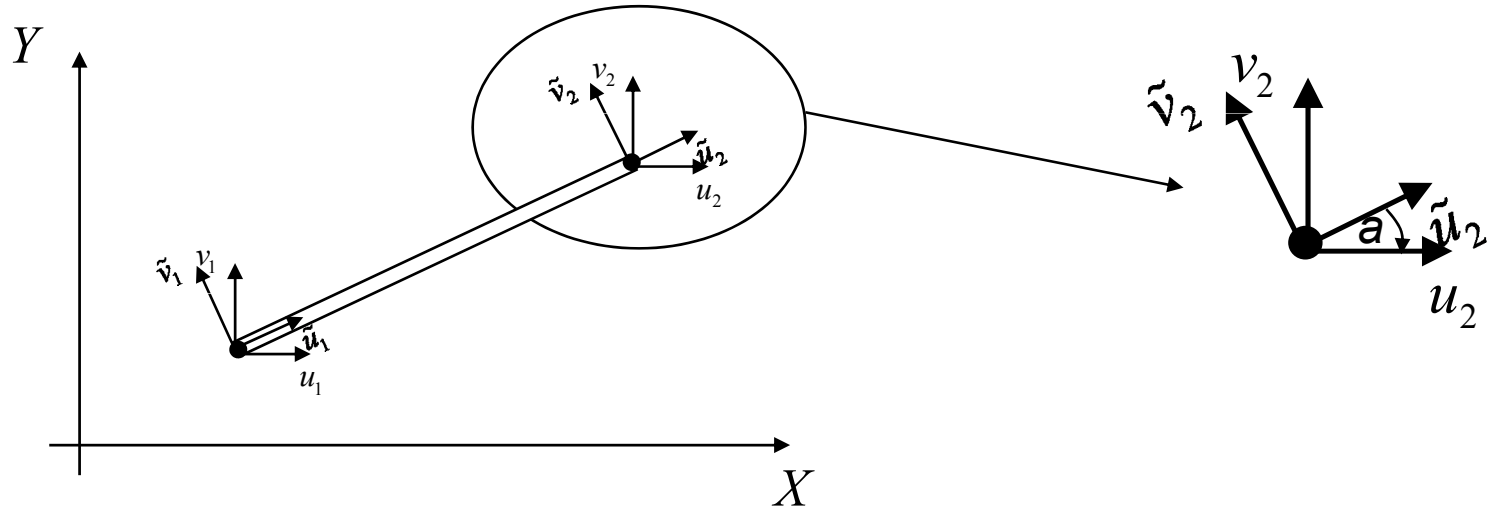
In this case we need to transform the element matrixes into global coordinates before we can assemble the global stiffness relation

$$\tilde{\mathbf{u}} = \mathbf{T}\hat{\mathbf{u}}$$



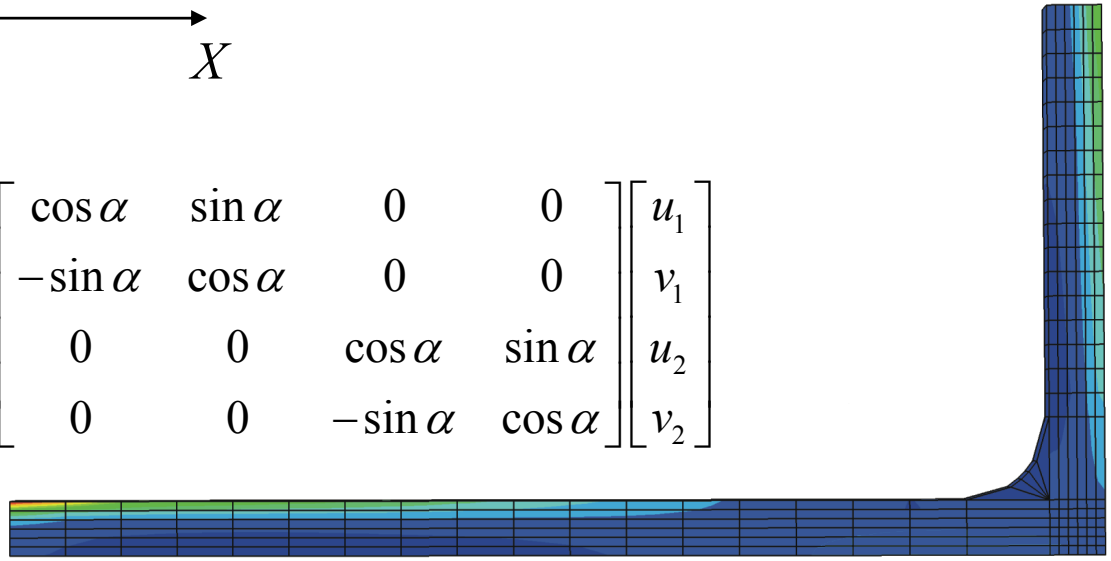
General Derivation of the FEE Equations

Local to global coordinate transformations:



$$\tilde{\mathbf{u}} = \mathbf{T}\hat{\mathbf{u}}$$

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{v}_1 \\ \tilde{u}_2 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$



General Derivation of the FEE Equations

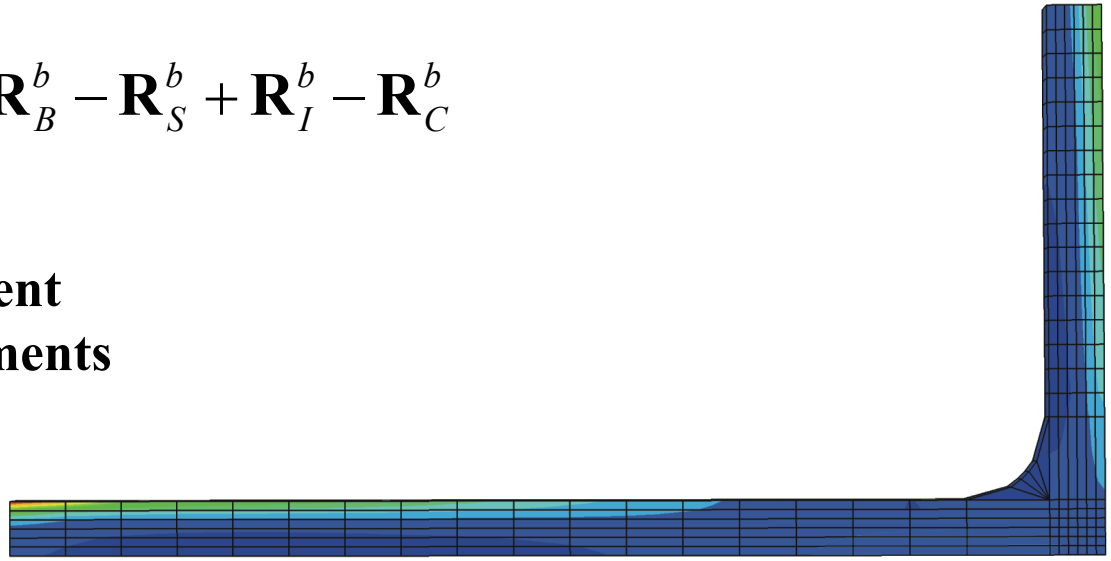
Imposition of displacement boundary conditions:

The boundary conditions in regard to displacements may be included in the equations system as:

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{U}_a \\ \mathbf{U}_b \end{bmatrix} = \begin{bmatrix} \mathbf{R}_a \\ \mathbf{R}_b \end{bmatrix} \Rightarrow \mathbf{K}_{aa}\mathbf{U}_a = \mathbf{R}_a - \mathbf{K}_{ab}\mathbf{U}_b$$

$$\mathbf{R}_b = \mathbf{K}_{ba}\mathbf{U}_a + \mathbf{K}_{bb}\mathbf{R}_b - \mathbf{R}_B^b - \mathbf{R}_S^b + \mathbf{R}_I^b - \mathbf{R}_C^b$$

U_a: unknown displacement
U_b: prescribed displacements
$R_{b:}$ reactions



General Derivation of the FEE Equations

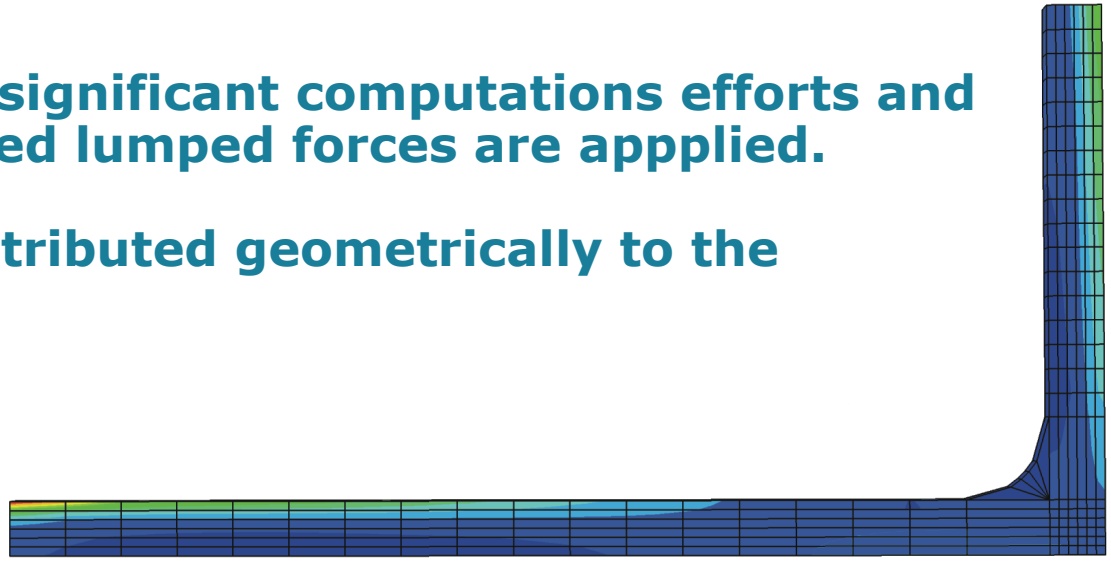
Lumping of structural properties and loads:

Using the formulations we have seen till now all forces acting on the volume are integrated up using the assumed displacement fields represented in terms of generalized coordinate models as weighing functions

The result of this is that the forces not already acting in the nodes are equivalated by nodal forces.

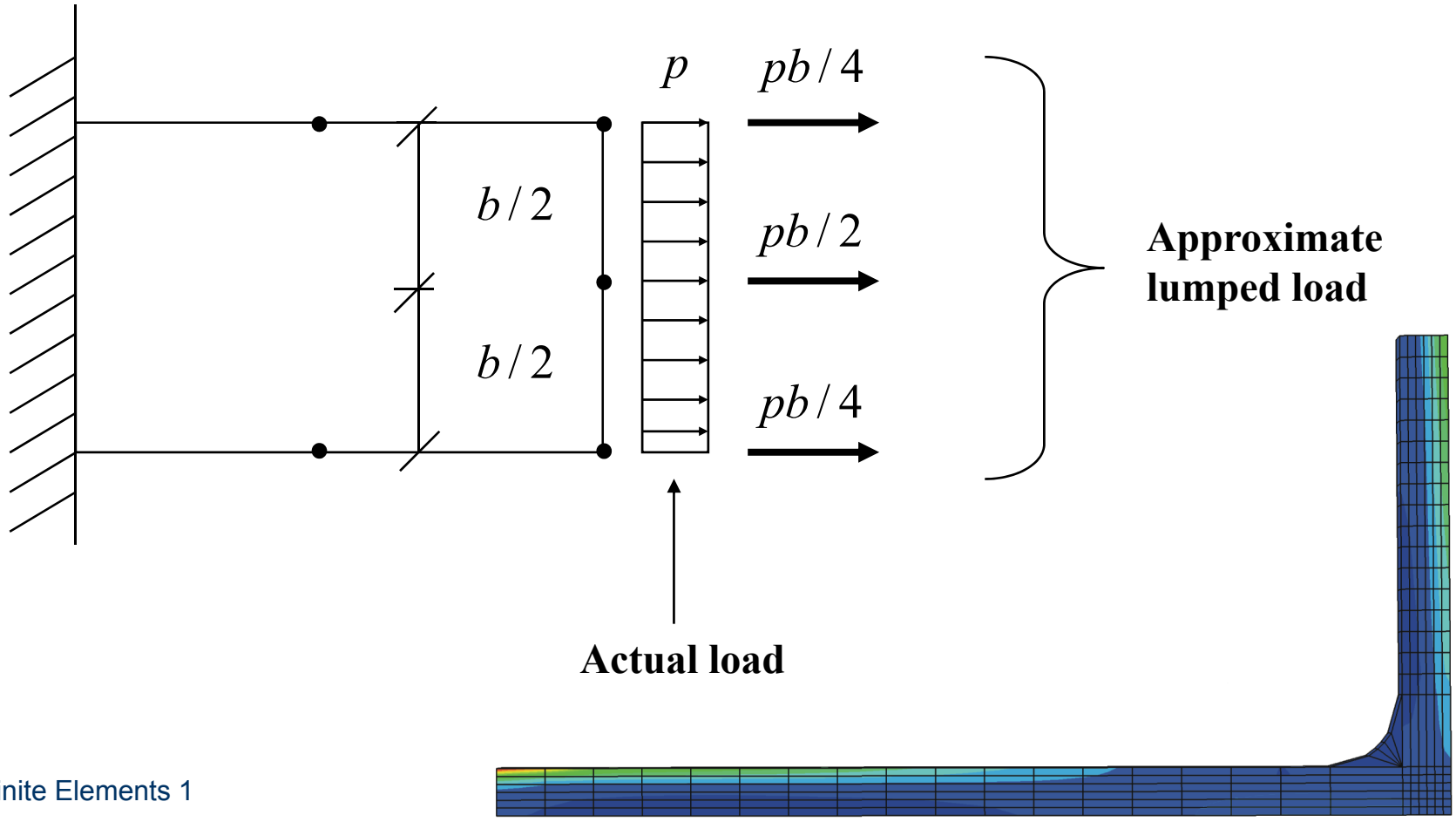
This process requires significant computations efforts and for this reason so-called lumped forces are appplied.

