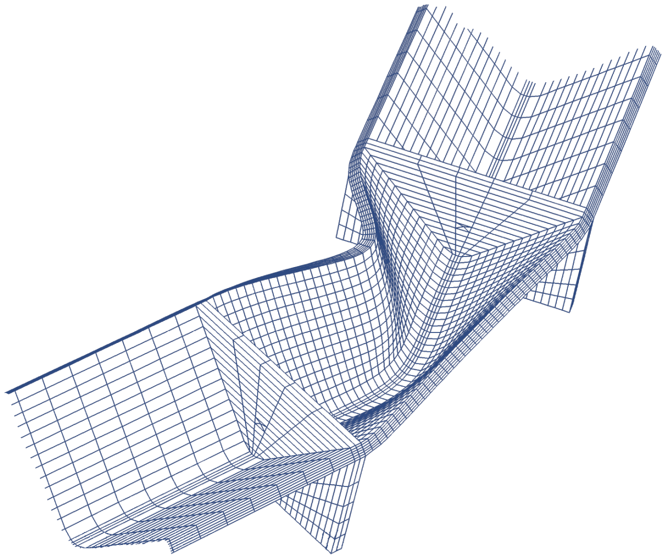
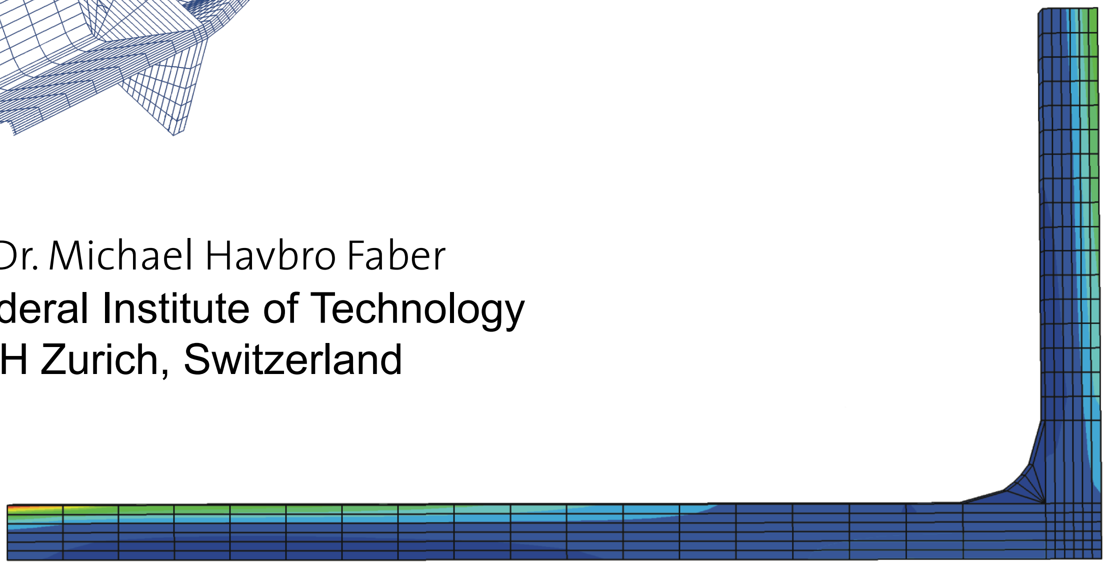
The Finite Element Method for the Analysis of Linear Systems

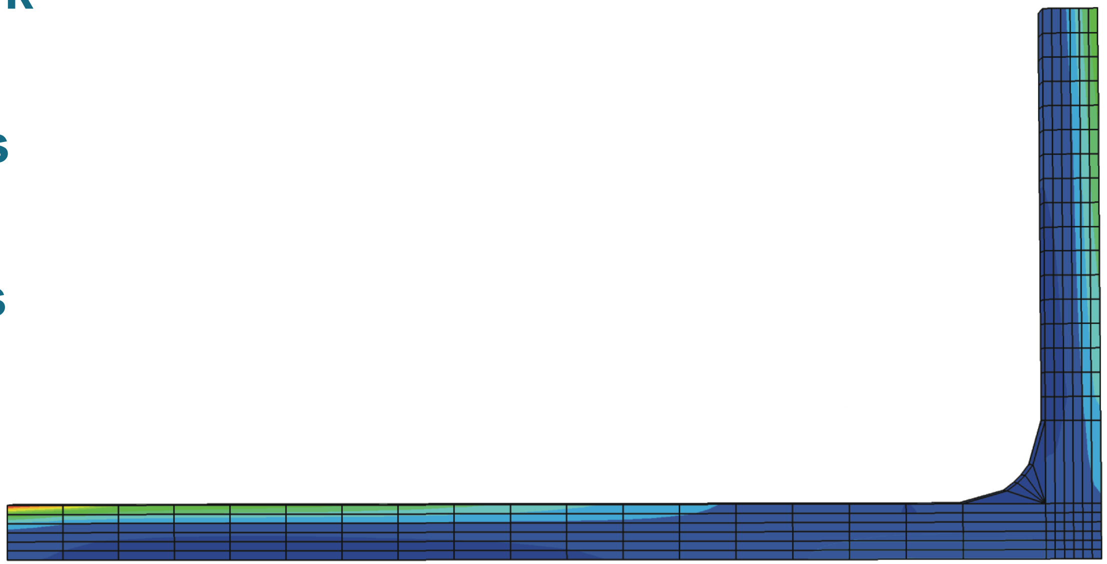


Prof. Dr. Michael Havbro Faber
Swiss Federal Institute of Technology
ETH Zurich, Switzerland



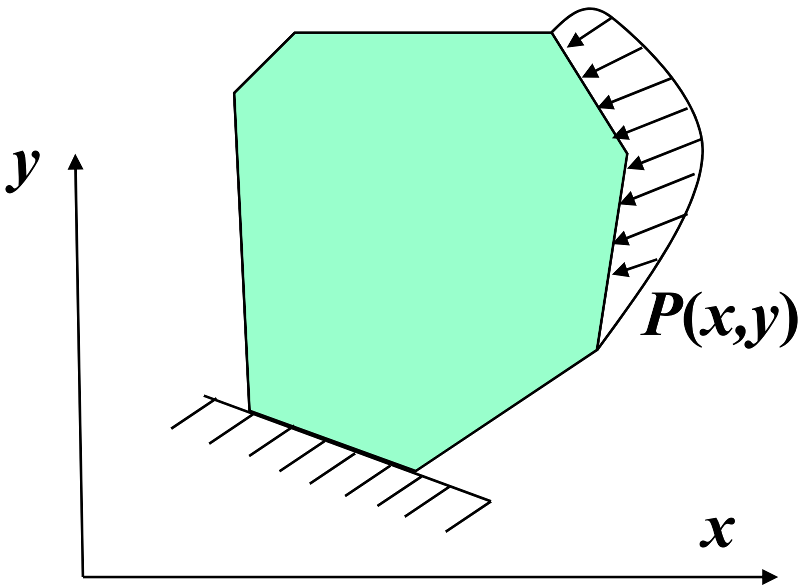
Contents of Today's Lecture

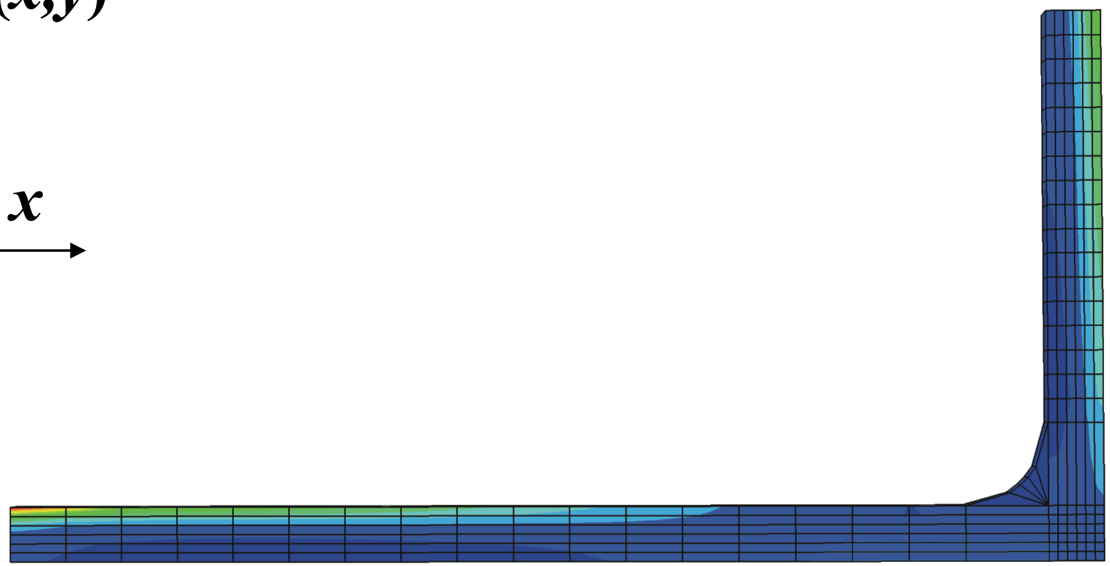
- Introduction
- Differential formulation
- Basic equations of the theory of elasticity
- Principles of virtual work
- Variational formulations
- Approximative methods
- Assignment 1



Introduction

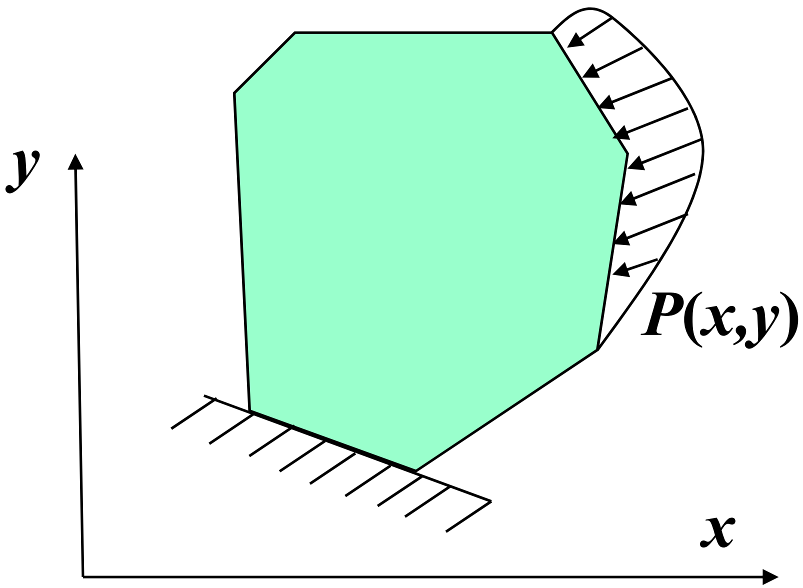
In principle the structures/systems we consider can be represented like show in the figure





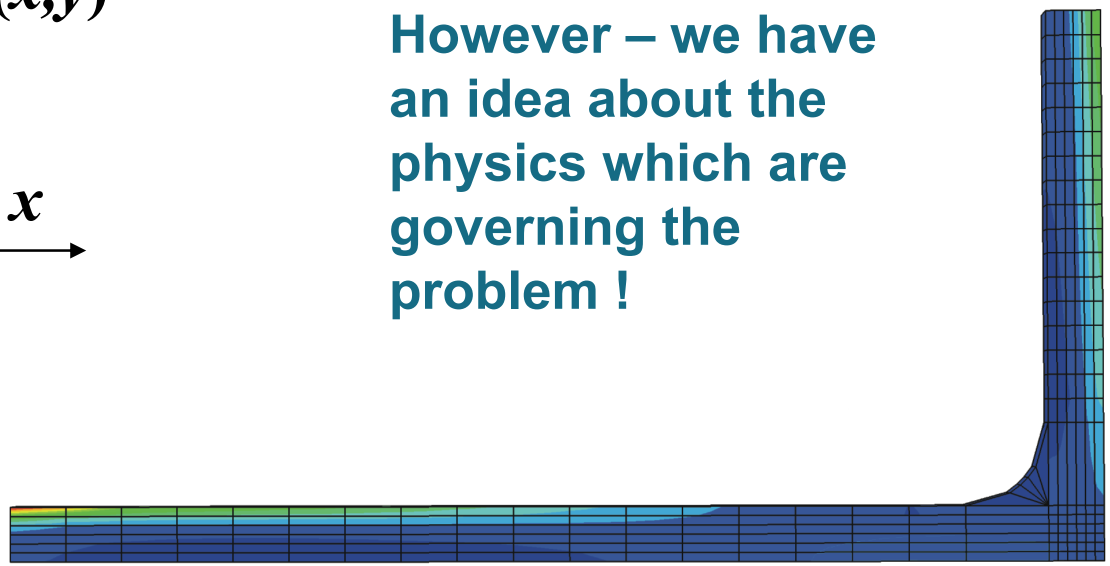
Introduction

We know that this type of problem can be analyzed taking basis in the governing differential equation



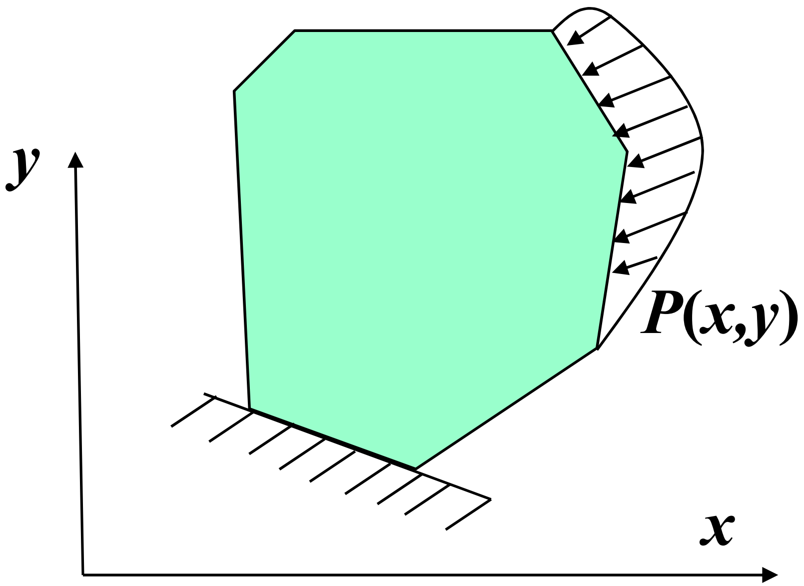
Only – the problem is that it is very difficult to find solutions for general cases

However – we have an idea about the physics which are governing the problem !

