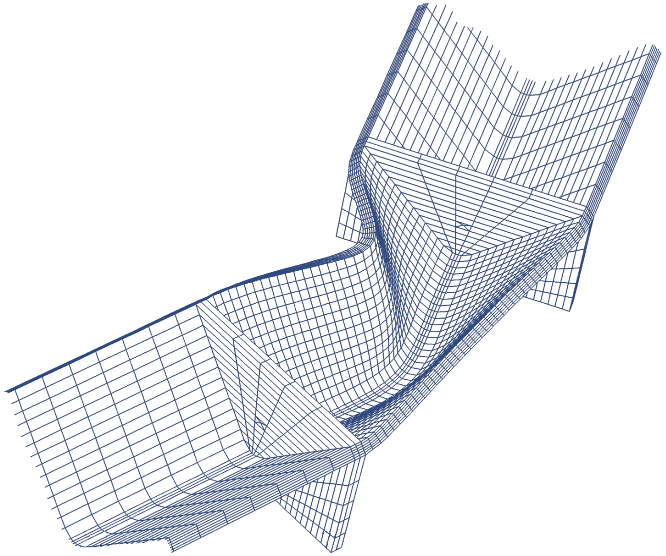
The Finite Element Method for the Analysis of Linear Systems



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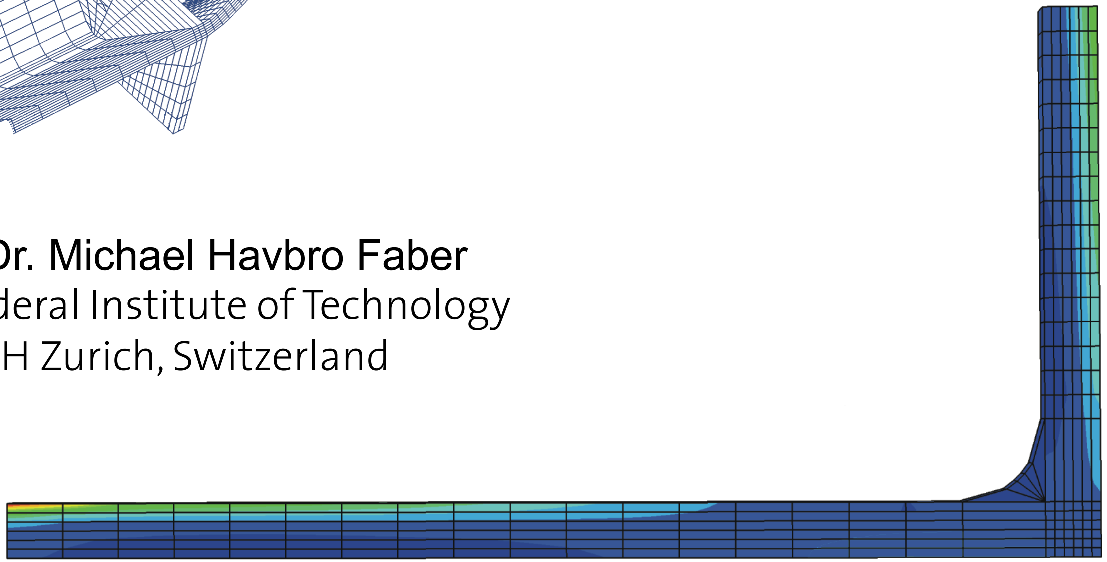
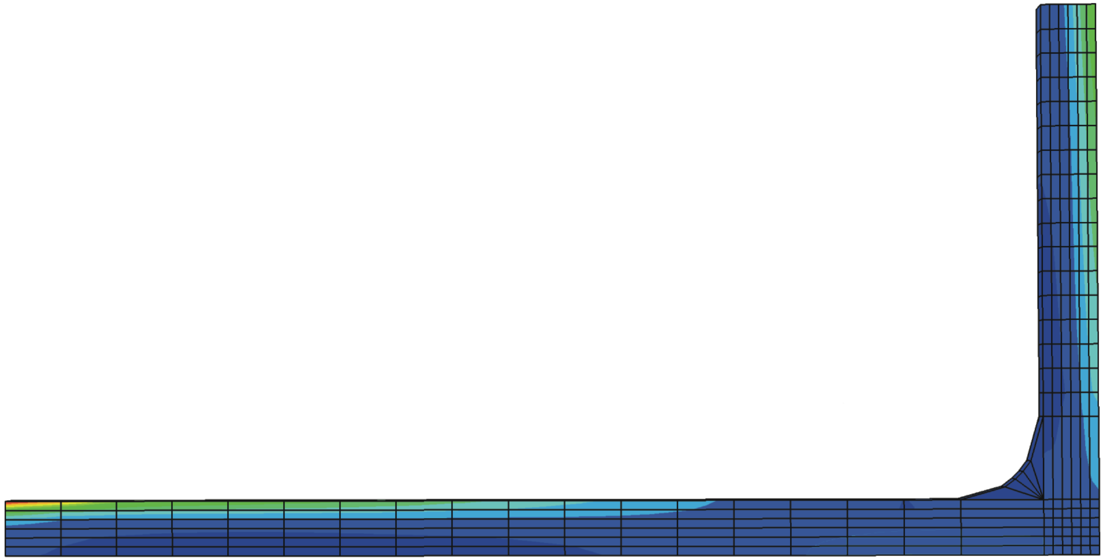


Contents of Today's Lecture

- Short summary
- Convergence of analysis results

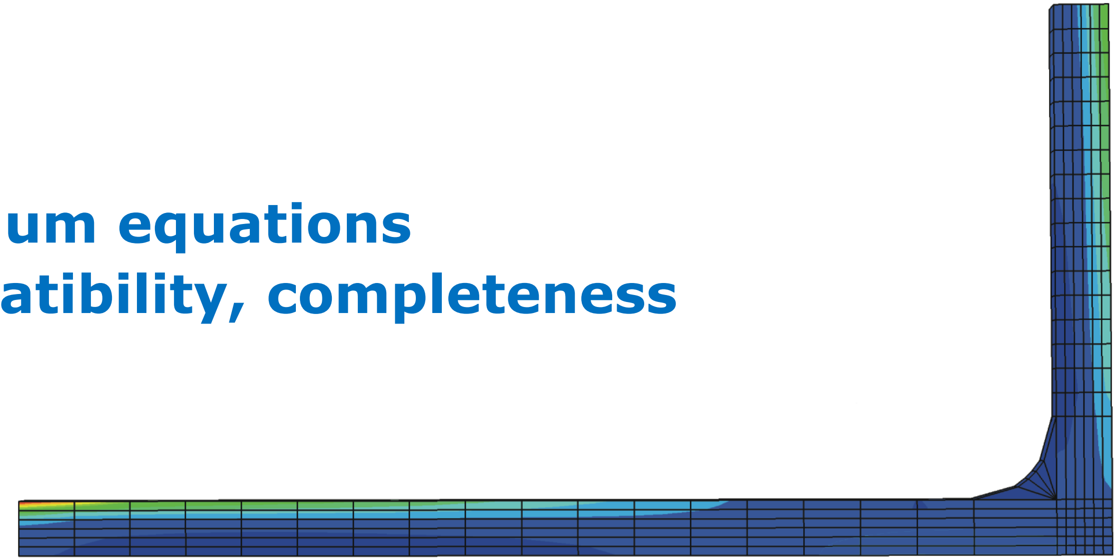
  - The model problem and a definition of convergence
  - Criteria for monotonic convergence
  - Properties of the Finite Element Solution
  - The “Patch Test”