Lumped forces are distributed geometrically to the adjacent nodes.



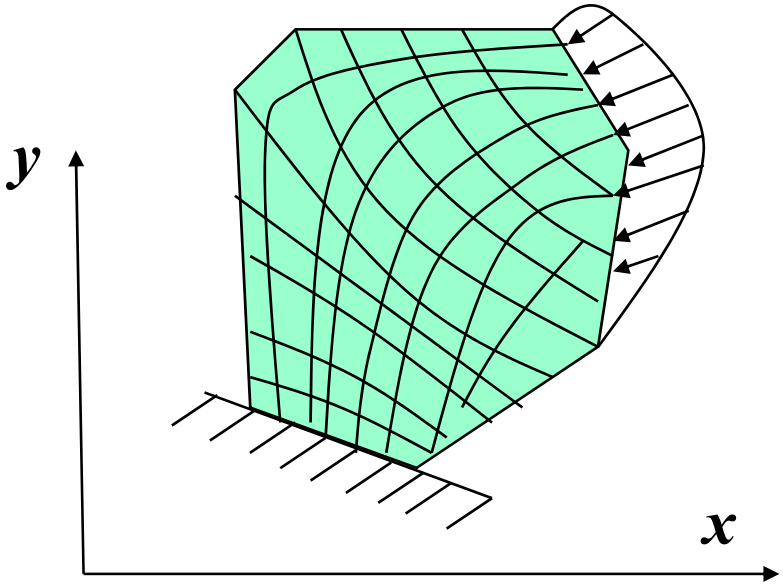
General Derivation of the FEE Equations

Lumping of structural properties and loads:

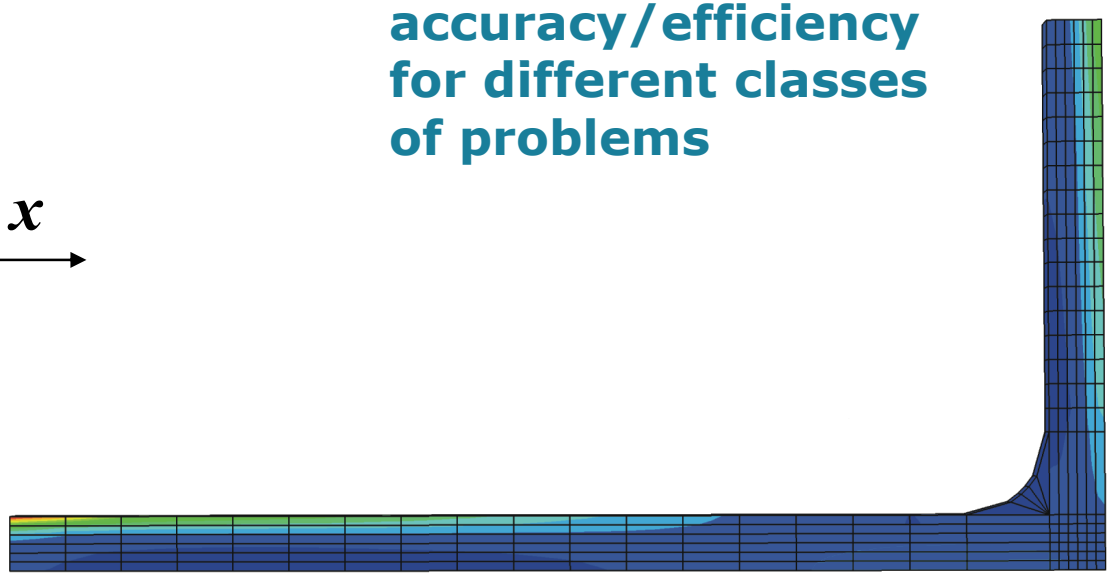


Generalized Coordinate Models

Having established the fundamental equations at element as well as system level

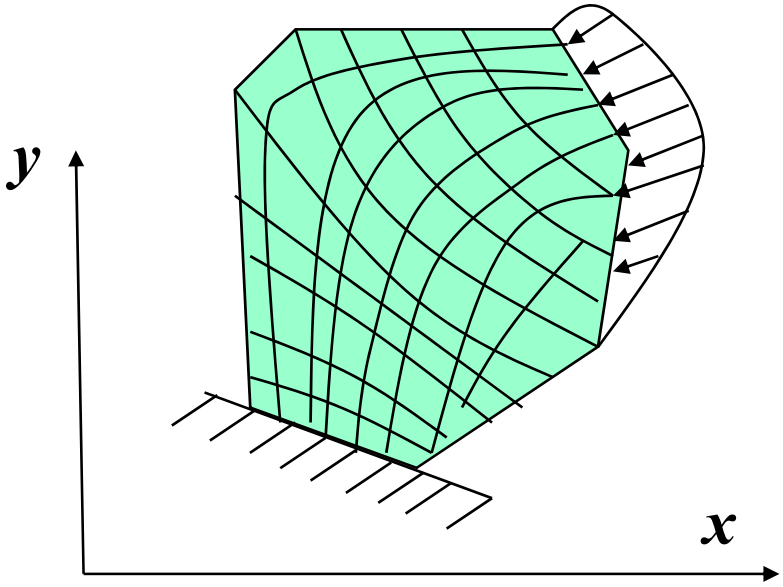


What remains is to establish the appropriate solutions at element level and to optimize these – in terms of accuracy/efficiency for different classes of problems

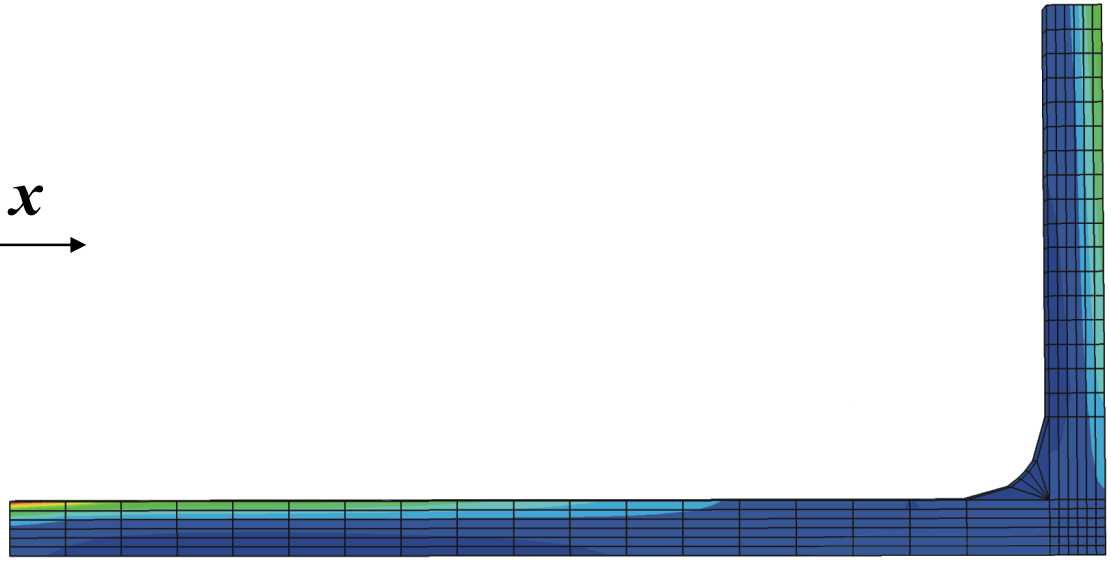


Generalized Coordinate Models

Fundamentally we need to identify the different cases to be considered – and their characteristics