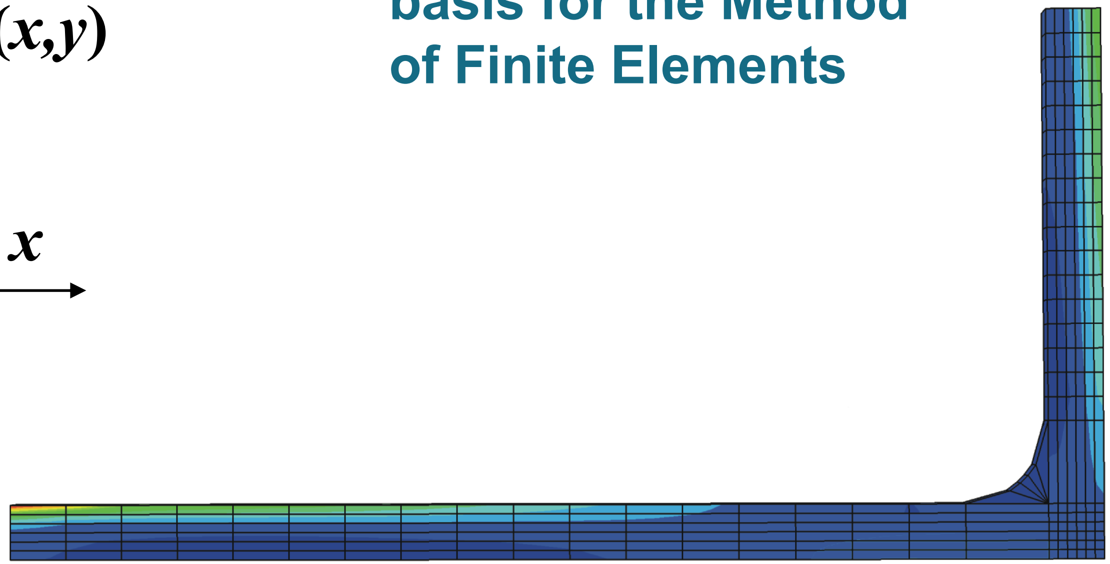


Introduction

General principles of mechanics on how to derive and solve the differential equations were developed by Ritz and Galerkin – taking basis in variational approaches

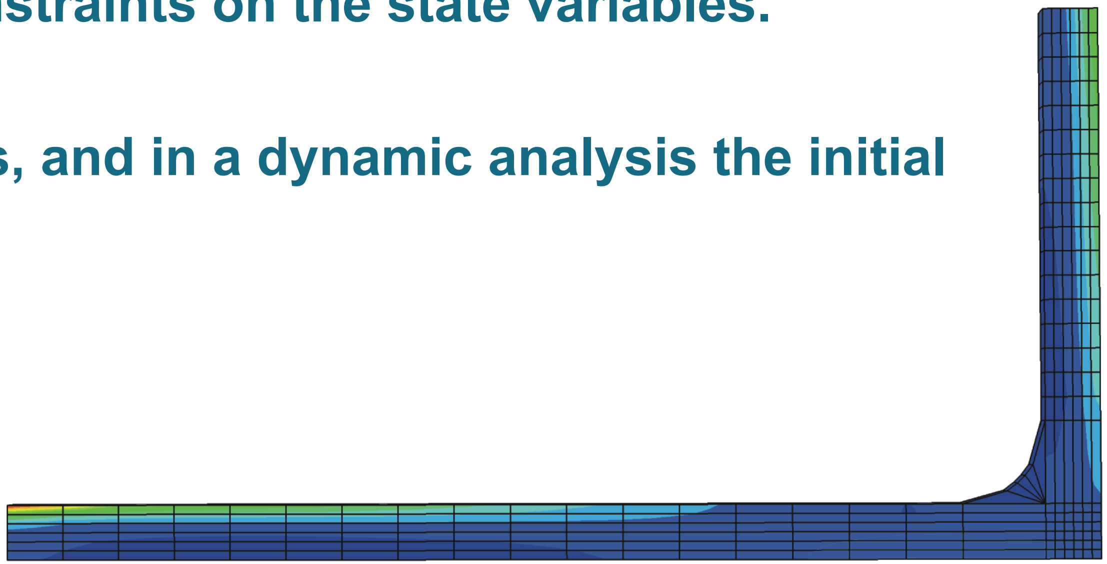


These developments led to the principle of virtual work - which essentially forms the basis for the Method of Finite Elements



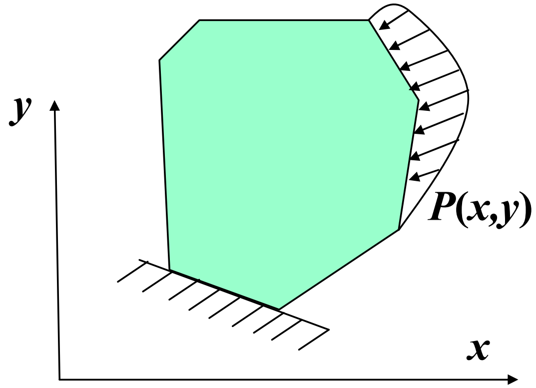
Differential formulation

- In the differential formulation, we establish the equilibrium and constitutive requirements of typical differential elements in terms of state variables.
- It is possible that all compatibility requirements are already contained in these differential equations. In general, the equations must be supplemented by additional differential equations that impose appropriate constraints on the state variables.
- All boundary conditions, and in a dynamic analysis the initial conditions, are stated.



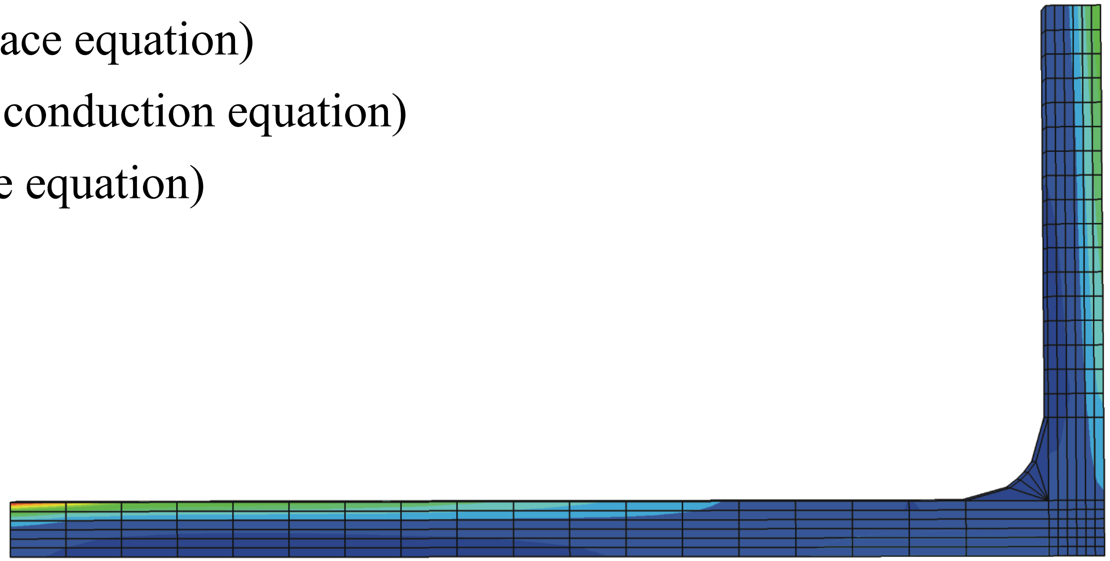
Differential formulation

The governing differential equation we consider in general have the form (second order differential equations)

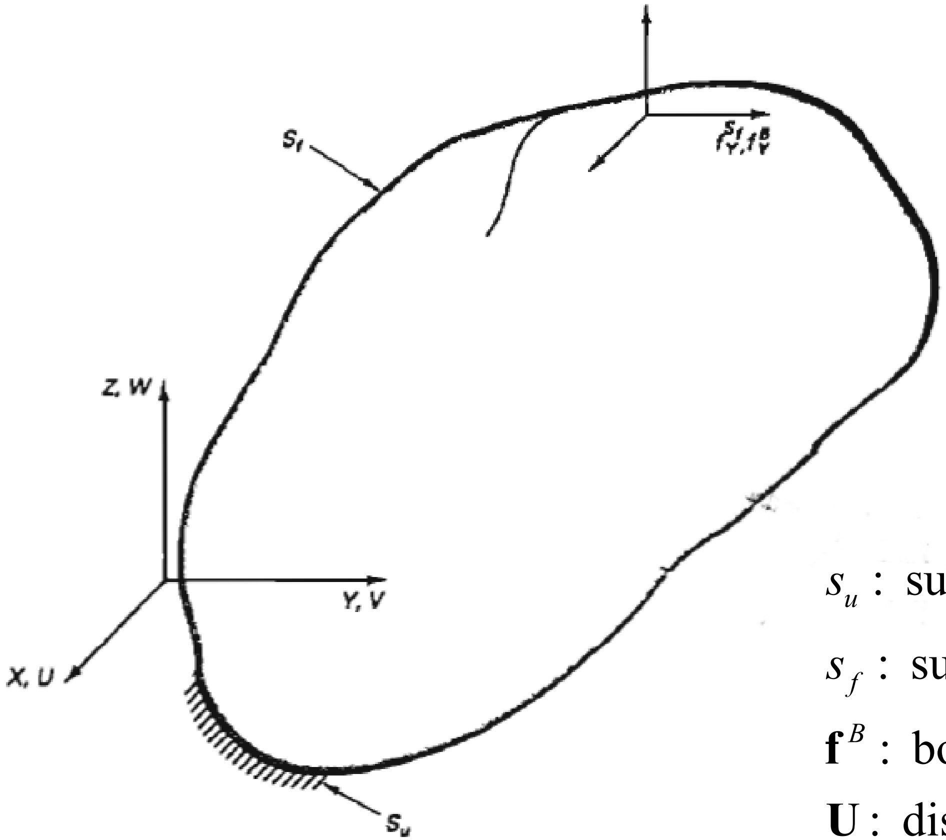


$$A(x,y)\frac{\partial^2 u}{\partial x^2}+2B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}=\phi\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$$

$$B^2-AC\begin{cases}<0 & \text{elliptic} \qquad \text{(Laplace equation)}\\ =0 & \text{parabolic} \qquad \text{(heat conduction equation)}\\ >0 & \text{hyperbolic} \qquad \text{(wave equation)}\end{cases}$$