- Discussion
- Mode of oral exam
- Closure ☺



Short Summary

- Introduction to the use of FEM
- Basic concepts of engineering analysis
- Displacement based FEM
- Formulation of Finite Elements
- Implementation
- Isoparametric finite element matrixes
- Quadrilateral elements
- Beam elements
- Plate elements
- Shell elements
- Solution of equilibrium equations
- Convergence, compatibility, completeness



Introduction to the use of finite element

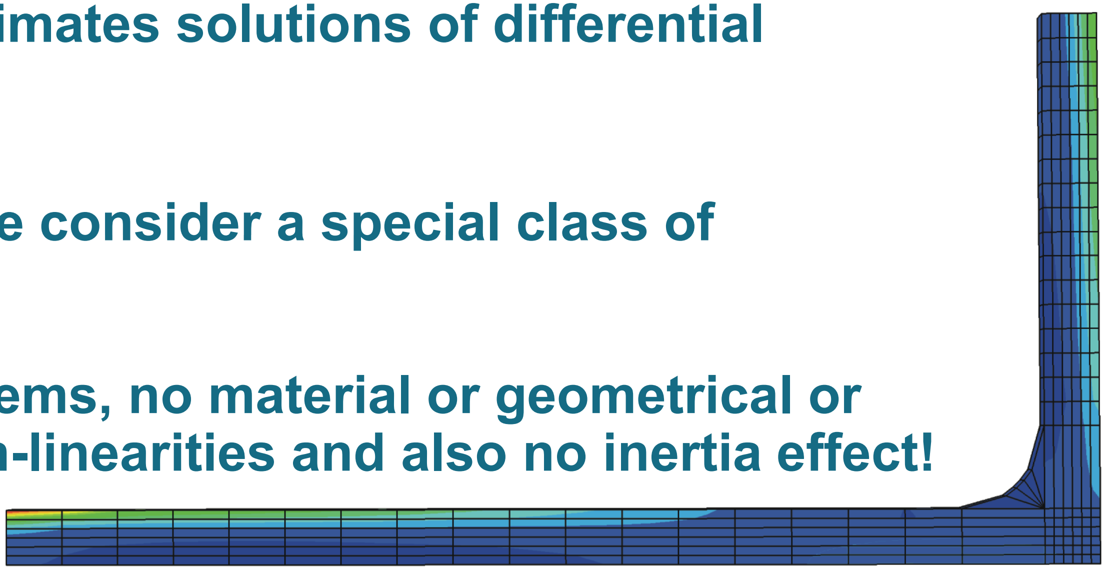
What we would like to establish is the response of a structure subject to "loading".

The Method of Finite Elements provides a framework for the analysis of such responses – however for very general problems.

The Method of Finite Elements provides a very general approach to the approximates solutions of differential equations.

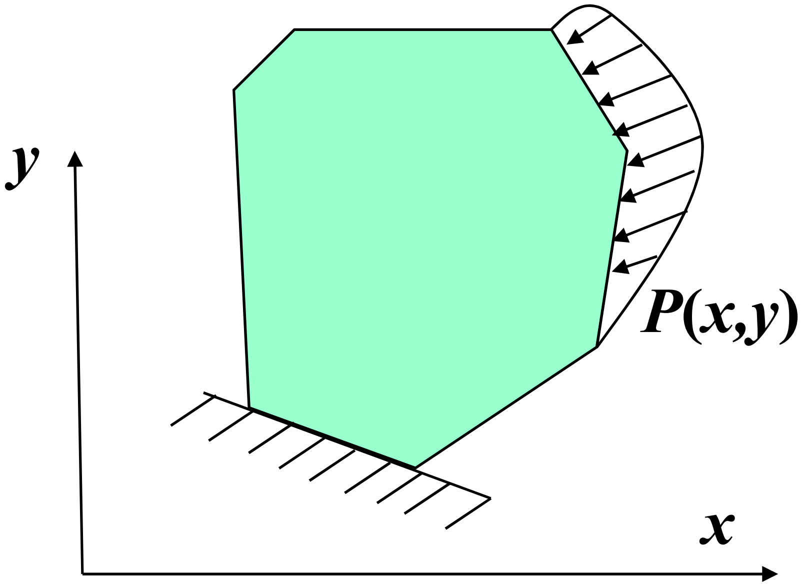
In the present course we consider a special class of problems, namely:

Linear quasi-static systems, no material or geometrical or boundary condition non-linearities and also no inertia effect!



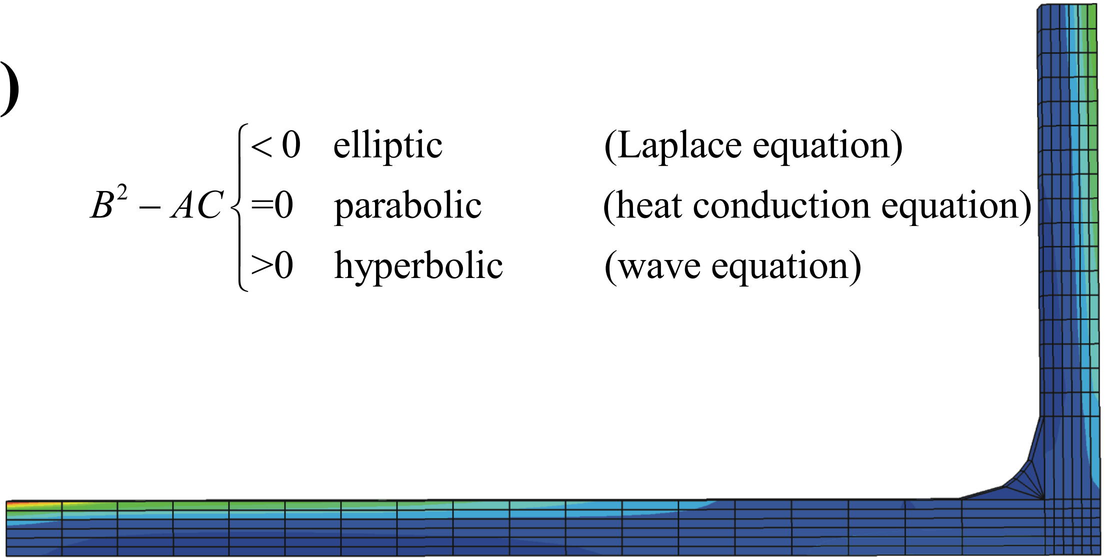
Basic concepts of engineering analysis

In principle the structures/systems we consider can be represented like show in the figure. This type of problem can be analyzed taking basis in the governing differential equation.