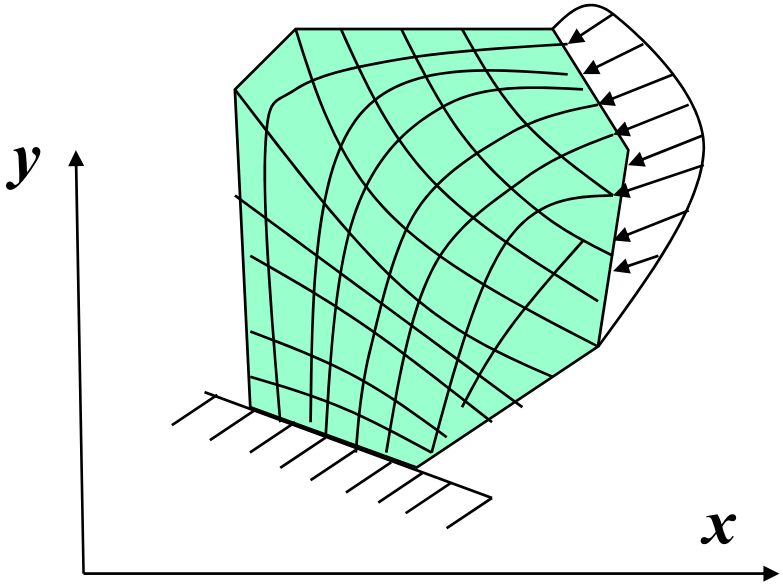


and then we need to solve these situation in the easiest way possible!

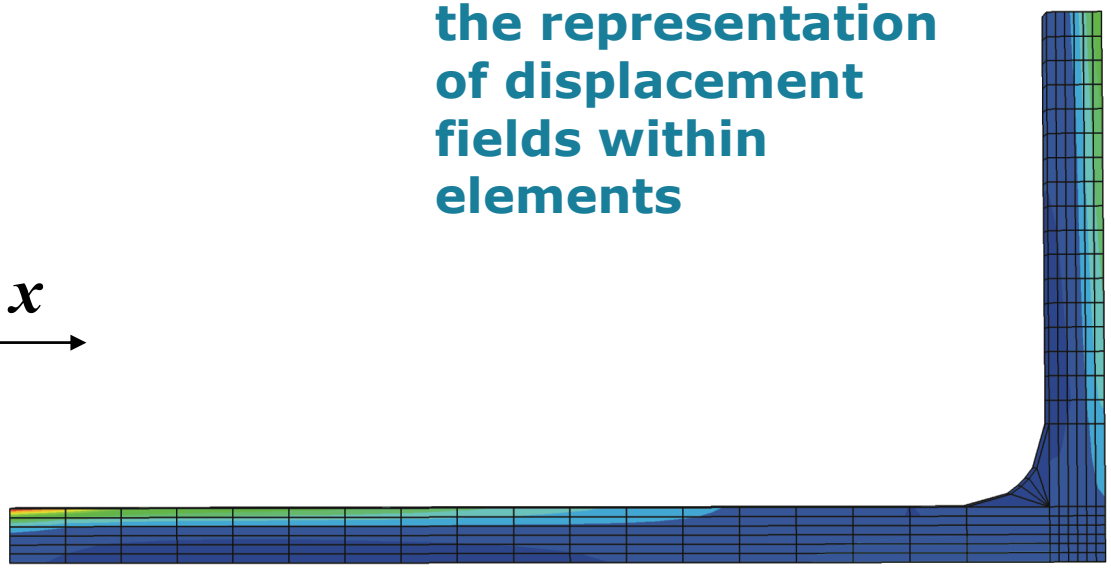


Generalized Coordinate Models

The fundamental idea is that we approximate the displacement fields within smaller parts of the system - elements – aiming to represent the real variation of these appropriately accurate in a simplified manner.



For this reason – a major part of the Method of Finite Elements concerns the representation of displacement fields within elements



Generalized Coordinate Models

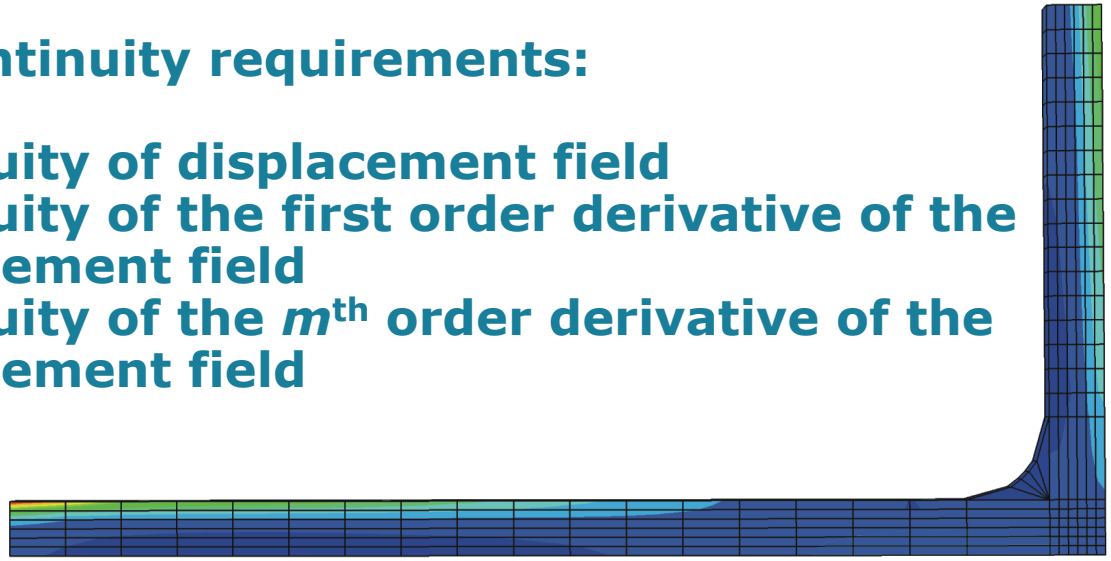
Requirements to shape functions:

First of all the functions which we want to represent (displacements and their derivatives) need to be able to represent the physics of the type of problem we are aiming to model within the individual elements.

Secondly we also need to be concerned with continuity over the borders of the elements.