Basic equations of the theory of elasticity



s_u : supported area with prescribed displacements $\mathbf{U}^{S_u}$

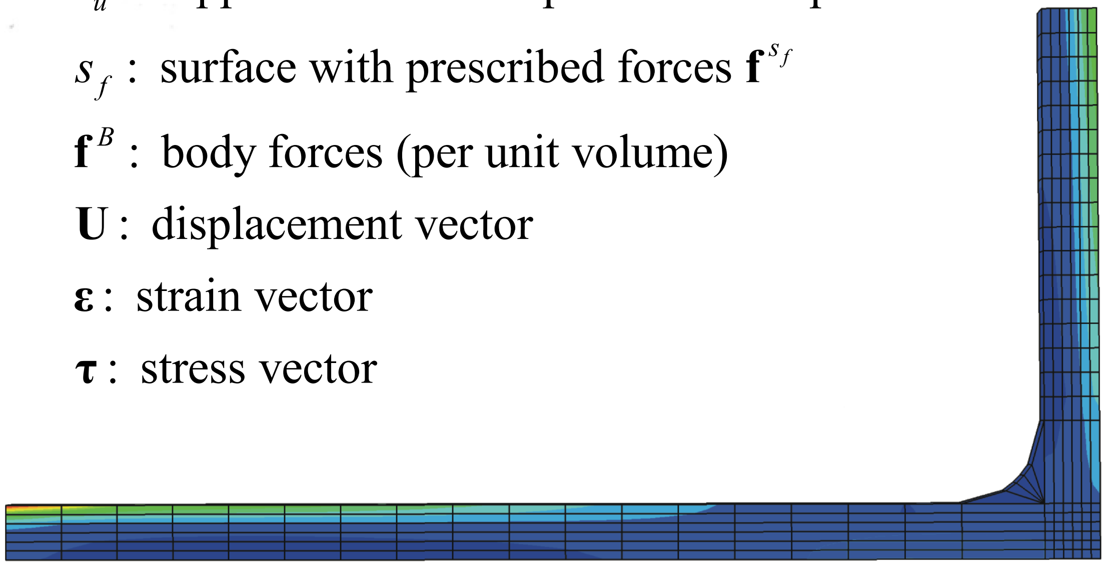
s_f : surface with prescribed forces $\mathbf{f}^{S_f}$

$\mathbf{f}^B$: body forces (per unit volume)

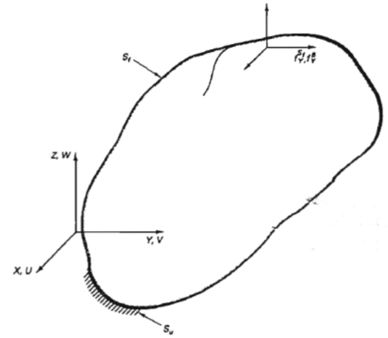
$\mathbf{U}$: displacement vector

$\boldsymbol{\varepsilon}$: strain vector

$\boldsymbol{\tau}$: stress vector



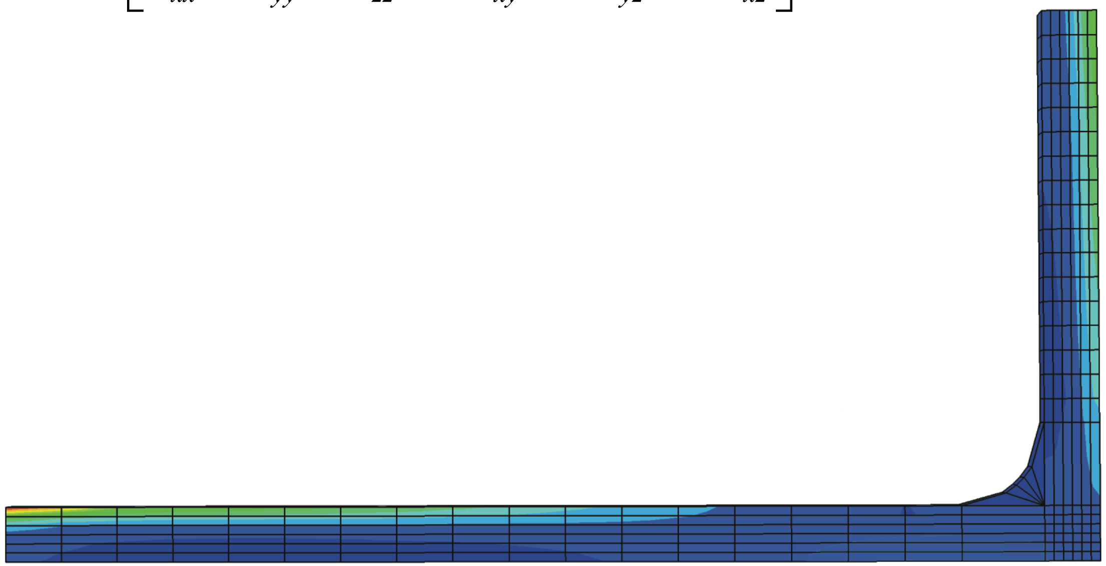
Basic equations of the theory of elasticity



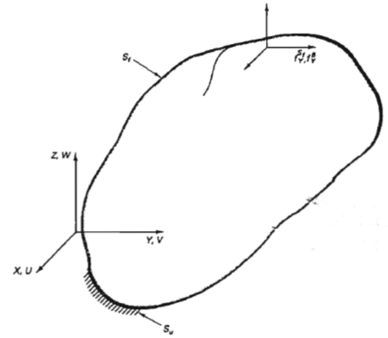
Kinematic relations

$$\mathbf{U}=\begin{bmatrix} U \\ V \\ W \end{bmatrix},\ \mathbf{L}=\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix},\ \boldsymbol{\varepsilon}^T=\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & 2\varepsilon_{xy} & 2\varepsilon_{yz} & 2\varepsilon_{xz} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}=\mathbf{LU}$$



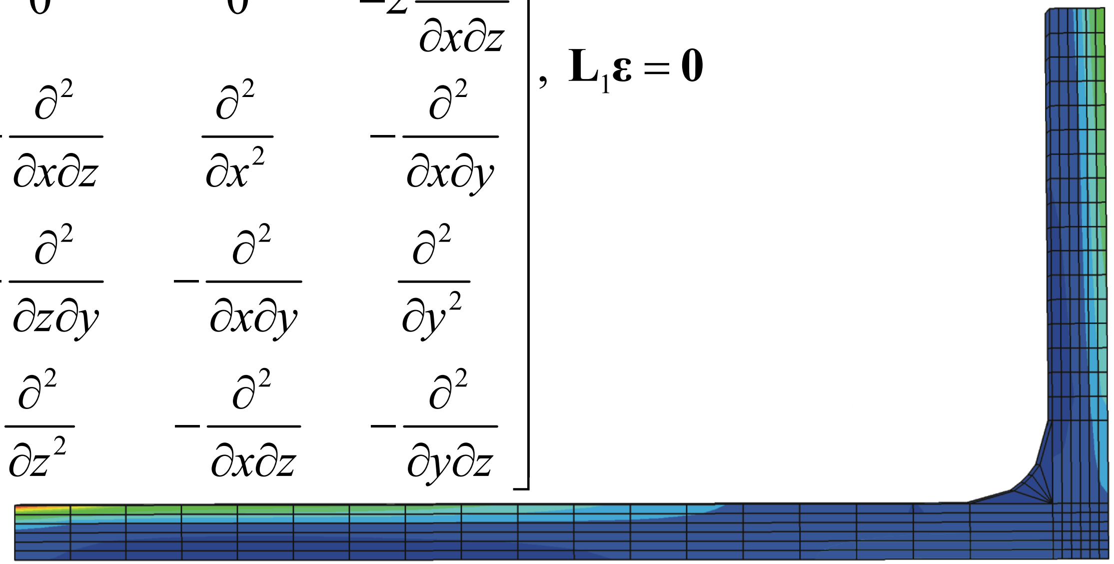
Basic equations of the theory of elasticity



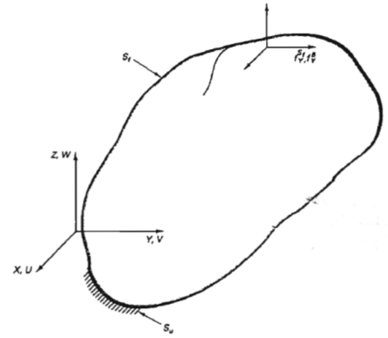
Kinematic relations

strain compatibility

$$\mathbf{L}_1 = \begin{bmatrix} \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x^2} & 0 & -2\frac{\partial^2}{\partial x \partial y} & 0 & 0 \\ 0 & \frac{\partial^2}{\partial z^2} & \frac{\partial^2}{\partial y^2} & 0 & -2\frac{\partial^2}{\partial y \partial z} & 0 \\ \frac{\partial^2}{\partial z^2} & 0 & \frac{\partial^2}{\partial x^2} & 0 & 0 & -2\frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial y \partial z} & 0 & 0 & -\frac{\partial^2}{\partial x \partial z} & \frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial x \partial y} \\ 0 & \frac{\partial^2}{\partial x \partial z} & 0 & -\frac{\partial^2}{\partial z \partial y} & -\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \\ 0 & 0 & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial z^2} & -\frac{\partial^2}{\partial x \partial z} & -\frac{\partial^2}{\partial y \partial z} \end{bmatrix}, \quad \mathbf{L}_1 \boldsymbol{\varepsilon} = \mathbf{0}$$



Basic equations of the theory of elasticity



Equilibrium equations

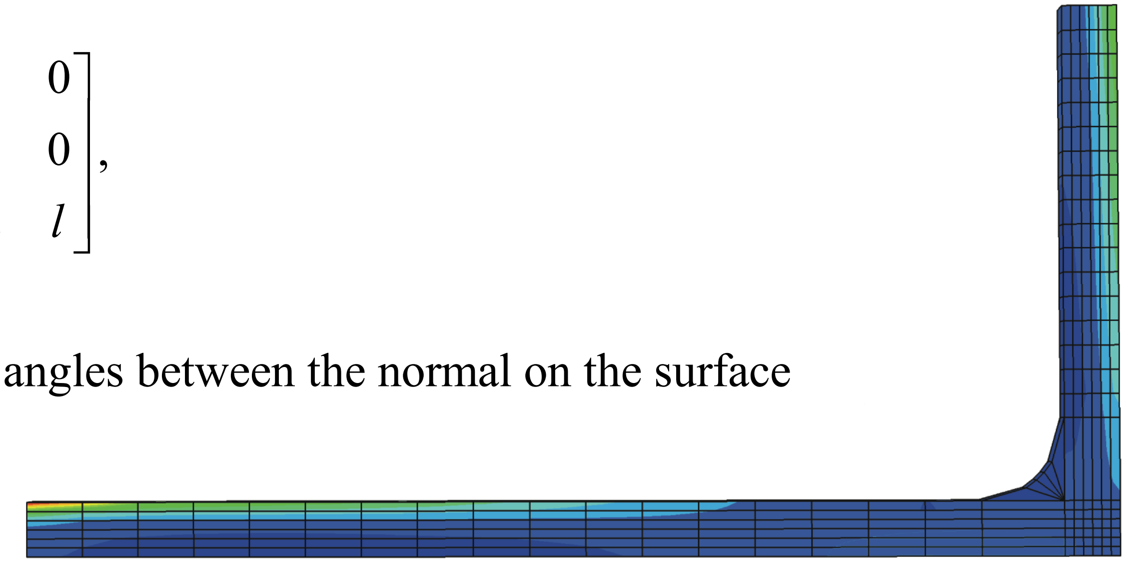
$$\mathbf{L}_2 \boldsymbol{\tau} + \mathbf{f}^B = \mathbf{0}$$

where $\boldsymbol{\tau}^T = \begin{bmatrix} \tau_{xx} & \tau_{yy} & \tau_{zz} & \tau_{xy} & \tau_{yz} & \tau_{zx} \end{bmatrix}$, $\mathbf{L}_2 = \mathbf{L}^T$

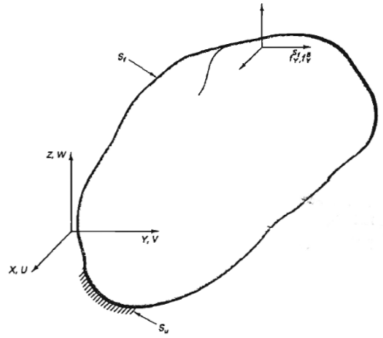
on s_f we have $\mathbf{N}\boldsymbol{\tau} - \mathbf{f}^{s_f} = \mathbf{0}$

where $\mathbf{N} = \begin{bmatrix} l & 0 & 0 & m & 0 & 0 \\ 0 & m & 0 & l & n & 0 \\ 0 & 0 & n & 0 & m & l \end{bmatrix}$,

l, m, and n are cosines of the angles between the normal on the surface and X, Y, and Z