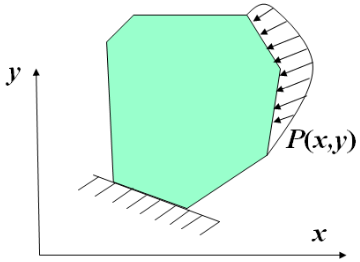


$$A(x,y)\frac{\partial^2 u}{\partial x^2}+2B(x,y)\frac{\partial^2 u}{\partial x \partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}=\phi(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$$

$$B^2-AC\begin{cases}<0 & \text{elliptic} & \text{(Laplace equation)}\\ =0 & \text{parabolic} & \text{(heat conduction equation)}\\ >0 & \text{hyperbolic} & \text{(wave equation)}\end{cases}$$



Displacement based FEM

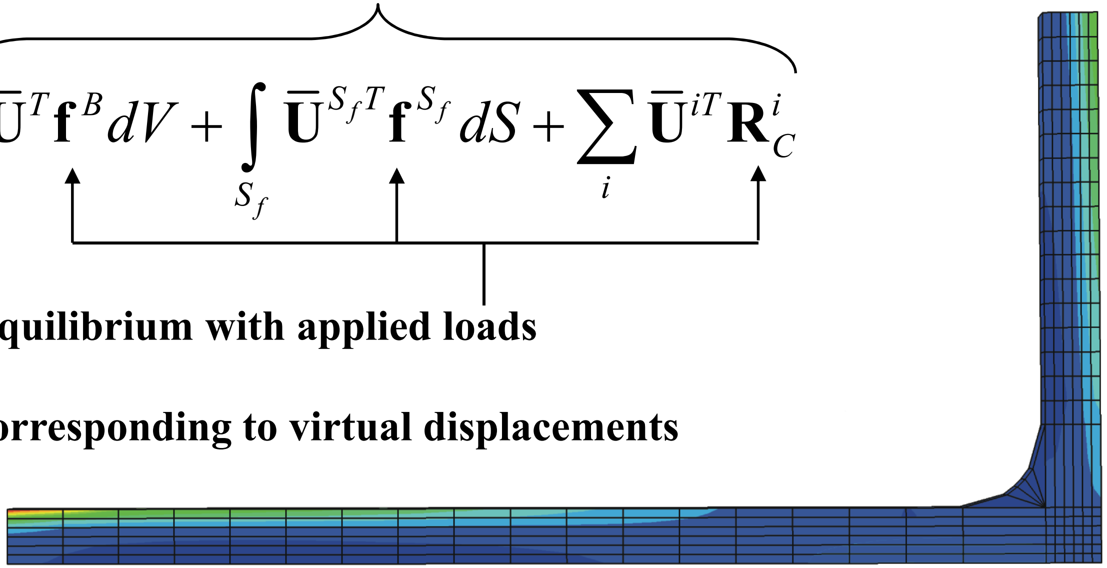


General principles of mechanics on how to derive and solve the differential equations were developed by Ritz and Galerkin – taking basis in variational approaches. These developments led to the principle of virtual displacements (also called as principle of virtual work) - which essentially forms the basis for the Method of Finite Elements.

$$\overbrace{\int_V \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\tau}\, dV}^{\text{Internal virtual work}} = \overbrace{\int_V \bar{\mathbf{U}}^T \mathbf{f}^B dV + \int_{S_f} \bar{\mathbf{U}}^{S_f T} \mathbf{f}^{S_f} dS + \sum_i \bar{\mathbf{U}}^{iT} \mathbf{R}_C^i}^{\text{External virtual work}}$$

Stresses in equilibrium with applied loads

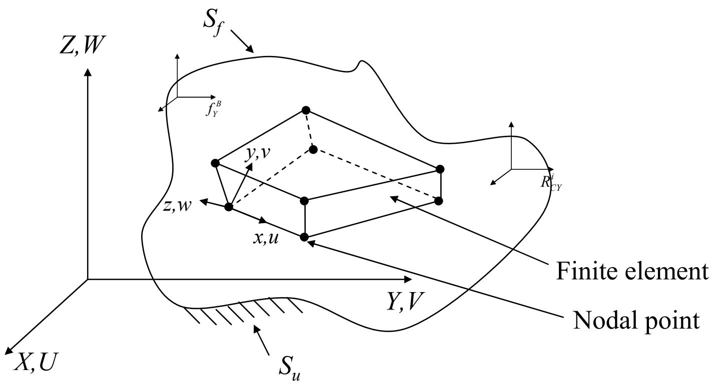
Virtual strains corresponding to virtual displacements



Formulation of Finite Elements

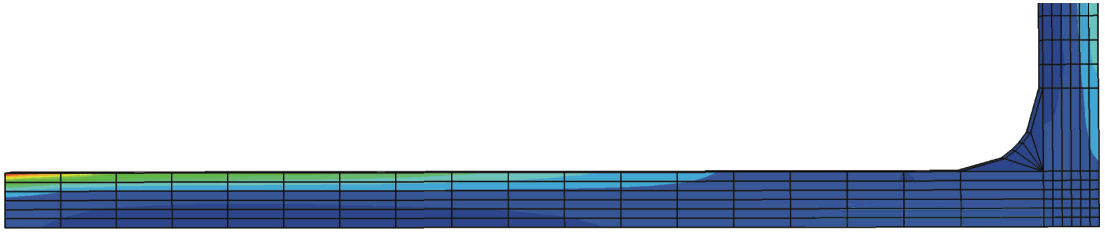
Finite Element Equations:

We now consider the volume modeled as an assemblage of *N* elements connected in the nodal points on the element boundaries

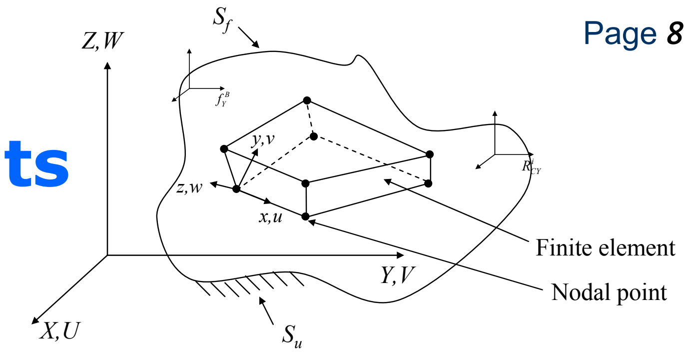


$$\hat{\bar{\mathbf{U}}}^T \left[\sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T} \mathbf{C}^{(m)} \mathbf{B}^{(m)} dV^{(m)} \right] \bar{\mathbf{U}} =$$

$$\hat{\bar{\mathbf{U}}}^T \left[\begin{array}{l} \displaystyle\sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{H}^{(m)T} \mathbf{f}^{B(m)} dV^{(m)} + \sum_{m=1}^{N} \int_{S_{f1}^{(m)}, S_{f2}^{(m)}, \ldots} \mathbf{H}^{(m)T} \mathbf{f}^{S_f(m)} dS^{(m)} \\ \displaystyle -\sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T} \boldsymbol{\tau}^{(m)} dV^{(m)} + \mathbf{R}_C \end{array} \right]$$



Formulation of Finite Elements



Finite Element Equations:

Now we may finally simplify as

$$\mathbf{K}\mathbf{U} = \mathbf{R}$$

$$\mathbf{K} = \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T}\mathbf{C}^{(m)}\mathbf{B}^{(m)} dV^{(m)}$$