We here introduce continuity requirements:

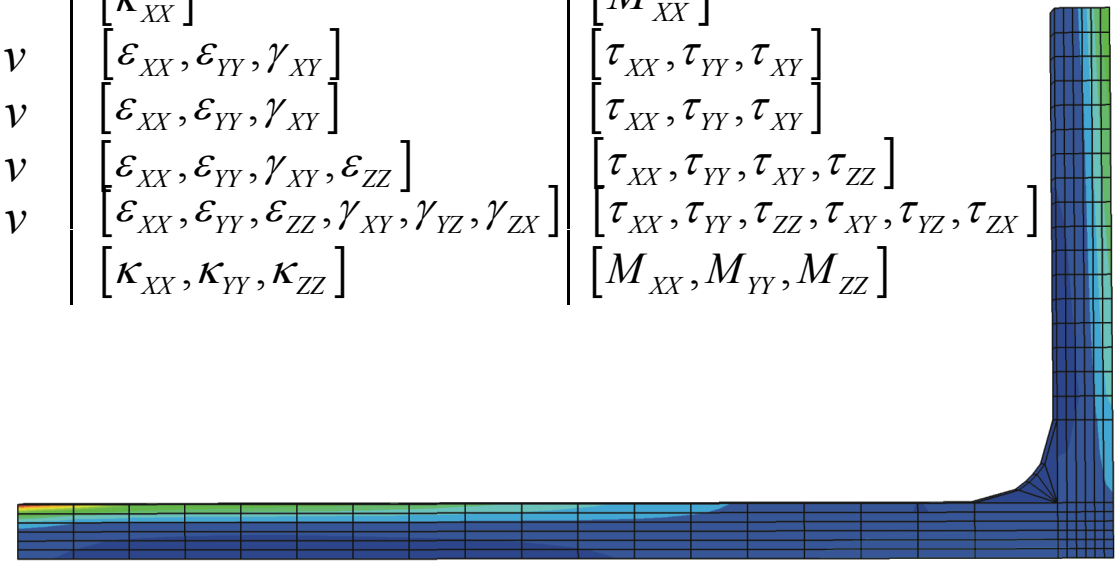
C^0 continuity	**Continuity of displacement field**
C^1 continuity	**Continuity of the first order derivative of the displacement field**
C^m continuity	**Continuity of the m^{th} order derivative of the displacement field**



Generalized Coordinate Models

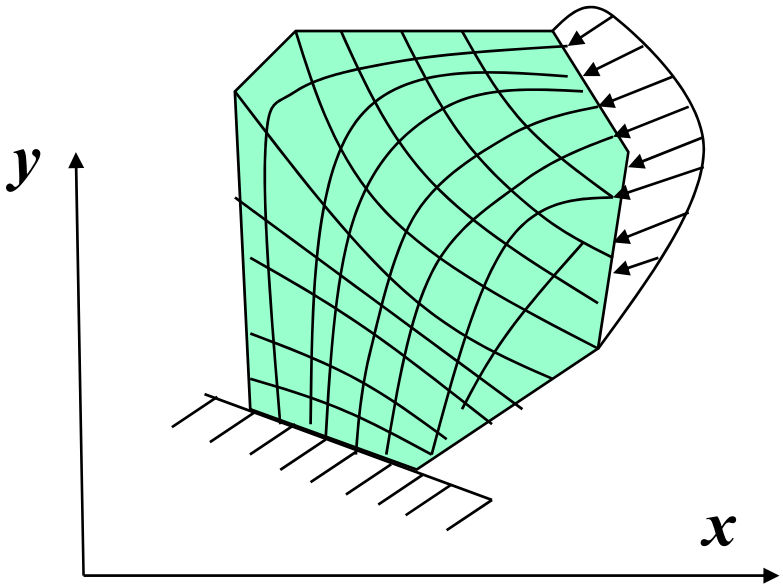
From the theory of elasticity we e.g. know what types of response is required to represent the characteristics of beams, plates, shells and solids under different loading conditions.

Element type		$\mathbf{u}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{\tau}$
1	Bar	u	$[\varepsilon_{XX}]$	$[\tau_{XX}]$
2	Beam	w	$[\kappa_{XX}]$	$[M_{XX}]$
3	Plane stress	u,v	$[\varepsilon_{XX},\varepsilon_{YY},\gamma_{XY}]$	$[\tau_{XX},\tau_{YY},\tau_{XY}]$
4	Plane strain	u,v	$[\varepsilon_{XX},\varepsilon_{YY},\gamma_{XY}]$	$[\tau_{XX},\tau_{YY},\tau_{XY}]$
5	Axisymmetric	u,v	$[\varepsilon_{XX},\varepsilon_{YY},\gamma_{XY},\varepsilon_{ZZ}]$	$[\tau_{XX},\tau_{YY},\tau_{XY},\tau_{ZZ}]$
6	Three-dimensional	u,v	$[\varepsilon_{XX},\varepsilon_{YY},\varepsilon_{ZZ},\gamma_{XY},\gamma_{YZ},\gamma_{ZX}]$	$[\tau_{XX},\tau_{YY},\tau_{ZZ},\tau_{XY},\tau_{YZ},\tau_{ZX}]$
7	Plate bending	w	$[\kappa_{XX},\kappa_{YY},\kappa_{ZZ}]$	$[M_{XX},M_{YY},M_{ZZ}]$



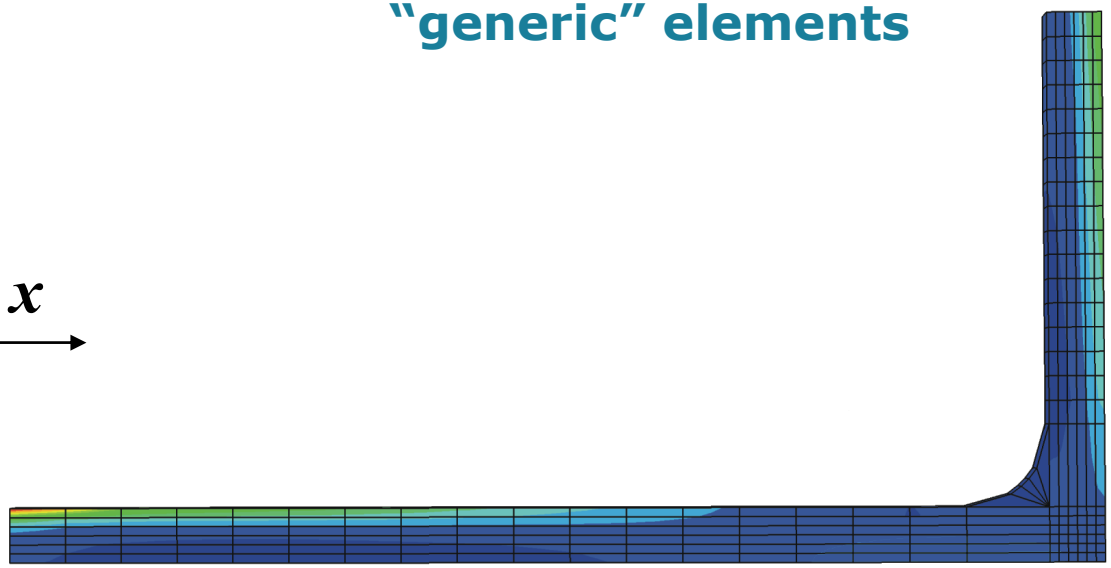
Generalized Coordinate Models

The real issue being to establish a cost efficient way of developing the element stiffness matrixes



To this end the general concept is to develop

"generic" elements



Generalized Coordinate Models

Generalized coordinate models:

The principle behind these models is:

Formulate displacement field in terms of polynomials

one-dimensional

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \cdots$$

two-dimensional

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 xy + \alpha_4 x^2 + \cdots$$