Basic equations of the theory of elasticity

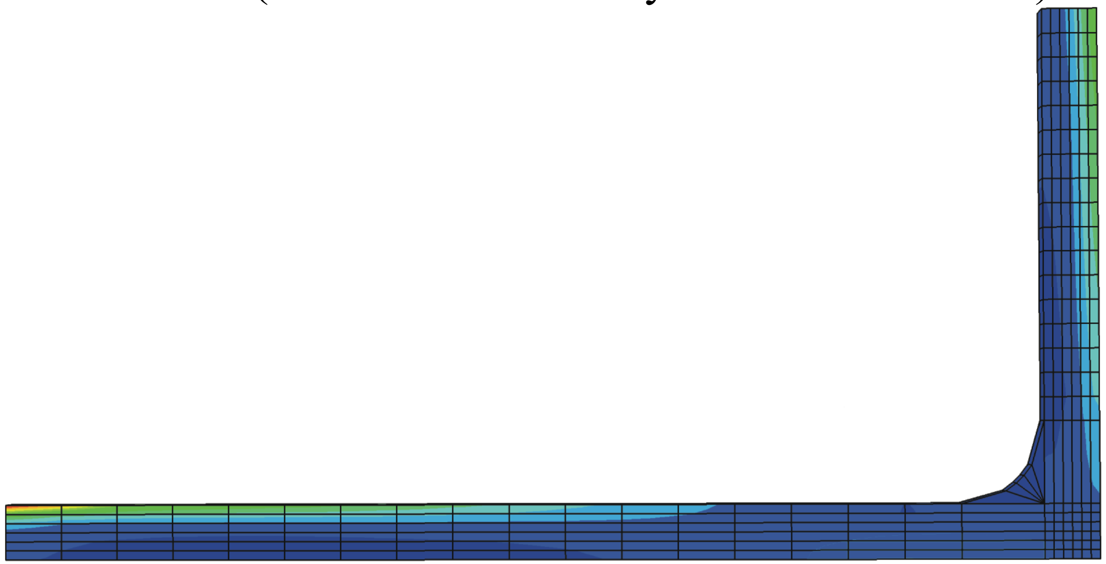


Constitutive law

$$\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\varepsilon}$$

where $\mathbf{C}$ is elasticity matrix

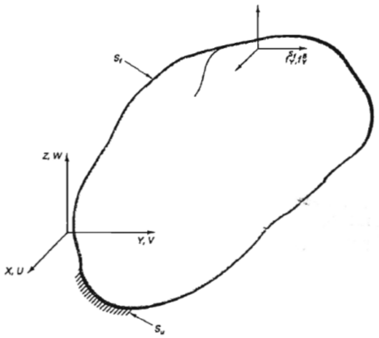
and depends on material properties E and ν (modulus of elasticity and Poisson's ratio)

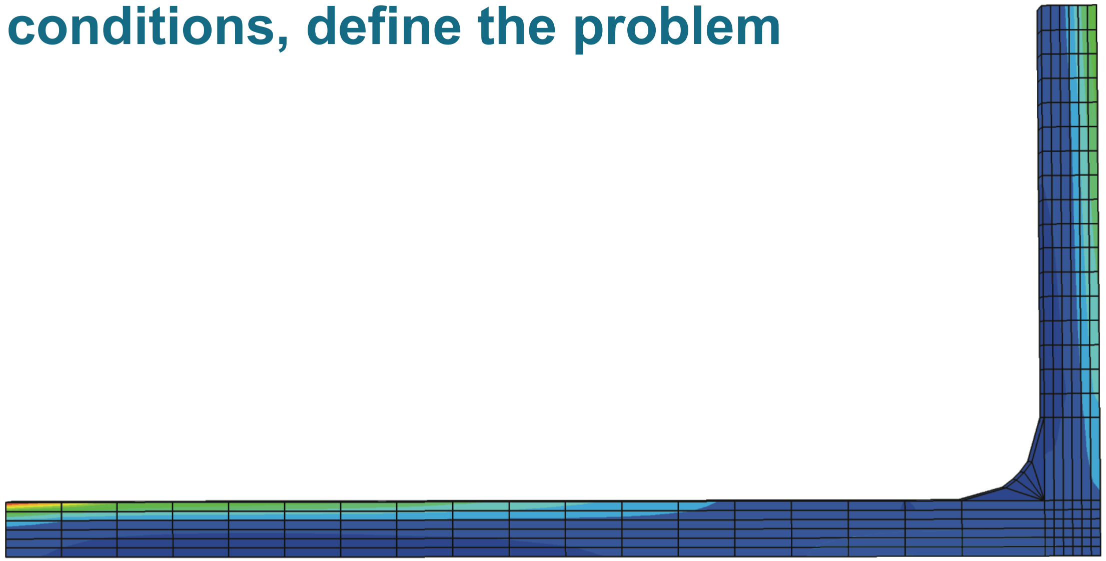


Basic equations of the theory of elasticity

Differential equations

- Stress and strain state is represented through 15 unknowns: 3 displacement components, 6 strain components, and 6 stress components.
- We also have 15 equations (9 differential and 6 algebraic). They, together with boundary conditions, define the problem





Principle of Virtual work

- The principle of virtual displacements: the virtual work of a system of equilibrium forces vanishes on compatible virtual displacements; the virtual displacements are taken in the form of variations of the real displacements

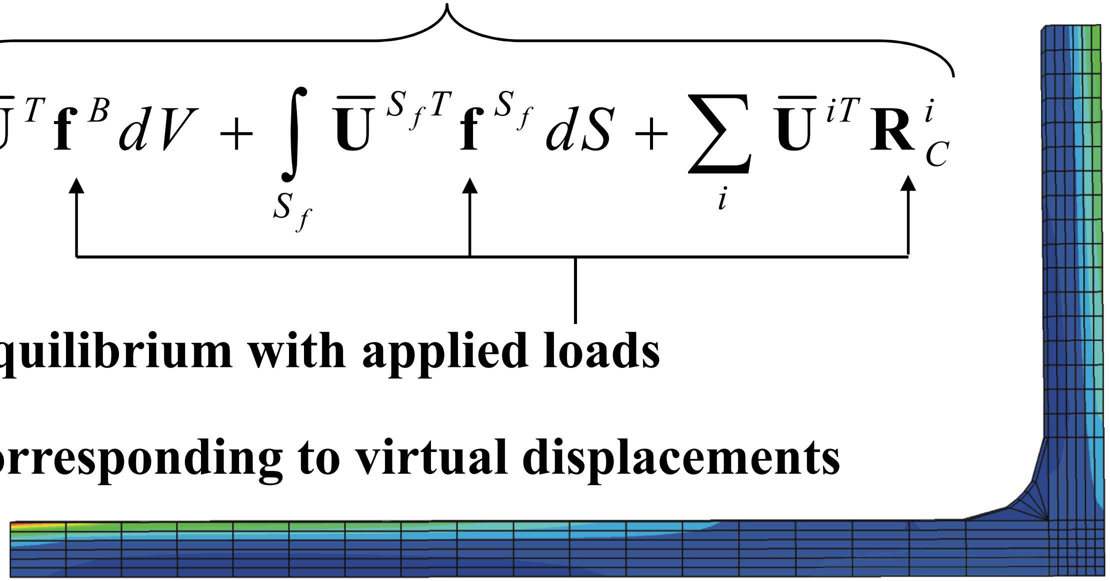
- Equilibrium is a consequence of vanishing of a virtual work

Internal virtual work **External virtual work**

$$\int_V \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\tau}\, dV = \int_V \bar{\mathbf{U}}^T \mathbf{f}^B dV + \int_{S_f} \bar{\mathbf{U}}^{S_f T} \mathbf{f}^{S_f} dS + \sum_i \bar{\mathbf{U}}^{iT} \mathbf{R}_C^i$$

Stresses in equilibrium with applied loads

Virtual strains corresponding to virtual displacements

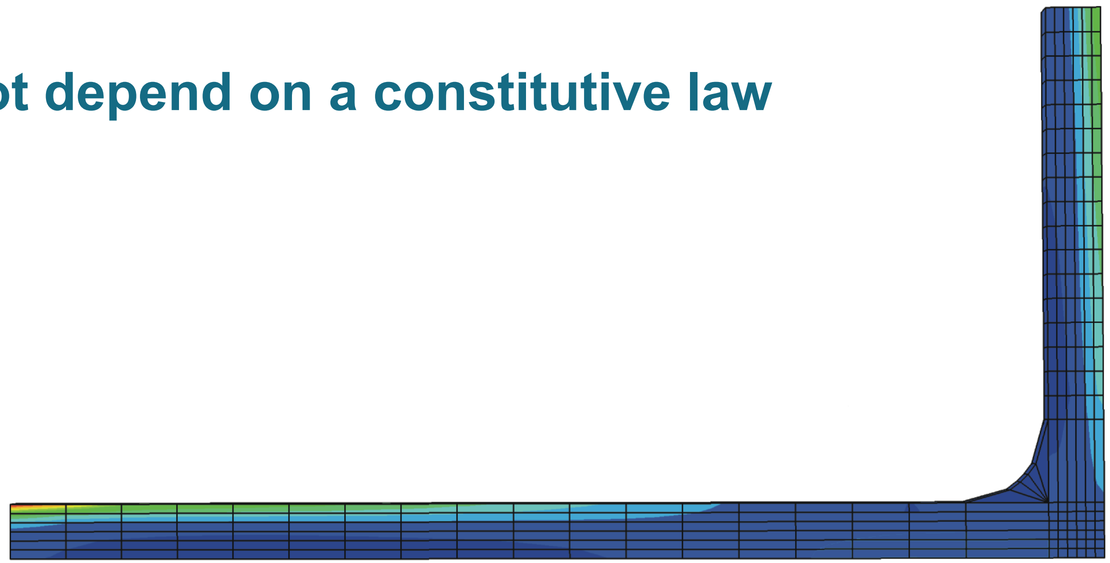


Principle of Virtual work

- The principle of virtual forces: virtual work of equilibrium variations of the stresses and the forces on the strains and displacements vanishes; the stress field considered is a statically admissible field of variation

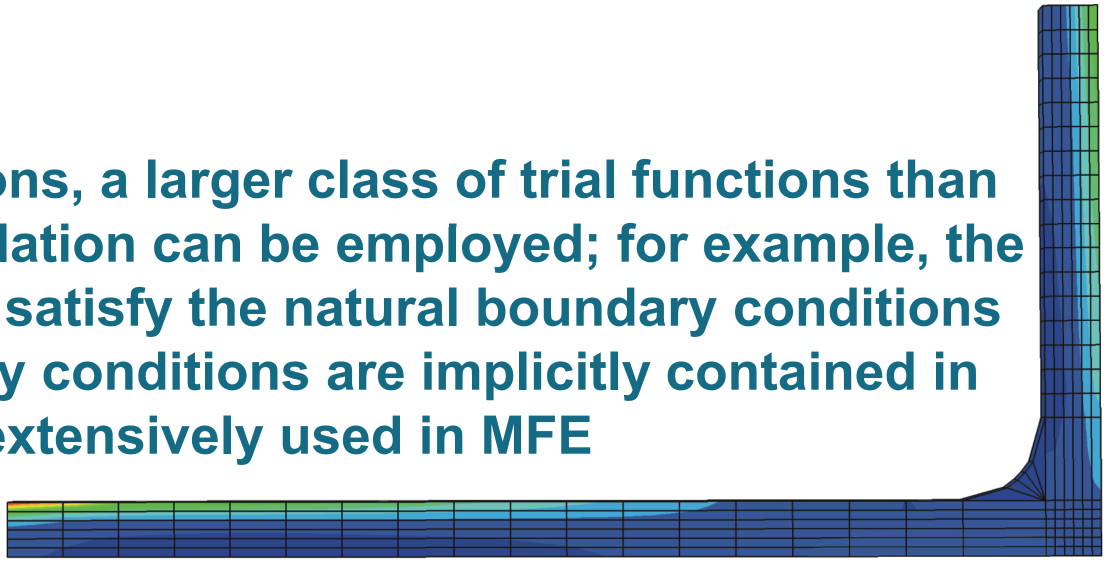
- Equilibrium is assumed to hold a priori and the compatibility of deformation is a consequence of vanishing of a virtual work

- Both principles does not depend on a constitutive law



Variational formulation

- **Based on the principle of stationarity of a functional, which is usually potential or complementary energy**
- **Two classes of the boundary conditions: essential (geometric) and natural (force) boundary conditions**
- **Scalar quantities (energies, potentials) are considered rather than vector quantities**
- **For approximate solutions, a larger class of trial functions than in the differential formulation can be employed; for example, the trial functions need not satisfy the natural boundary conditions because these boundary conditions are implicitly contained in the functional – this is extensively used in MFE**