These are the finite element equations to be solved ☺

We need efficient approaches to solve these integrals

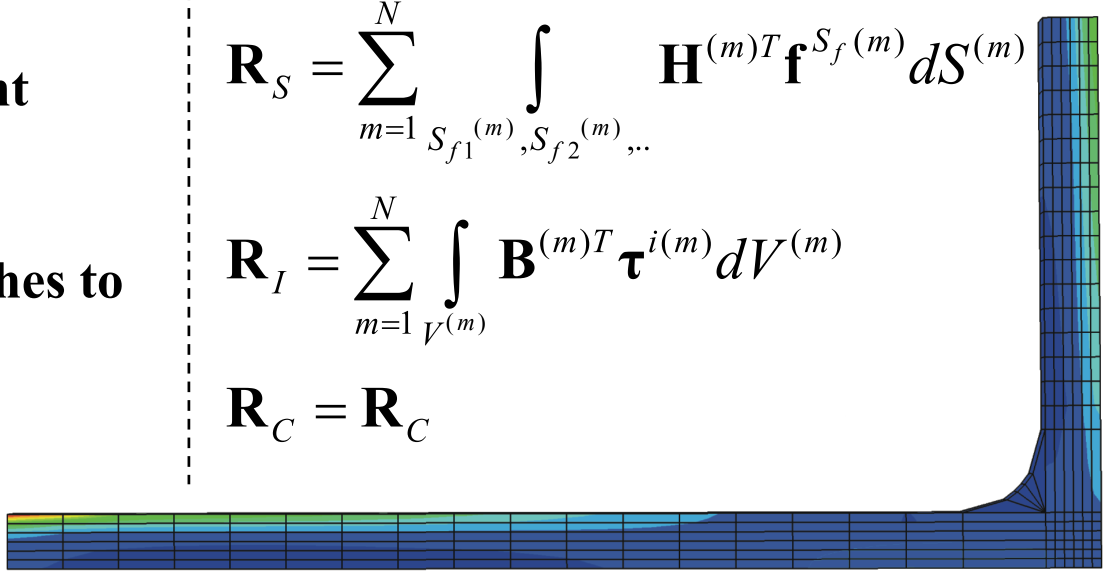
$$\mathbf{R} = \mathbf{R}_B + \mathbf{R}_S - \mathbf{R}_I + \mathbf{R}_C$$

$$\mathbf{R}_B = \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{H}^{(m)T}\mathbf{f}^{B(m)} dV^{(m)}$$

$$\mathbf{R}_S = \sum_{m=1}^{N} \int_{S_{f1}^{(m)},S_{f2}^{(m)},..} \mathbf{H}^{(m)T}\mathbf{f}^{S_f(m)} dS^{(m)}$$

$$\mathbf{R}_I = \sum_{m=1}^{N} \int_{V^{(m)}} \mathbf{B}^{(m)T}\boldsymbol{\tau}^{i(m)} dV^{(m)}$$

$$\mathbf{R}_C = \mathbf{R}_C$$



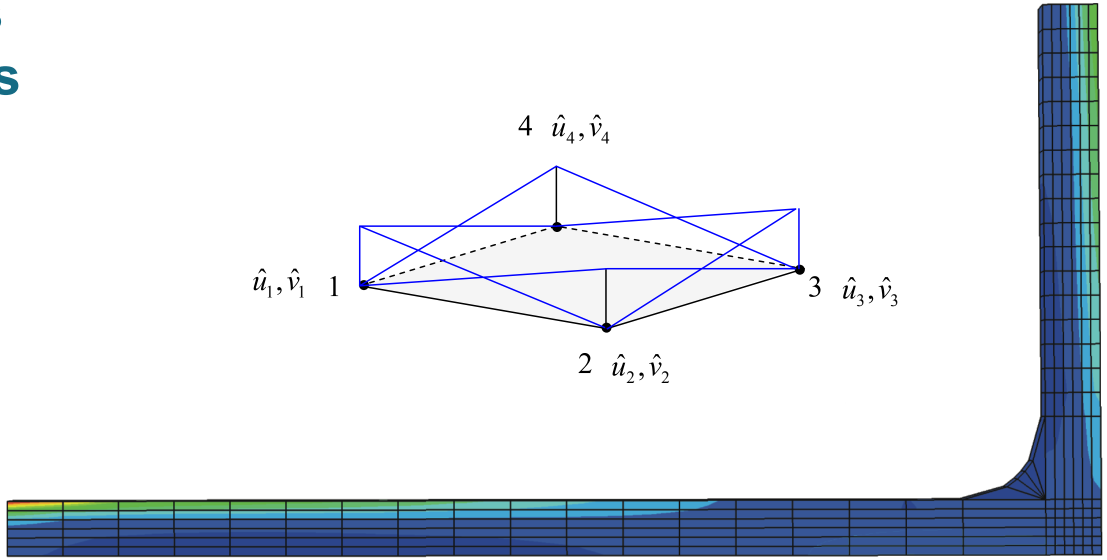
Implementation

Shape functions:

- Requirements to shape functions

- On the choice of shape functions

  Polynomials are usually applied for the development of shape functions (polynomials are easily differentiated analytically)
  - Langrange polynomials
  - Serendipity polynomials
  - Hermitian polynomials



Implementation

Implementation of FEM

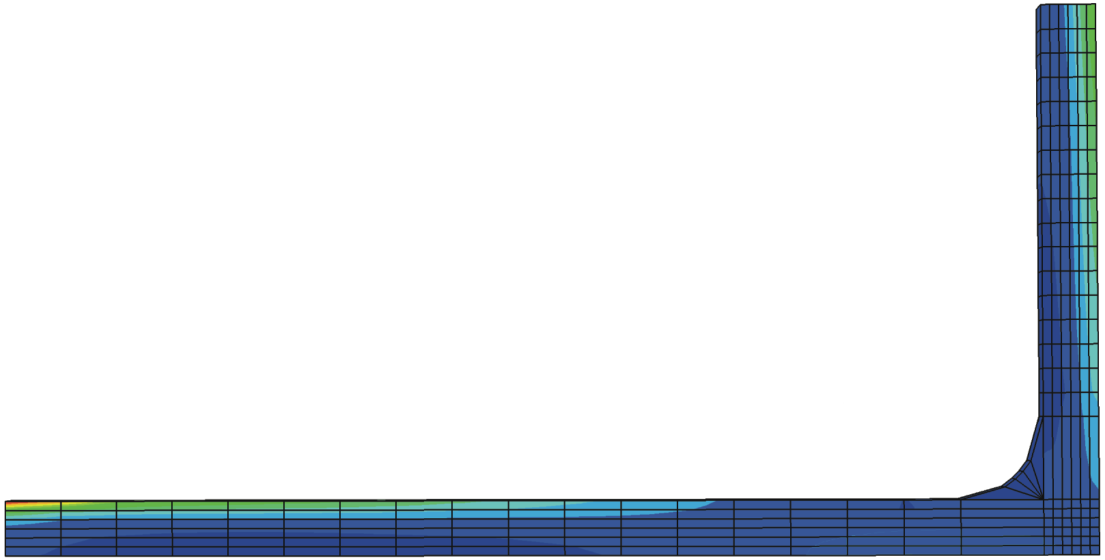
- Integration of “matrixes”
- Interpolation using a polynomial
- Newton Cotes integration
- Gauss integration

In practice we may solve the integrals in terms of sums

$$\int \mathbf{F}(r)dr = \sum_{i} \alpha_i \mathbf{F}(r_i) + \mathbf{R}_n,$$

$$\int \mathbf{F}(r,s)drds = \sum_{i,j} \alpha_{ij} \mathbf{F}(r_i,s_j) + \mathbf{R}_n,$$

$$\int \mathbf{F}(r,s,t)drdsdt = \sum_{i,j,k} \alpha_{ijk} \mathbf{F}(r_i,s_j,t_k) + \mathbf{R}_n$$