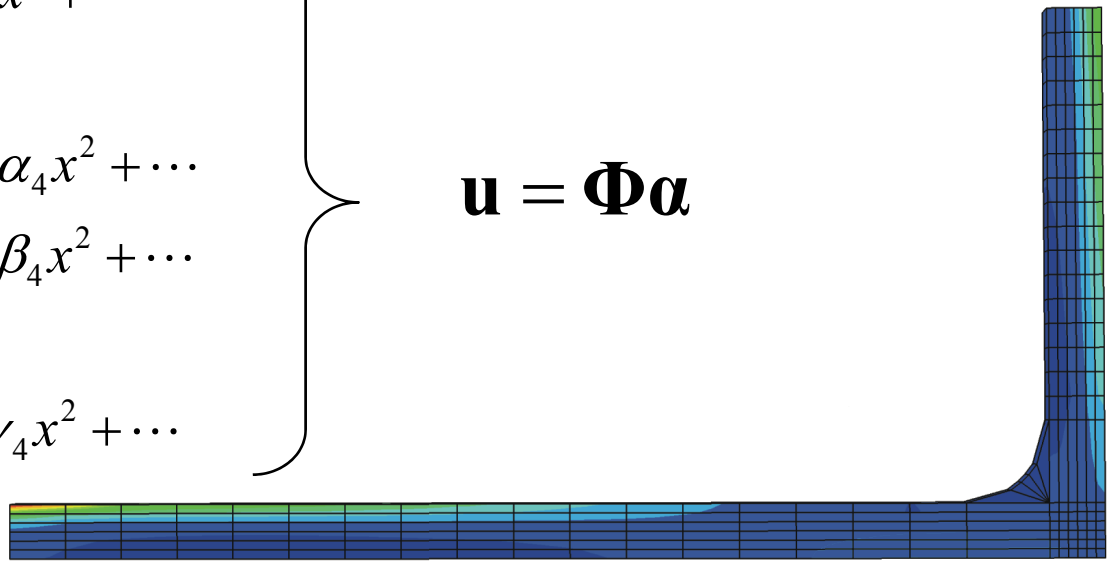
$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 xy + \beta_4 x^2 + \cdots$$

plate bending

$$w(x,y) = \gamma_1 + \gamma_2 x + \gamma_3 xy + \gamma_4 x^2 + \cdots$$

$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$: Genenralized coordinates

$$\mathbf{u} = \boldsymbol{\Phi}\boldsymbol{\alpha}$$



Generalized Coordinate Models

Generalized coordinate models:

Next step is to relate the generalized coordinates to the nodal displacements:

We insert the nodal coordinates into $\mathbf{u} = \Phi\boldsymbol{\alpha}$

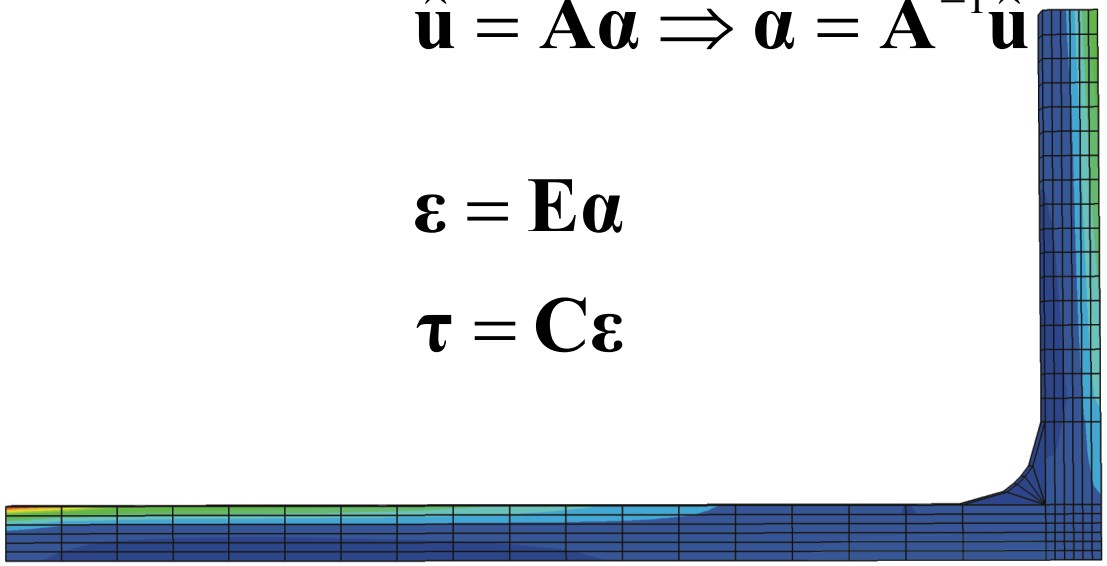
and get:

$$\hat{\mathbf{u}} = \mathbf{A}\boldsymbol{\alpha} \Rightarrow \boldsymbol{\alpha} = \mathbf{A}^{-1}\hat{\mathbf{u}}$$

Now we can obtain

$$\boldsymbol{\varepsilon} = \mathbf{E}\boldsymbol{\alpha}$$

$$\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\varepsilon}$$



Generalized Coordinate Models

Shape functions:

In general we may write the approximate relation between the field representation (displacements/strains) and the nodal displacements as:

$$u(x,y) = \mathbf{H}^T(x,y)\hat{\mathbf{u}}$$

We consider an element with n nodes

Displacements in the u-direction

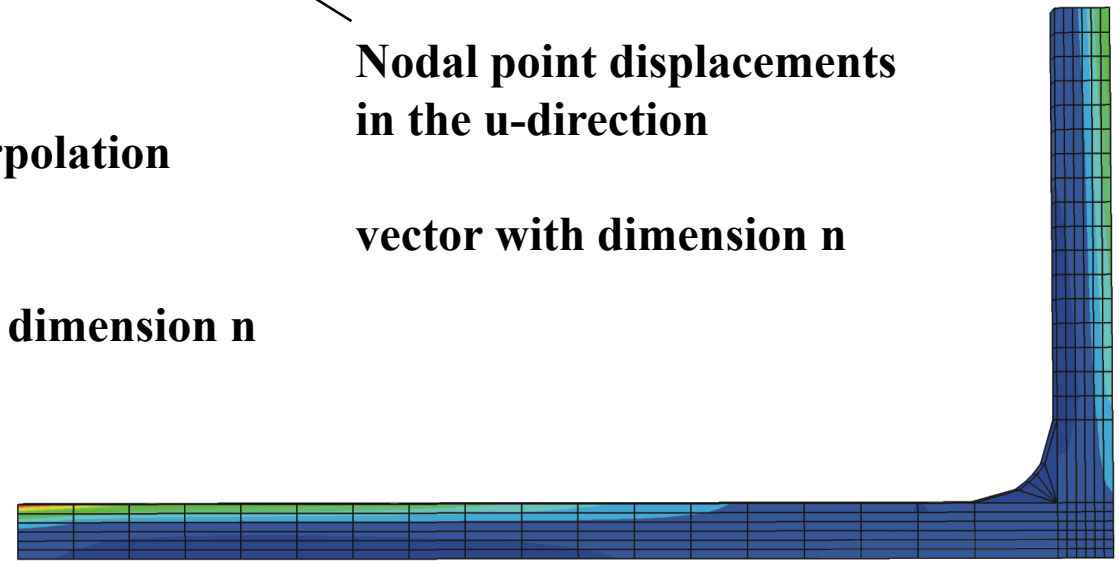
scalar

Shape/interpolation functions

vector with dimension n

Nodal point displacements in the u-direction

vector with dimension n



Generalized Coordinate Models

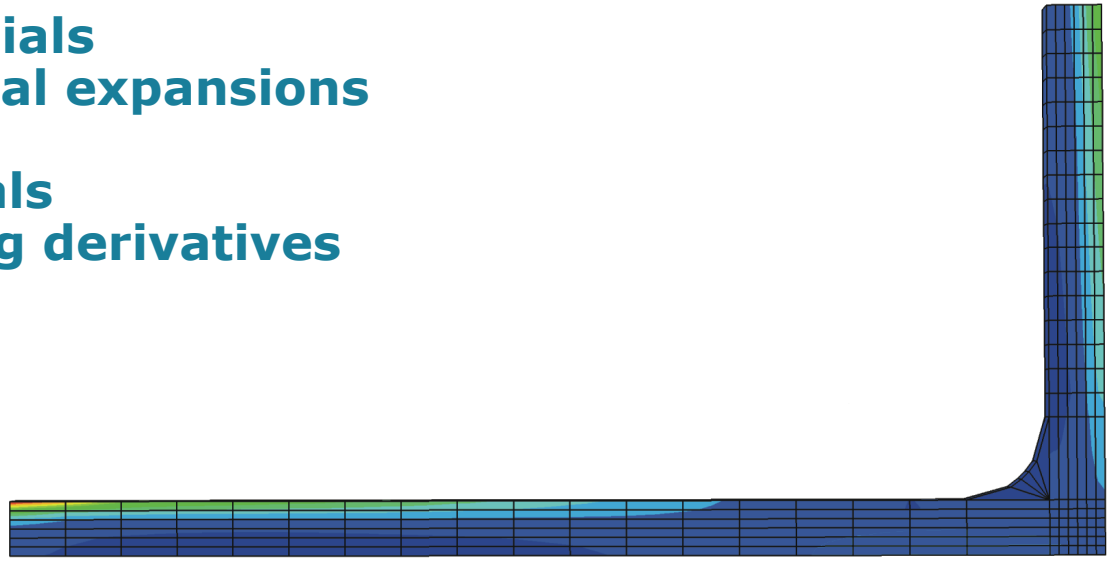
Shape functions:

Polynomials are usually applied for the development of shape functions (polynomials are easily differentiated analytically)

- **Langrange polynomials**
  complete polynomial expansions

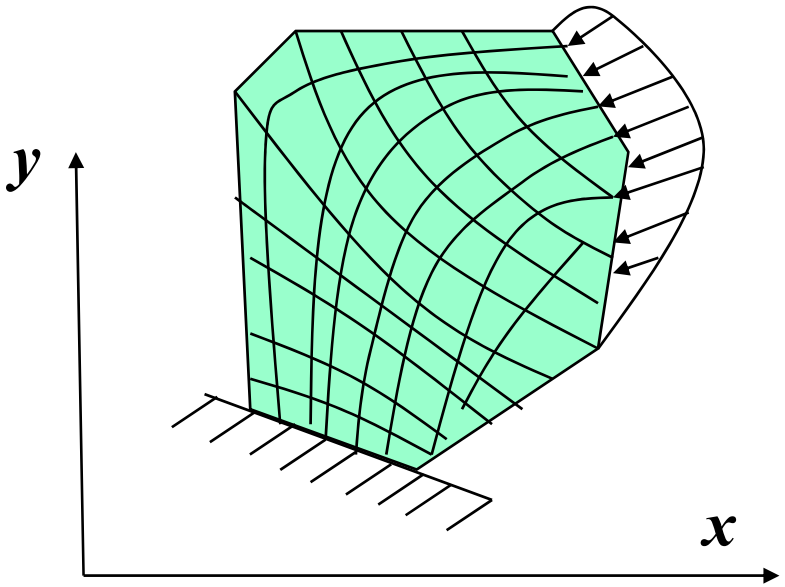
- **Serendipity polynomials**
  incomplete polynomial expansions

- **Hermitian polynomials**
  polynomials including derivatives

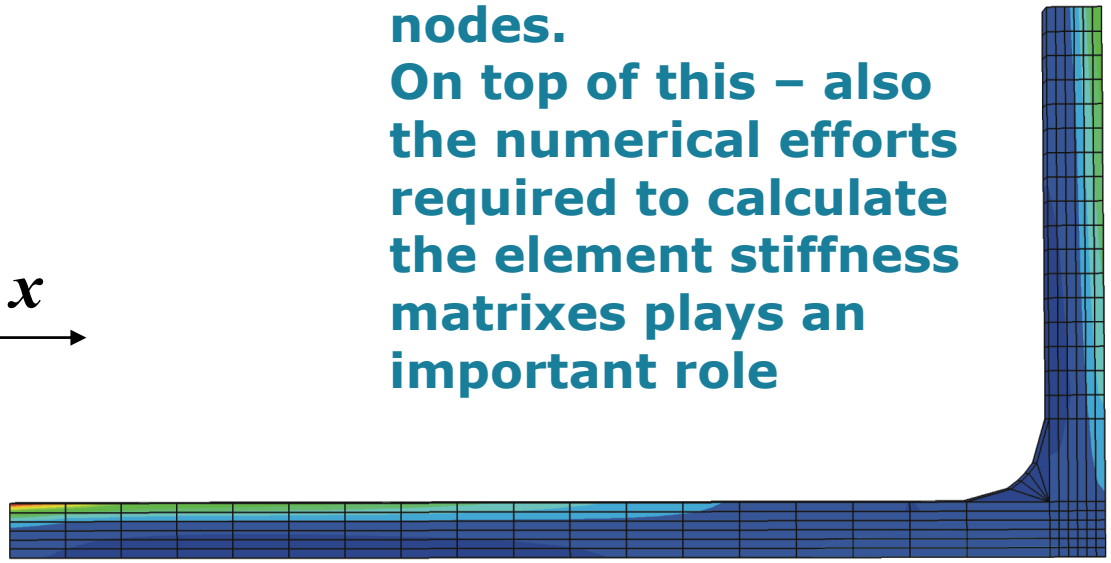


Generalized Coordinate Models

To achieve "generic" elements is important – but also numerical efficiency is important.