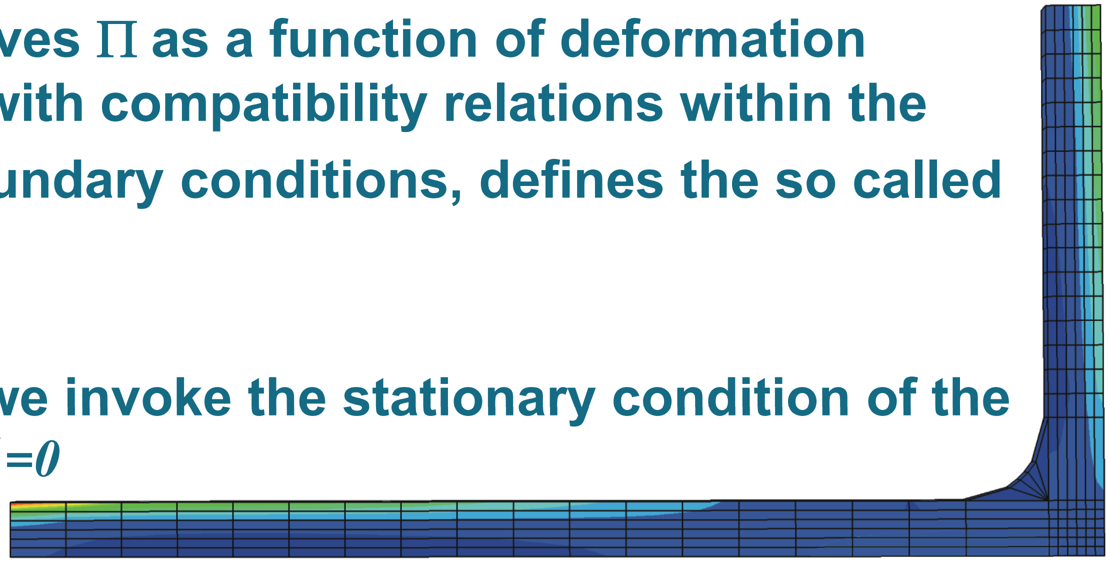
Variational formulation

- For elastic problems (linear and non-linear) a special case of the principle of virtual work – principle of minimum total potential energy can be applied

- Total potential energy is a sum of strain energy and potential of loads, $\Pi = \mathcal{U} - \mathcal{W}$

- This equation, which gives Π as a function of deformation components, together with compatibility relations within the solid and geometric boundary conditions, defines the so called Lagrange functional

- Applying the variation we invoke the stationary condition of the functional $\delta\Pi = \delta\mathcal{U} - \delta\mathcal{W} = 0$

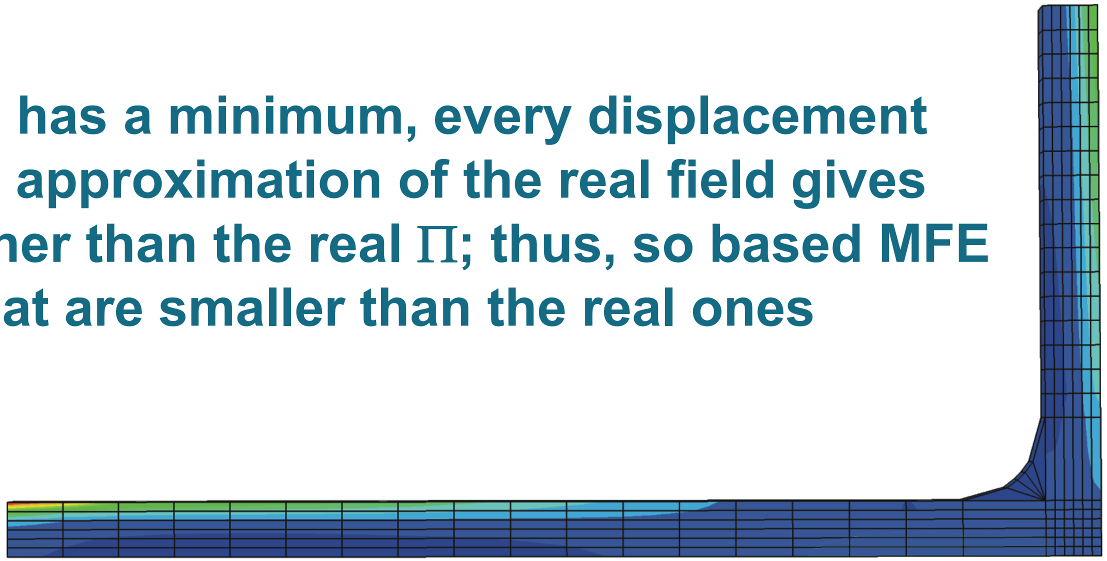


Variational formulation

- It can be shown that functional Π has a minimum since $\delta^2\Pi = \delta^2 \mathcal{U} > 0$ (due to the fact that the elasticity matrix is a positive definite matrix)

- If $\delta\Pi = 0$ holds and the variations of the displacements satisfy the essential boundary conditions and the compatibility relations, the element will be in equilibrium

- Since the total potential has a minimum, every displacement field that is used for the approximation of the real field gives values of Π that are higher than the real Π; thus, so based MFE yields displacements that are smaller than the real ones



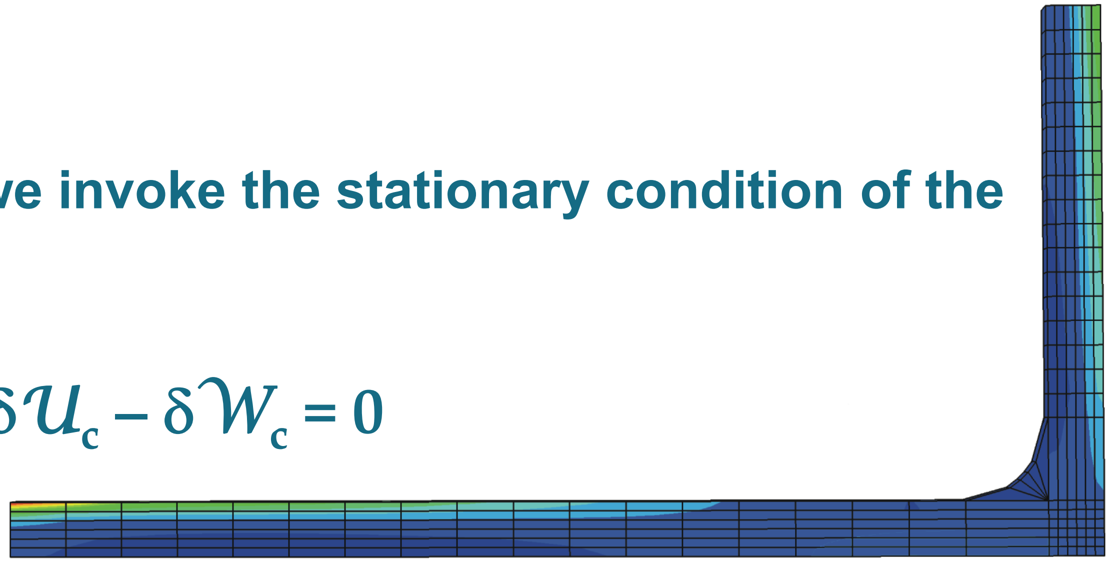
Variational formulation

- **Complementary potential energy is a sum of a complementary strain energy and potential (complementary work) of loads,**
  $\Pi_c = \mathcal{U}_c - \mathcal{W}_c$

- **This equation, which gives Π_c as a function of stress components, together with equilibrium relations within the solid and static boundary conditions, defines the so called Castigliano functional**

- **Applying the variation we invoke the stationary condition of the functional Π_c**

$$\delta\,\Pi_c = \delta\,\mathcal{U}_c - \delta\,\mathcal{W}_c = 0$$

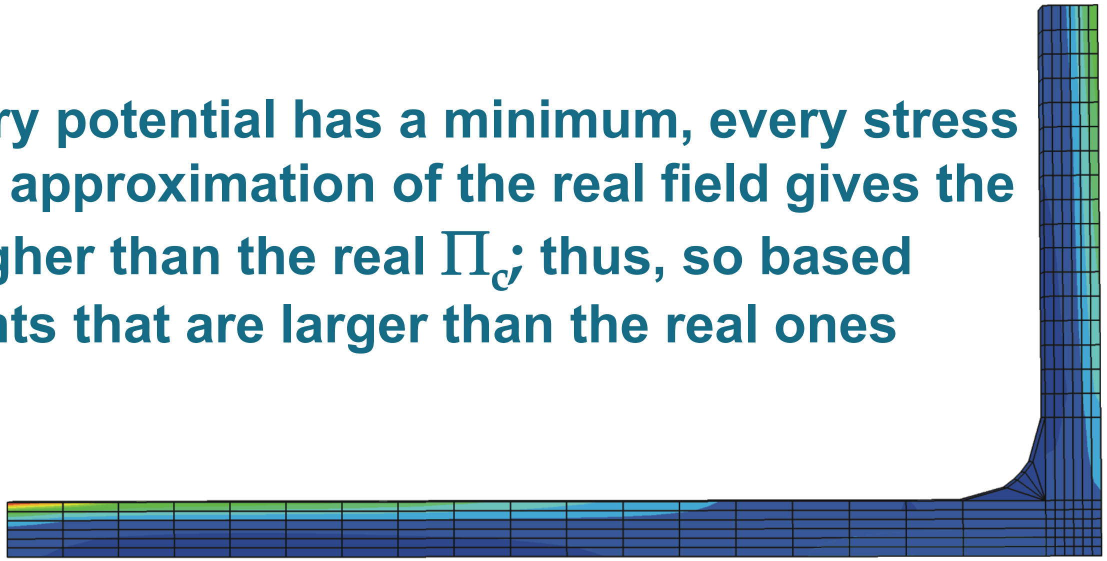


Variational formulation

- **It can be shown that functional Π_c has a minimum since $\delta^2\Pi_c = \delta^2\mathcal{U}_c > 0$ (due to the fact that the elasticity matrix is a positive definite matrix)**

- **If $\delta\Pi_c = 0$ holds and the variations of the stresses satisfy equilibrium and the natural boundary conditions, the deformation (displacement) field will be compatible**

- **Since the complementary potential has a minimum, every stress field that is used for the approximation of the real field gives the values of Π_c that are higher than the real Π_c; thus, so based MFE yields displacements that are larger than the real ones**



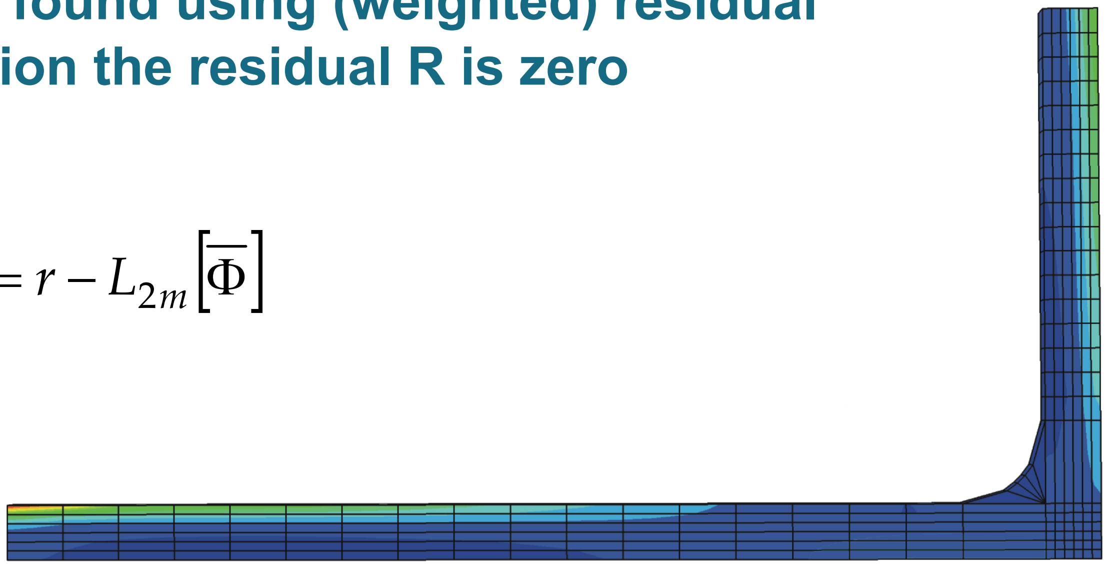
Approximative methods

- **Solution in the form of (series of) trial functions**

$$\overline{\Phi} = \sum_{i=1}^{n} a_i f_i$$

- **Solution of the problem, which is given by differential formulation $L_{2m}[\phi] = r$, is found using (weighted) residual methods; for exact solution the residual R is zero**