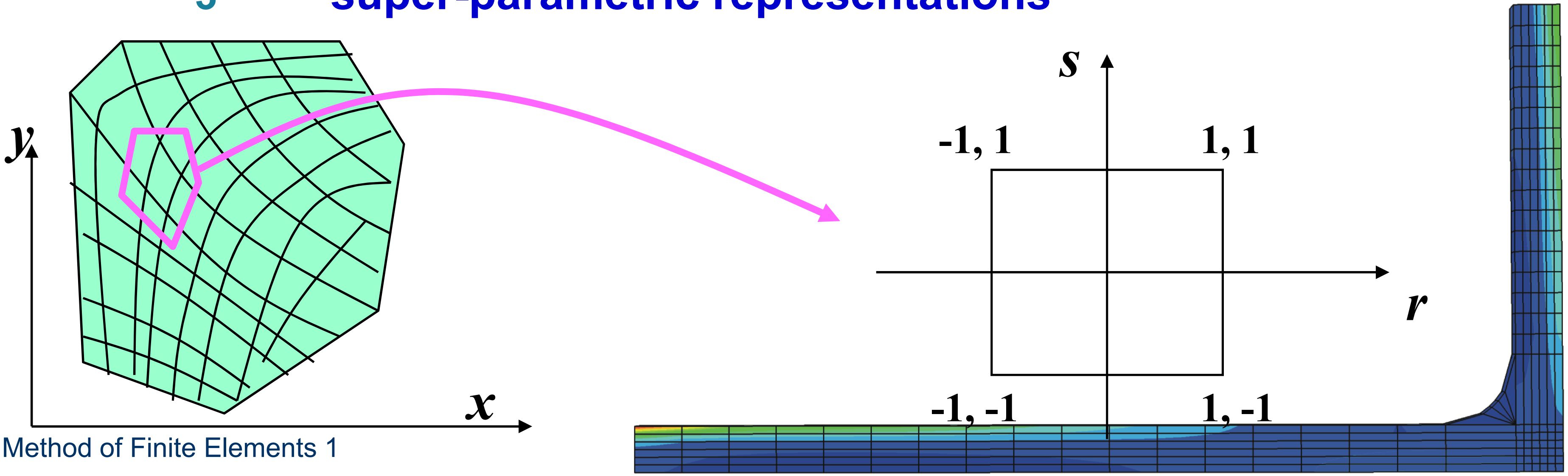


Isoparametric Elements

For the purpose of standardizing the process of developing the element matrixes it is convenient to introduce the so-called **natural coordinate system**.

Different schemes exist for establishing such transformations:

1 **sub-parametric representations**
2 **iso-parametric representations**
3 **super-parametric representations**



Isoparametric Elements

Transformation from natural to global coordinates :

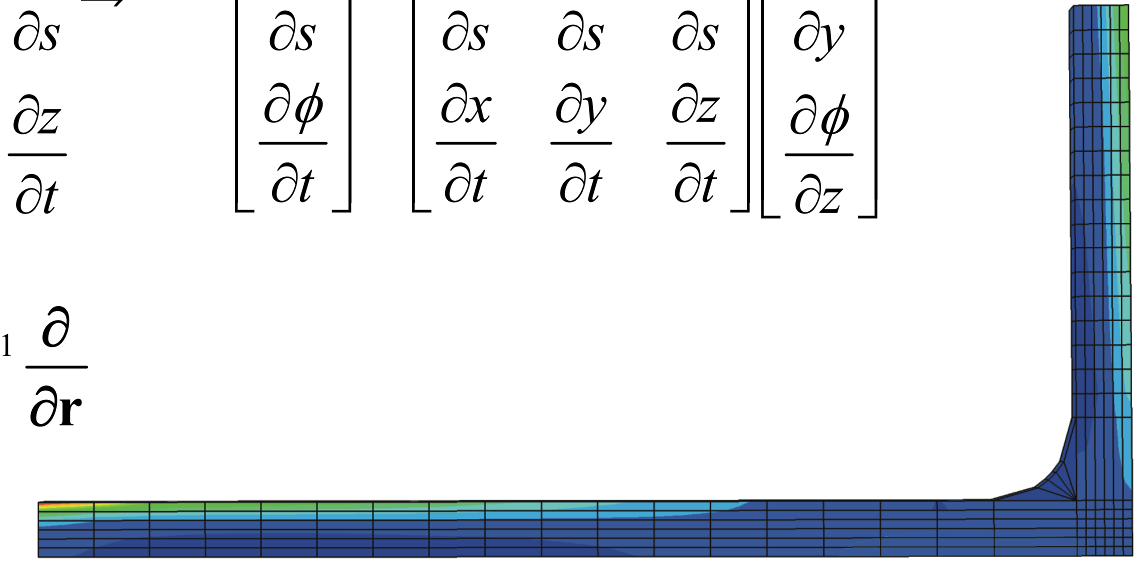
Considering the general three-dimensional case there is:

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial z}{\partial s} \Rightarrow \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial s} \\ \frac{\partial \phi}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial}{\partial \mathbf{r}} = \mathbf{J}\frac{\partial}{\partial \mathbf{x}} \quad \Rightarrow \quad \frac{\partial}{\partial \mathbf{x}} = \mathbf{J}^{-1}\frac{\partial}{\partial \mathbf{r}}$$



Quadrilateral elements

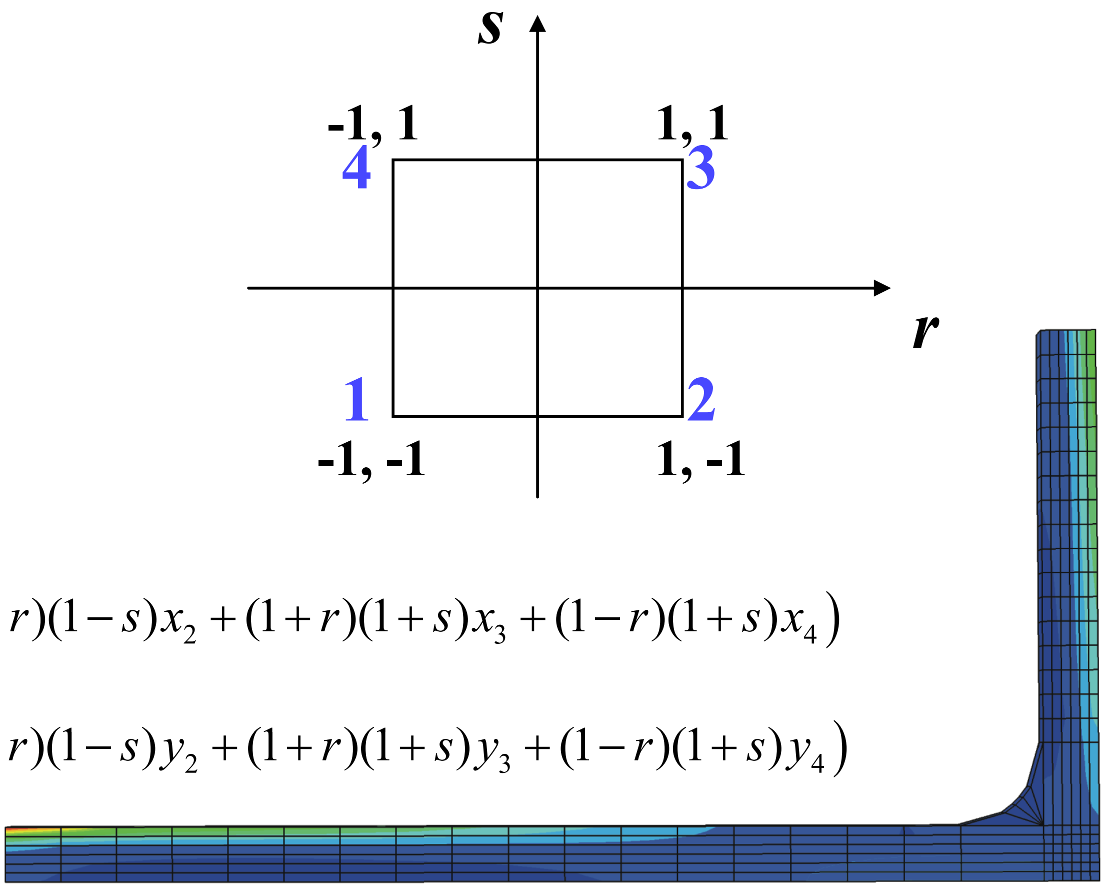
For the bi-linear four node element the shape functions in this coordinate system become:

$$h_1 = \frac{1}{2}(1-r)\frac{1}{2}(1-s)$$

$$h_2 = \frac{1}{2}(1+r)\frac{1}{2}(1-s)$$

$$h_3 = \frac{1}{2}(1+r)\frac{1}{2}(1+s)$$

$$h_4 = \frac{1}{2}(1-r)\frac{1}{2}(1+s)$$



$$x(r,s) = \frac{1}{4}\left((1-r)(1-s)x_1 + (1+r)(1-s)x_2 + (1+r)(1+s)x_3 + (1-r)(1+s)x_4\right)$$

$$y(r,s) = \frac{1}{4}\left((1-r)(1-s)y_1 + (1+r)(1-s)y_2 + (1+r)(1+s)y_3 + (1-r)(1+s)y_4\right)$$

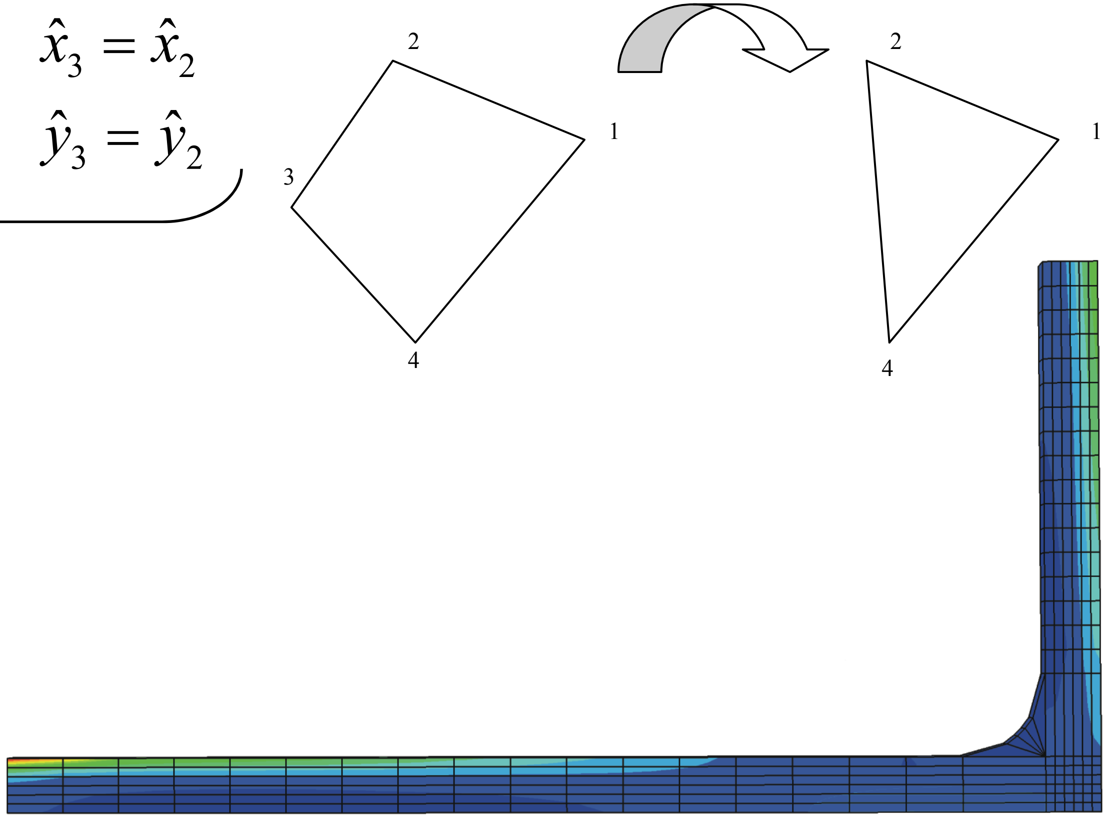
Quadrilateral elements

We can also construct the triangular element directly from the quadrilateral element – by so-called collapsing:

$$\underbrace{\begin{array}{ll} x = h_1\hat{x}_1 + h_2\hat{x}_2 + h_3\hat{x}_3 + h_4\hat{x}_4 & \quad \hat{x}_3 = \hat{x}_2 \\ y = h_1\hat{y}_1 + h_2\hat{y}_2 + h_3\hat{y}_3 + h_4\hat{y}_4 & \quad \hat{y}_3 = \hat{y}_2 \end{array}}$$

$$x = h_1\hat{x}_1 + (h_2 + h_3)\hat{x}_2 + h_4\hat{x}_4$$

$$y = h_1\hat{y}_1 + (h_2 + h_3)\hat{y}_2 + h_4\hat{y}_4$$



Beam Elements

- **Straight beam elements**

  - Straight beam elements: neglecting shear effects (Bernoulli beams)

  - Straight beam elements: including shear effects (Timoshenko beams)