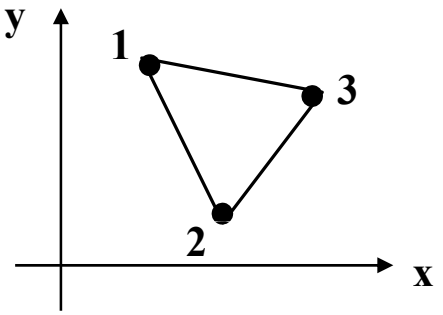
Numerical efficiency relates to the degree to which the elements are able to represent the real displacement fields measured in terms of number of nodes.
On top of this – also the numerical efforts required to calculate the element stiffness matrixes plays an important role



Generalized Coordinate Models

Shape functions (example):

Three node triangle element



we assume a complete first order polynomial

$$u(x,y) = a_1 + a_2 x + a_3 y = \mathbf{\Phi\alpha}$$

we can now relate this displacement field to the nodal displacements

$$\hat{\mathbf{u}} = \hat{\mathbf{\Phi}}\boldsymbol{\alpha}$$

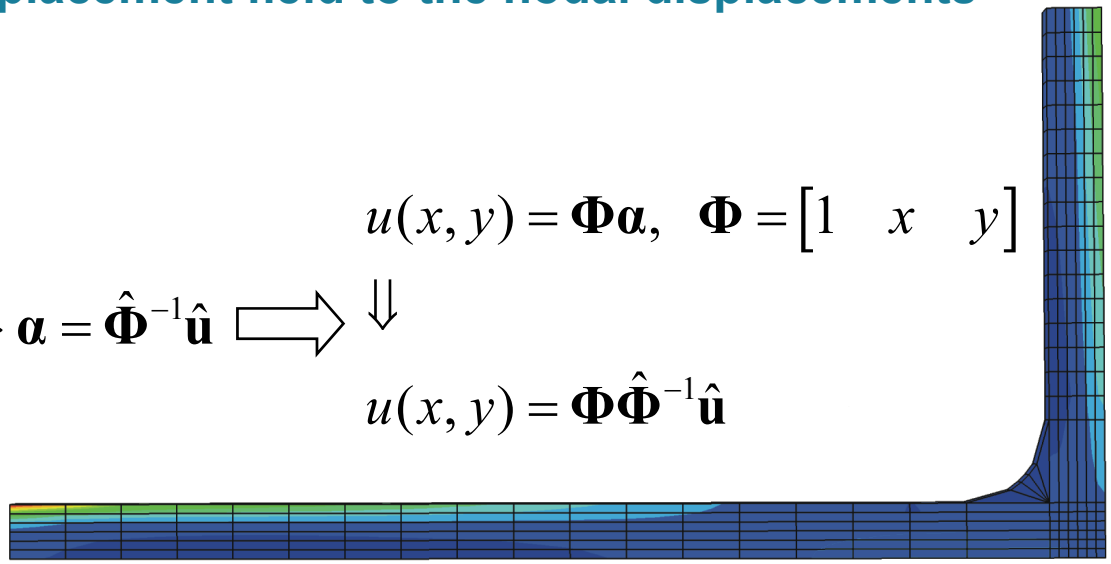
$$\Downarrow$$

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Rightarrow \boldsymbol{\alpha} = \hat{\mathbf{\Phi}}^{-1}\hat{\mathbf{u}} \Longrightarrow$$

$$u(x,y) = \mathbf{\Phi\alpha}, \quad \mathbf{\Phi} = \begin{bmatrix} 1 & x & y \end{bmatrix}$$

$$\Downarrow$$

$$u(x,y) = \mathbf{\Phi}\hat{\mathbf{\Phi}}^{-1}\hat{\mathbf{u}}$$



Generalized Coordinate Models

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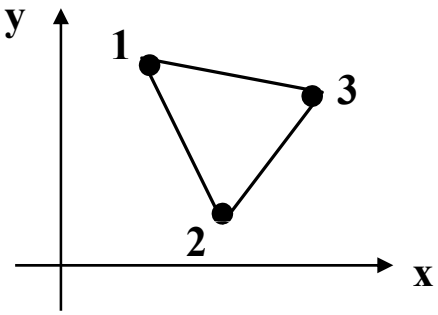

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$$\Downarrow$$

$$u(x,y) = \mathbf{\Phi}\hat{\mathbf{\Phi}}^{-1}\hat{\mathbf{u}}$$

$$\Downarrow$$

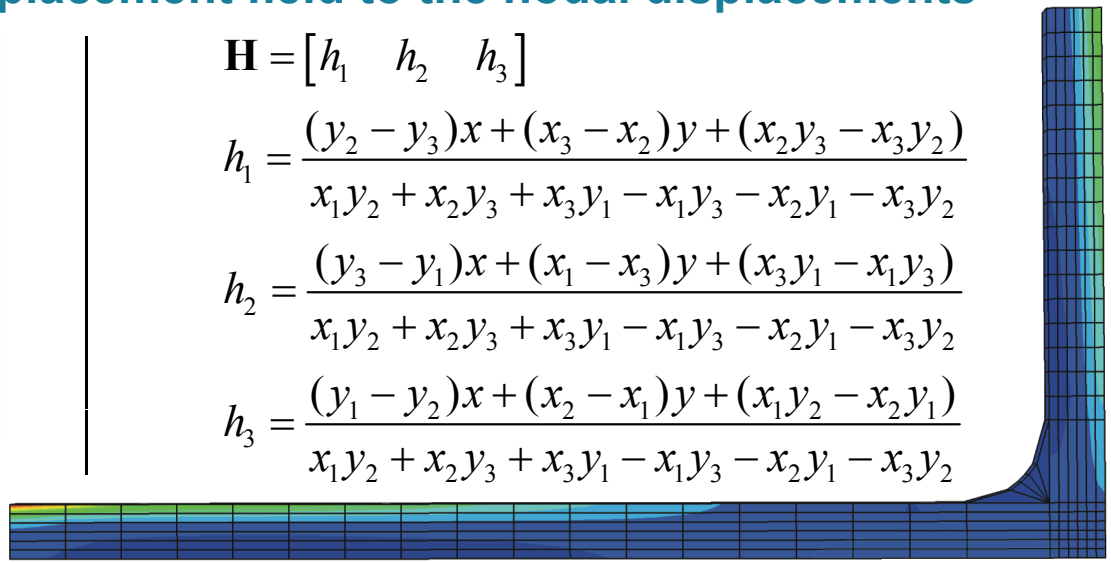
$$u(x,y) = \mathbf{H}\hat{\mathbf{u}}$$

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}$$

$$h_1 = \frac{(y_2 - y_3)x + (x_3 - x_2)y + (x_2 y_3 - x_3 y_2)}{x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_2 y_1 - x_3 y_2}$$

$$h_2 = \frac{(y_3 - y_1)x + (x_1 - x_3)y + (x_3 y_1 - x_1 y_3)}{x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_2 y_1 - x_3 y_2}$$

$$h_3 = \frac{(y_1 - y_2)x + (x_2 - x_1)y + (x_1 y_2 - x_2 y_1)}{x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_2 y_1 - x_3 y_2}$$



Generalized Coordinate Models

Choice of shape functions:

So far we have only addressed the representation of displacements and their derivatives within the elements by stating that we need to establish some approximations for these – e.g. through polynomials which might be directly related (polynomial coefficients) to the nodal displacements.

Of course we need these fields in order to establish the element stiffness matrixes as well as the nodal point forces equivalating the surface and the body forces.

In the Method of Finite Elements this step is a major one – in principle using „brutal force" the approaches we have seen are sufficient – however, considering the associated required numerical efforts it is not efficient. For this reason we will later go into more details on how element stiffness matrixes might be developed generically and efficiently.