$$R = r - L_{2m}\left[\overline{\Phi}\right]$$



Approximative methods

- Galerkin method, n equations for a_i i=1,2,…n

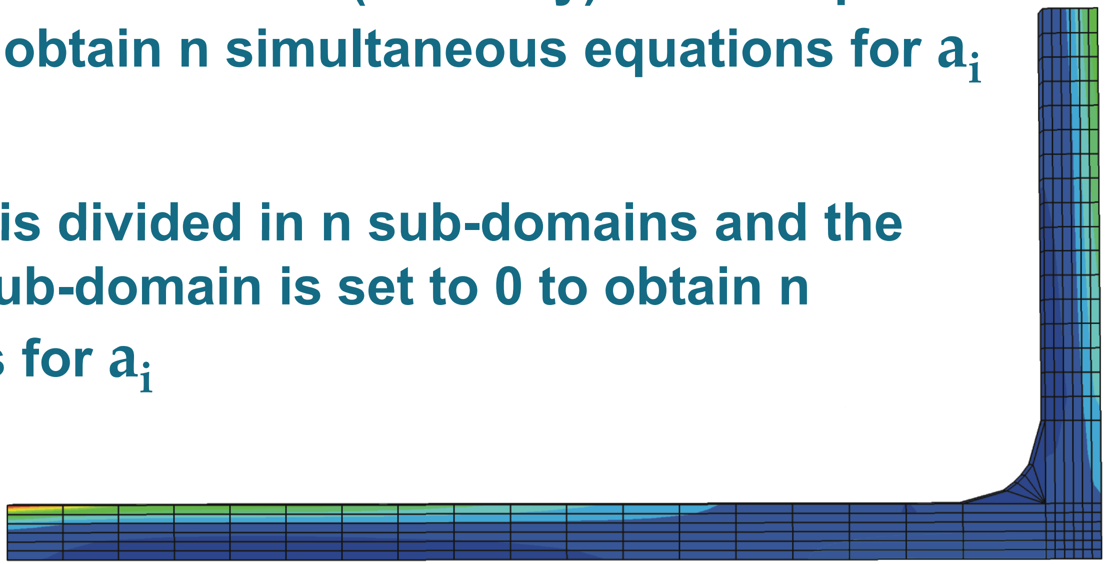
$$\int_D f_i R dD = 0$$

- Least square method, i=1,2,…n

$$\frac{\partial}{\partial a_i}\int_D R^2 dD = \int_D R L_{2m}[f_i] dD = 0$$

- Collocation method: R is set to 0 in n (arbitrary) discrete points in solution domain D to obtain n simultaneous equations for a_i

- Sub-domain method: D is divided in n sub-domains and the integral of R over that sub-domain is set to 0 to obtain n simultaneous equations for a_i

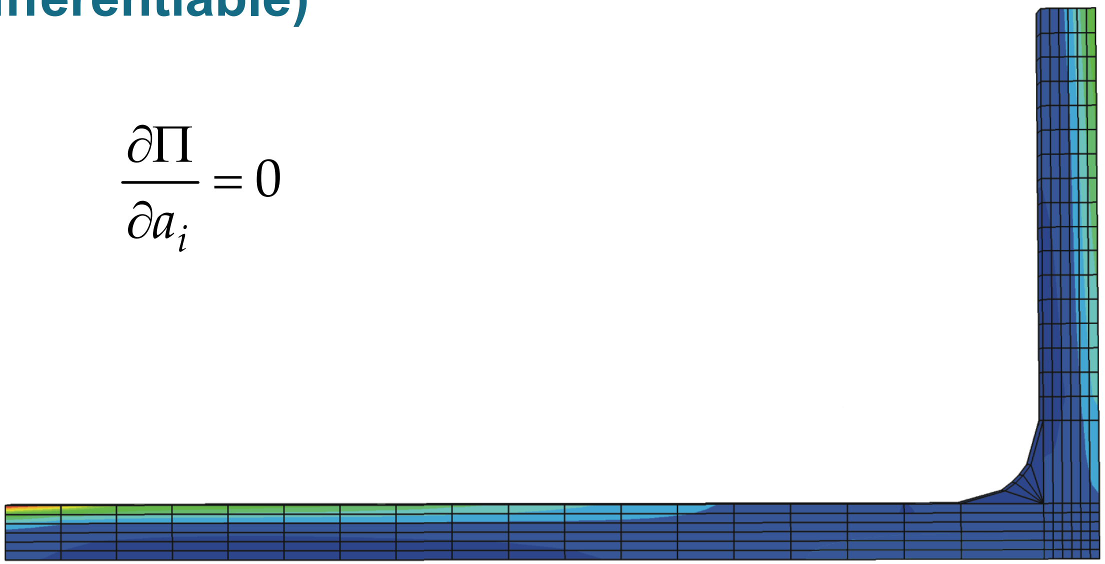


Approximative methods

- **Solution of varionatial problem**

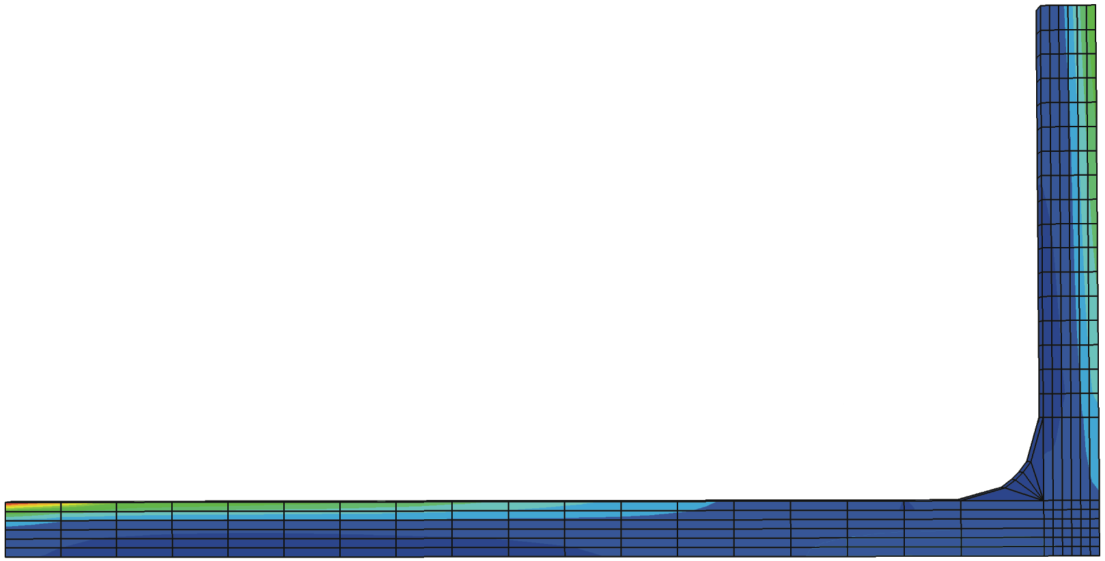
Trial function is substituted in functional Π and using a stationery condition $\delta\Pi = 0$ a set of n equations for the parameters a_i is obtained, i=1,2,…n (for $2m$ rank problem f_i must be only m-times differentiable)

$$\frac{\partial \Pi}{\partial a_i} = 0$$



Ritz method

- This method operates on the functional corresponding to the problem.
- In our case we choose potential energy as a functional.



Ritz method

The trial function is in the form

$$\Pi = \sum_{i=1}^{n} a_i f_i$$

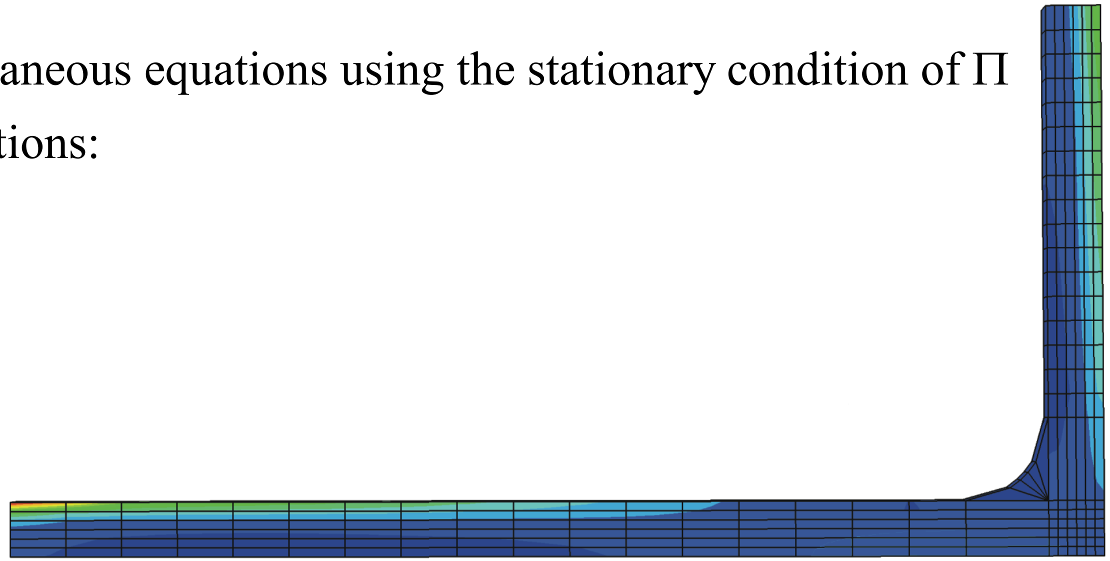
where f_i is independent trial functions

and a_i is multipliers to be determined in the solution.

a_is are obtained from the simultaneous equations using the stationary condition of Π

i.e. $\delta\Pi = 0$ which yields n equations:

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad i = 1, 2, \ldots, n$$



Ritz method

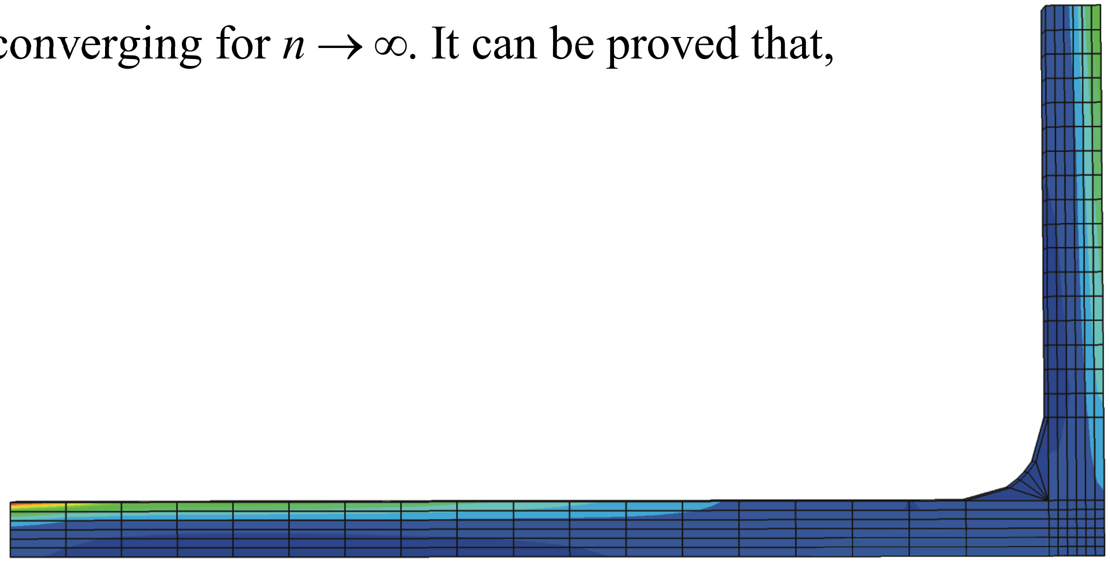
The trial function is in the form

$$\Pi = \sum_{i=1}^{n} a_i f_i$$

f_i need to satisfy only the essential (geometric) boundary conditions and not the natural (force) boundary conditions.

Ritz approximation method is converging for $n \to \infty$. It can be proved that, for one dimensional space

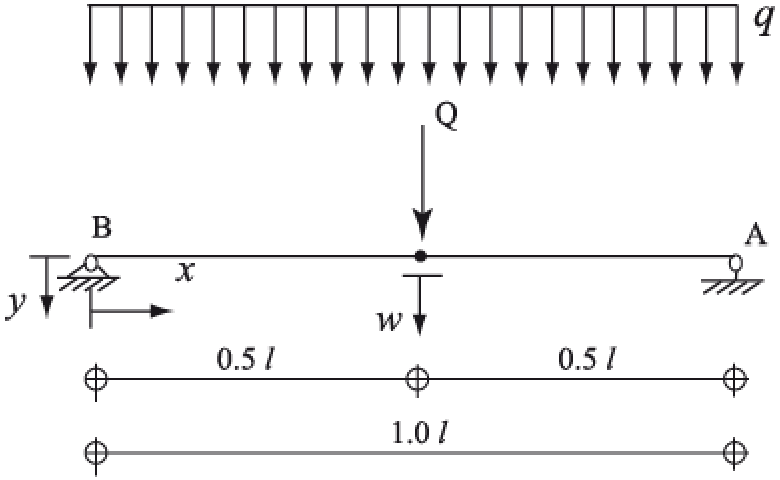
$$\lim_{n\to\infty} \int_a^b \left[\Pi - \sum_{i=1}^{n} a_i f_i(x) \right]^2 dx \to 0$$



Ritz method: Example

Let us consider a simple beam loaded by a) uniformly distributed load *q* and b) concentrated force *Q*.