  - Phenomena of shear locking

- **General curved beam elements**

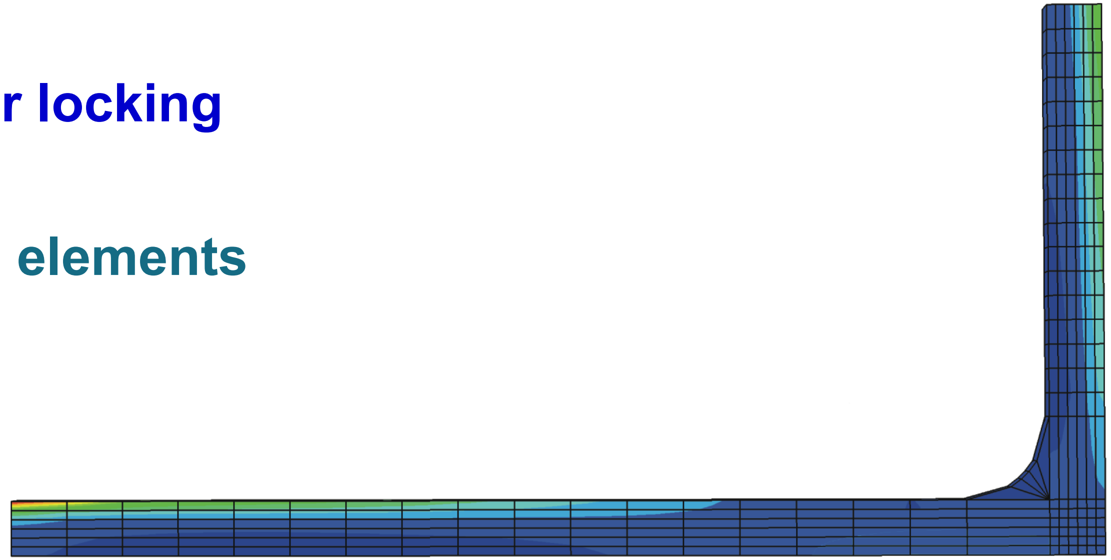
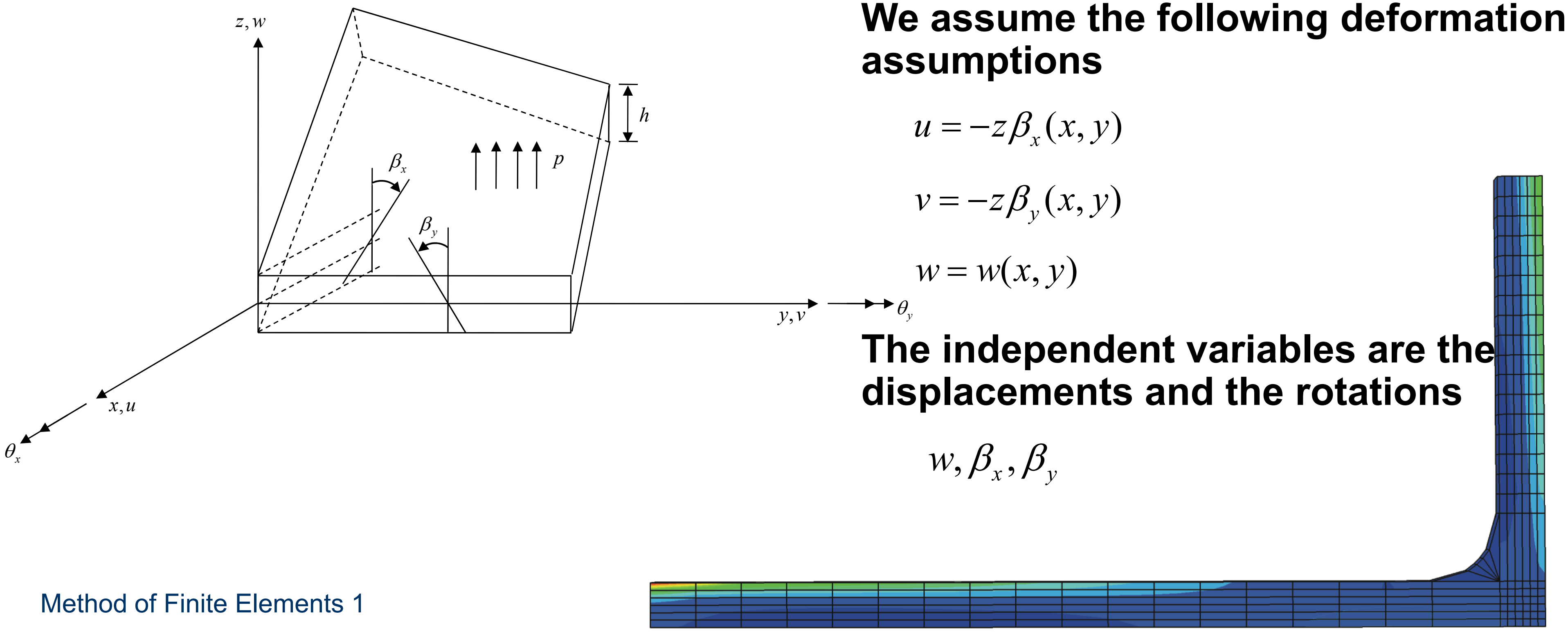


Plate Elements

The Reissner-Midlin plate theory

- Pure displacement based formulation
- Mixed interpolation elements (MITC*n*)
- Performance considerations



We assume the following deformation assumptions

$$u = -z\beta_x(x, y)$$

$$v = -z\beta_y(x, y)$$

$$w = w(x, y)$$

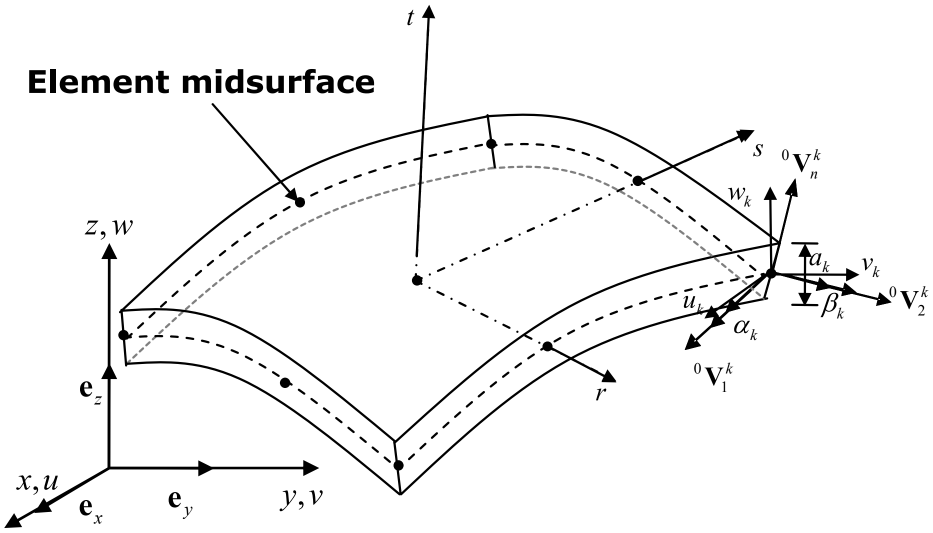
The independent variables are the displacements and the rotations

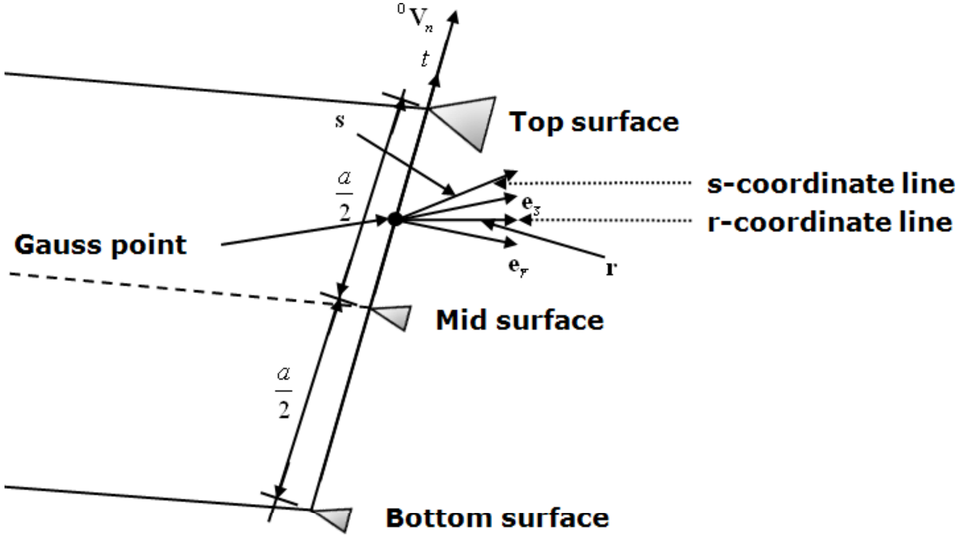
$$w, \beta_x, \beta_y$$

Shell Elements

General shell elements

- Pure displacement based formulation
- Mixed interpolation elements (MITC*n*)