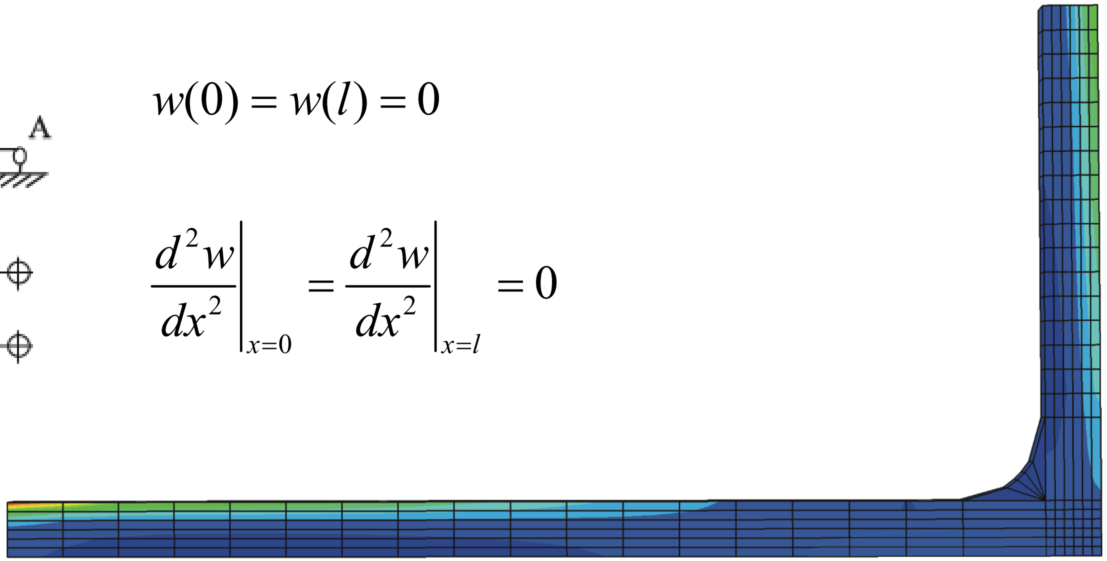
We use the Ritz method to calculate the deflection *w* and bending moment *M* at the midspan.



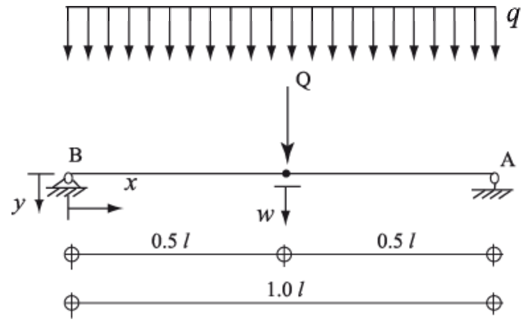
Boudary conditions at $x = 0$ (B) and $x = l$ (A):

$$w(0) = w(l) = 0$$

$$\left.\frac{d^2w}{dx^2}\right|_{x=0} = \left.\frac{d^2w}{dx^2}\right|_{x=l} = 0$$



Ritz method: Example



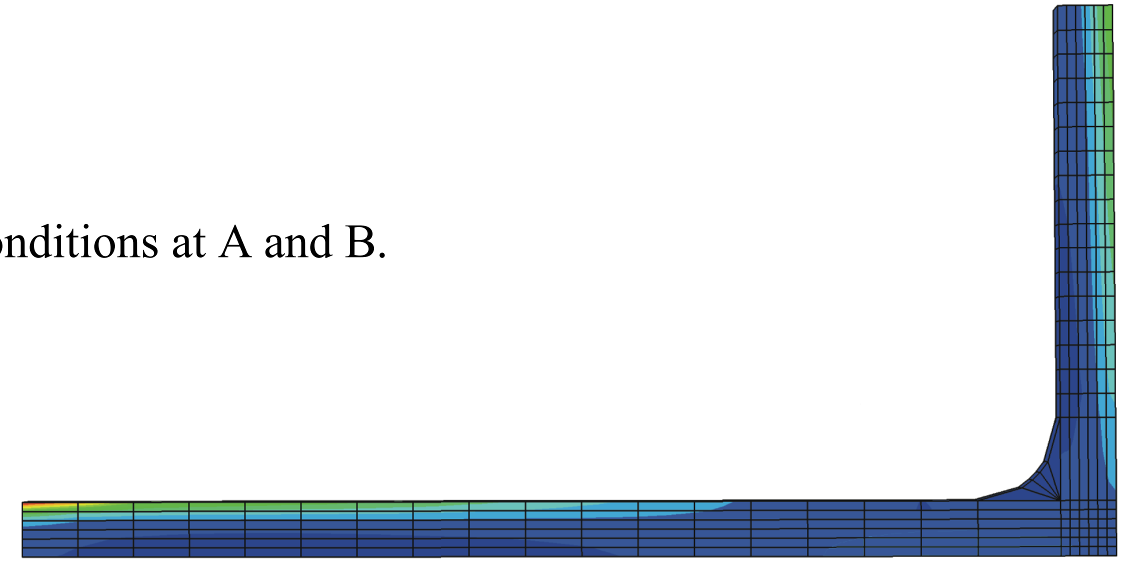
Potential energy functional is obtained from:

$$\Pi = \int_0^l \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 dx - \int_0^l q w dx - Q\, w\Big|_{\frac{l}{2}}$$

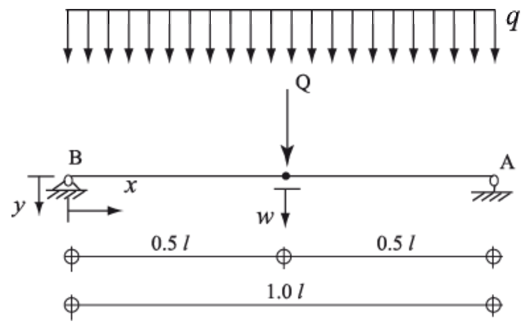
Trial function is choosen as

$$w(x) = \sum_{i=1}^{n} a_i \sin \frac{i\pi x}{l}$$

which satisfies the boundary conditions at A and B.



Ritz method: Example



We consider only the first term, thus

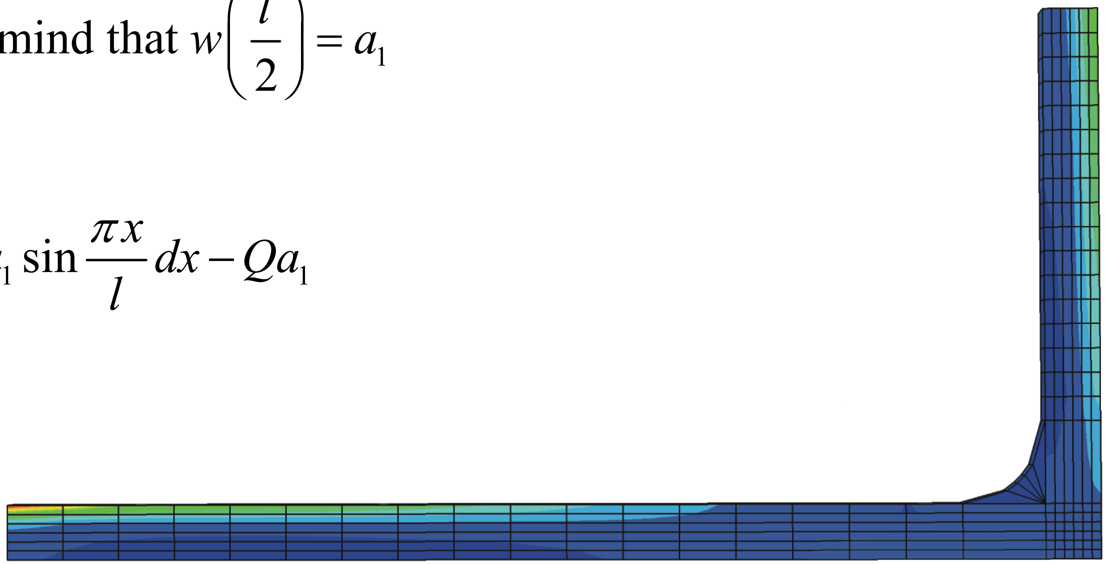
$$w(x) = a_1 \sin \frac{\pi x}{l}$$

$$\frac{dw}{dx} = a_1 \frac{\pi}{l} \cos \frac{\pi x}{l} \quad \frac{d^2 w}{dx^2} = -a_1 \frac{\pi^2}{l^2} \sin \frac{\pi x}{l}$$

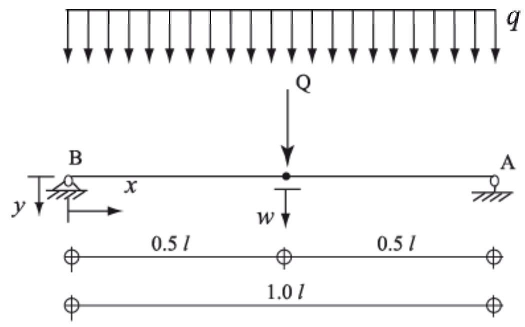
Now, we calculate Π keeping in mind that $w\left(\frac{l}{2}\right) = a_1$

$$\Pi = \int_0^l \frac{1}{2} EI a_1^2 \frac{\pi^4}{l^4} \sin^2 \frac{\pi x}{l} dx - \int_0^l q a_1 \sin \frac{\pi x}{l} dx - Q a_1$$

$$= \frac{EI\pi^4}{4l^3} a_1^2 - \left(\frac{2ql}{\pi} + Q \right) a_1$$



Ritz method: Example



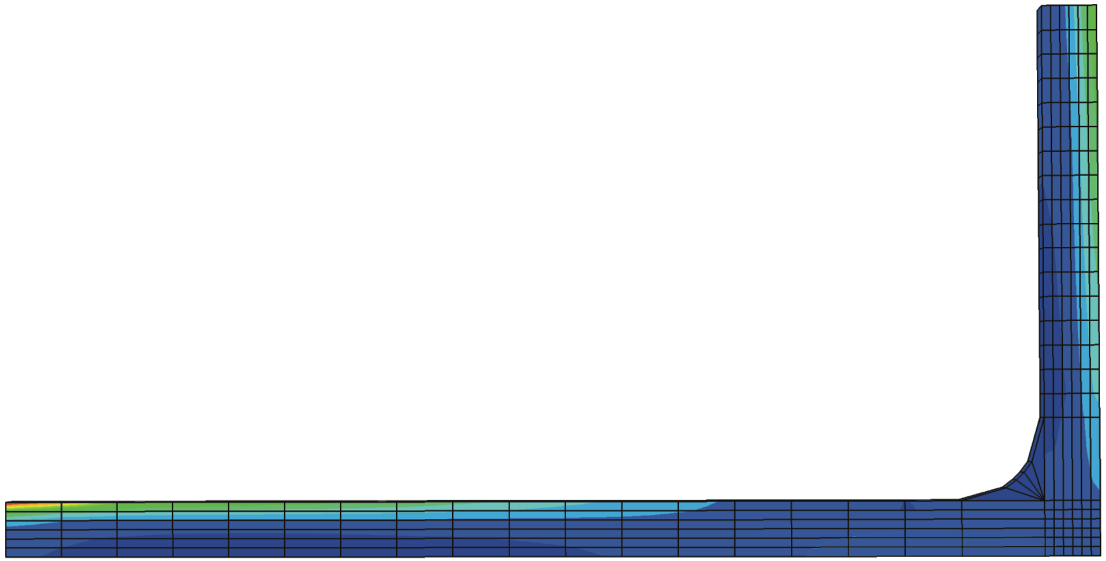
We invoke stationarity of Π

$$\frac{d\Pi}{da_1} = \frac{EI\pi^4}{2l^3} a_1 - \left(\frac{2ql}{\pi} + Q \right) = 0$$

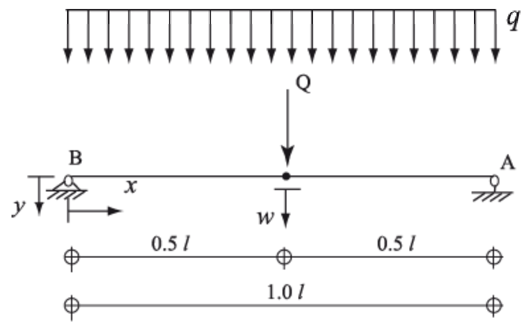
$$a_1 = \left(\frac{2ql}{\pi} + Q \right) \frac{2l^3}{EI\pi^4}$$

which leads to

$$w(x) = \left(\frac{2ql}{\pi} + Q \right) \frac{2l^3}{EI\pi^4} \sin \frac{\pi x}{l}$$



Ritz method: Example



Deflection at midspan

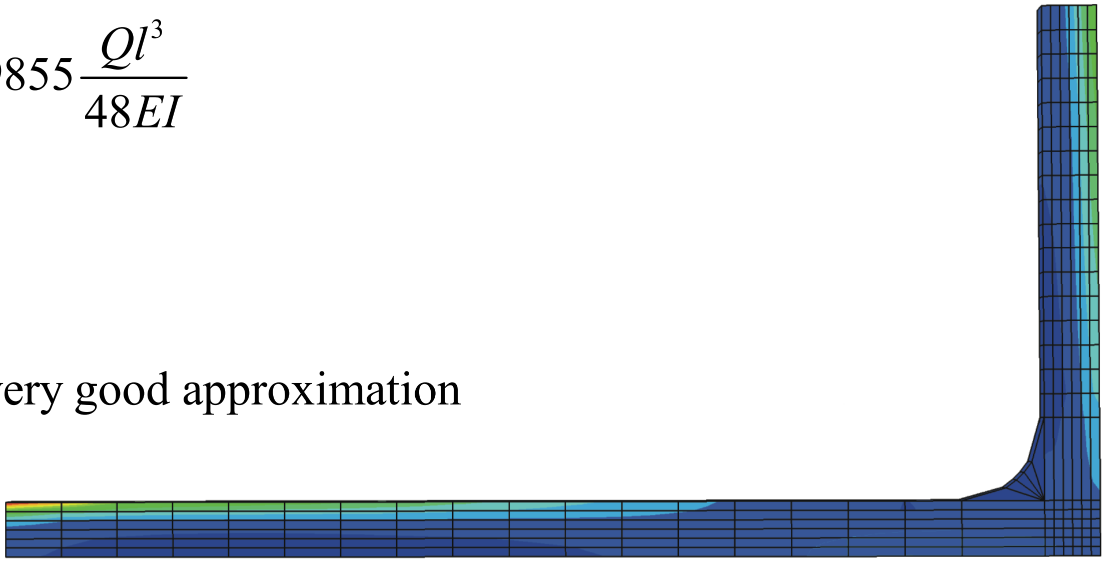
$$\text{a) } w\left(\frac{l}{2}\right) = \frac{2ql}{\pi}\frac{2l^3}{EI\pi^4}\sin\frac{\pi\frac{l}{2}}{l} = 1.0039\frac{5ql^4}{384EI}$$

$$\text{exact solution: } \frac{5ql^4}{384EI}, \ \Delta \approx 0$$

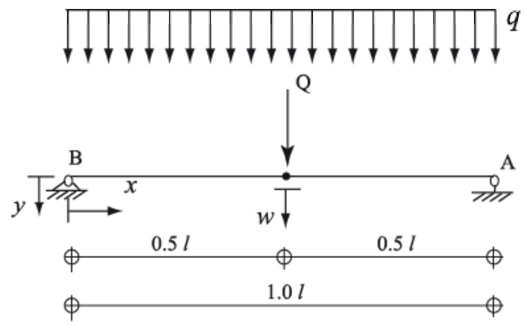
$$\text{b) } w\left(\frac{l}{2}\right) = Q\frac{2l^3}{EI\pi^4}\sin\frac{\pi\frac{l}{2}}{l} = 0.9855\frac{Ql^3}{48EI}$$

$$\text{exact solution: } \frac{Ql^3}{48EI}, \ \Delta \approx 1.4\%$$

It can be noticed that we reach very good approximation



Ritz method: Example



Bending moment at midspan

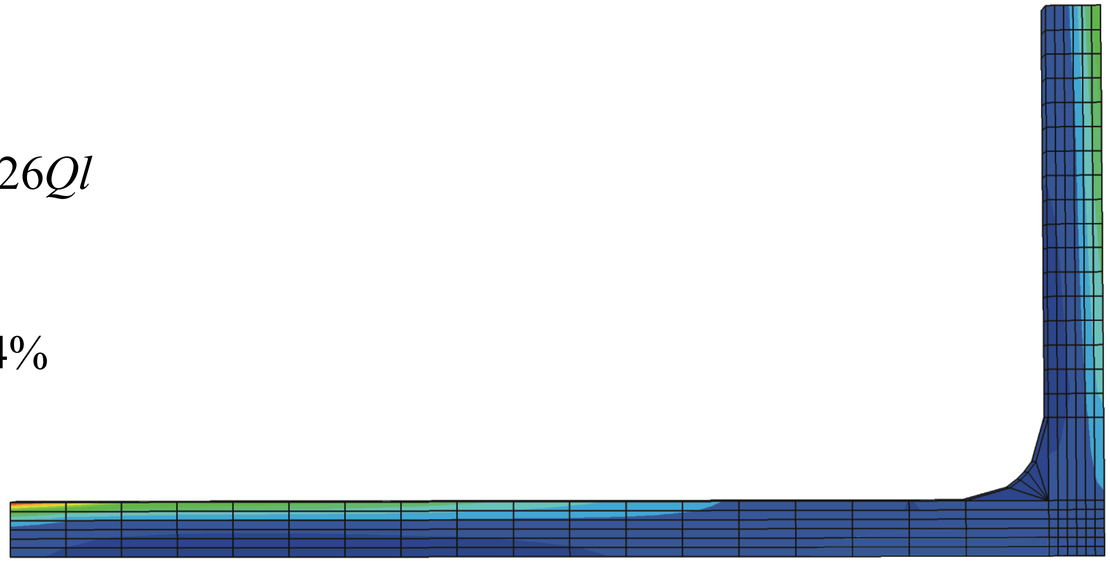
$$M = -EI\frac{d^2w}{dx^2} = \left(\frac{2ql}{\pi} + Q\right)\frac{2l}{\pi^2}\sin\frac{\pi x}{l}$$

$$\text{a) } M\left(\frac{l}{2}\right) = \frac{2l}{\pi^2}\frac{2ql}{\pi}\sin\frac{\pi\frac{l}{2}}{l} = 0.129ql^2$$

exact solution: $0.125ql^2$, $\Delta \approx 3.2\%$

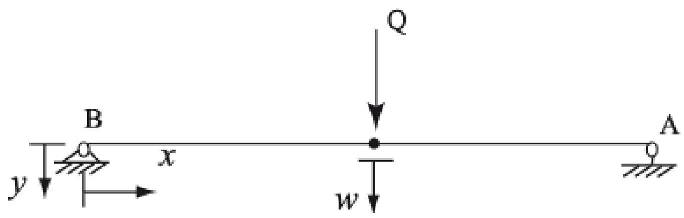
$$\text{b) } M\left(\frac{l}{2}\right) = \frac{2l}{\pi^2}Q\sin\frac{\pi\frac{l}{2}}{l} = 0.2026Ql$$

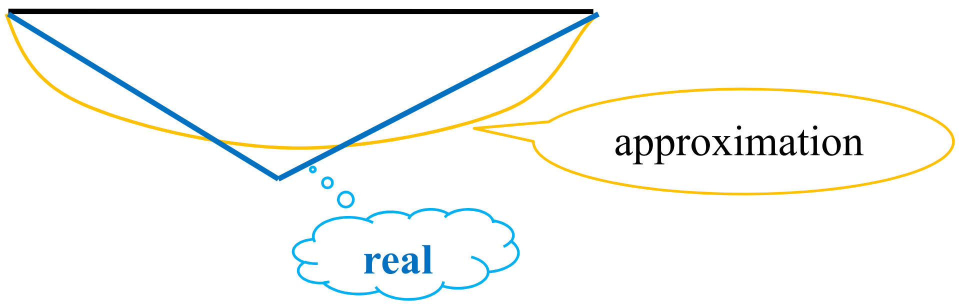
exact solution: $0.25Ql$, $\Delta \approx 23.4\%$



Ritz method: Example

Note the relatively poor approximation for the bending moment value due to the concentrated force. This is because the difference between the real moment distribution (linear) and our approximation (sin).





Assignment 1

1. Cantilever Beam

Using the variational approach calculate the vertical displacement w at point A and the bending moment distribution $M(x)$ for a cantilever beam (Figure 1) subjected to

a) a uniform distributed load with q

b) a concentrated load Q at point A.

Here, EI is assumed to be a constant. Approximate the displacement $w(x)$ by a third-order polynomial.

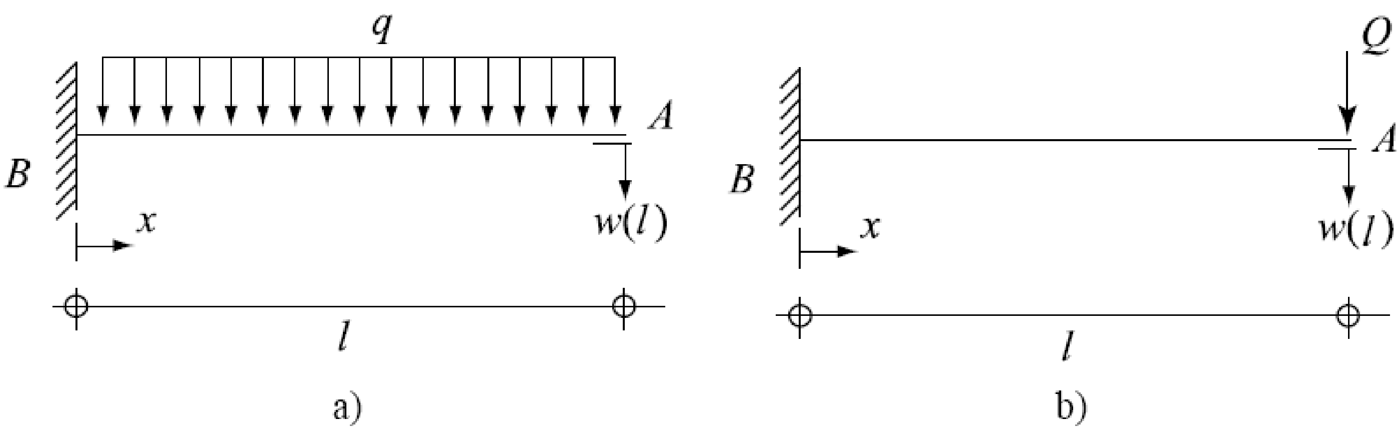
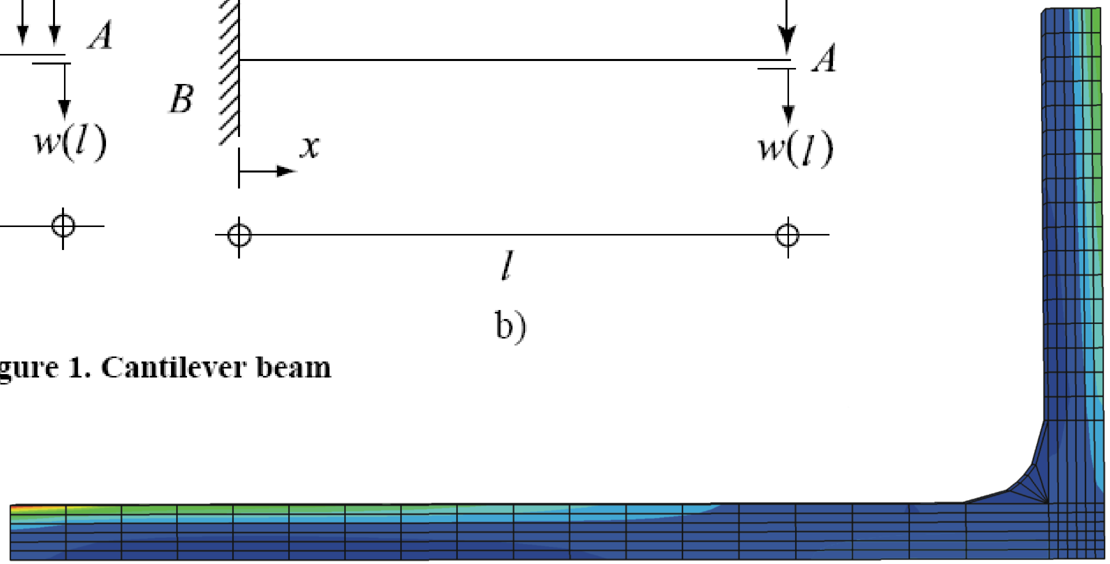


Figure 1. Cantilever beam



Assignment 1

Two classes of the boundary conditions: essential (geometric) and natural (force) boundary conditions

For approximate solutions, a larger class of trial functions than in the differential formulation can be employed; for example, the trial functions need not satisfy the natural boundary conditions