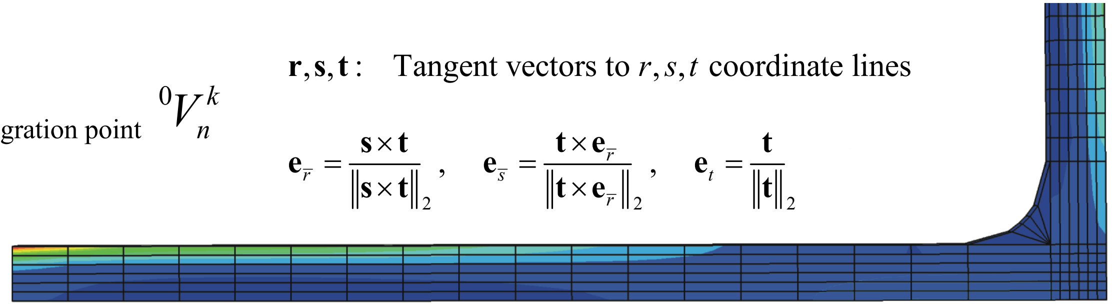




$$\frac{a}{2}\,{}^0V_n\Big|_{\text{at Gauss integration point}} = \sum_k \frac{a}{2} h_k\Big|_{\text{at Gauss integration point}}\,{}^0V_n^k$$

$\mathbf{r}, \mathbf{s}, \mathbf{t}$: Tangent vectors to r, s, t coordinate lines

$$\mathbf{e}_{\bar{r}} = \frac{\mathbf{s}\times\mathbf{t}}{\|\mathbf{s}\times\mathbf{t}\|_2}\,, \quad \mathbf{e}_{\bar{s}} = \frac{\mathbf{t}\times\mathbf{e}_{\bar{r}}}{\|\mathbf{t}\times\mathbf{e}_{\bar{r}}\|_2}\,, \quad \mathbf{e}_{t} = \frac{\mathbf{t}}{\|\mathbf{t}\|_2}$$



Solution of equilibrium equations

- Gauss elimination

- LDL^T solution

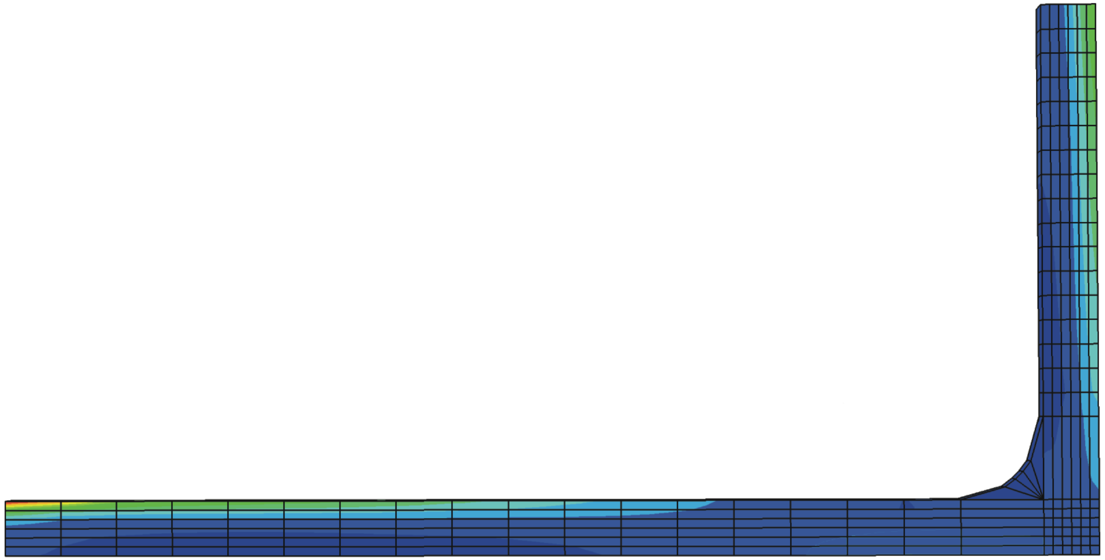
$$\mathbf{K} = \mathbf{L}\mathbf{D}\mathbf{L}^T$$

- Cholesky factorization

$$\mathbf{K} = \tilde{\mathbf{L}}\tilde{\mathbf{L}}^T$$

$$\text{where } \tilde{\mathbf{L}} = \mathbf{L}\mathbf{D}^{\frac{1}{2}}$$

- Solution errors
  - Truncation error
  - Round-off error



Convergence of analysis results

- **The model problem and a definition of convergence**

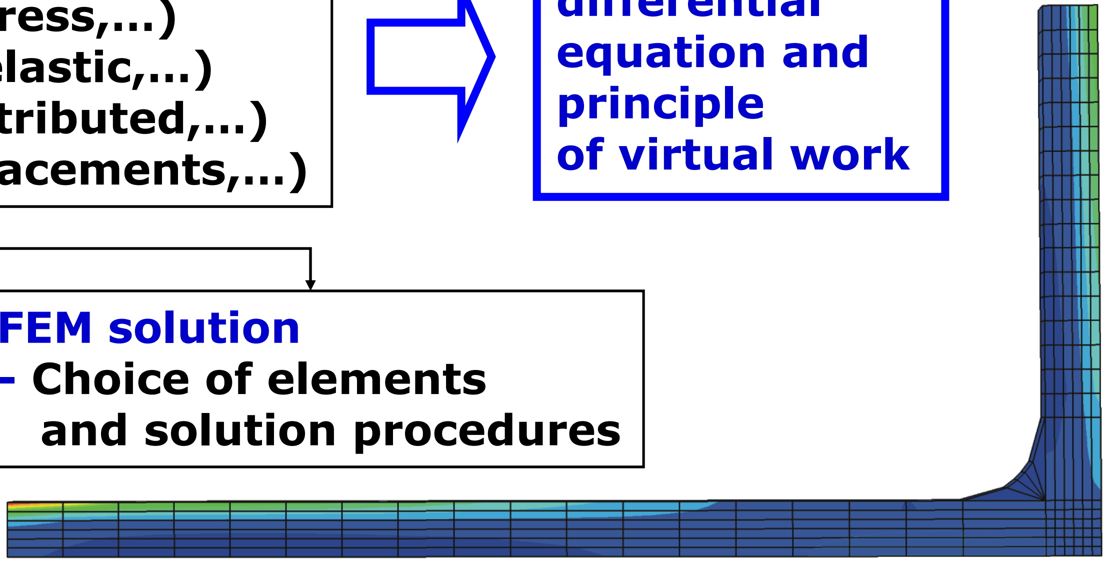
Physical problem
- Geometry
- Material
- Loading
- Boundary conditions

Mathematical model
- Kinematics (truss, plane stress,…)
- Material (isotropic linear, elastic,…)
- Loading (concentrated, distributed,…)
- Boundary conditions (displacements,…)

⇨ **Governing differential equation and principle of virtual work**

FEM solution
- Choice of elements and solution procedures

⇦ **Approximate solution of mathematical model**



Convergence of analysis results

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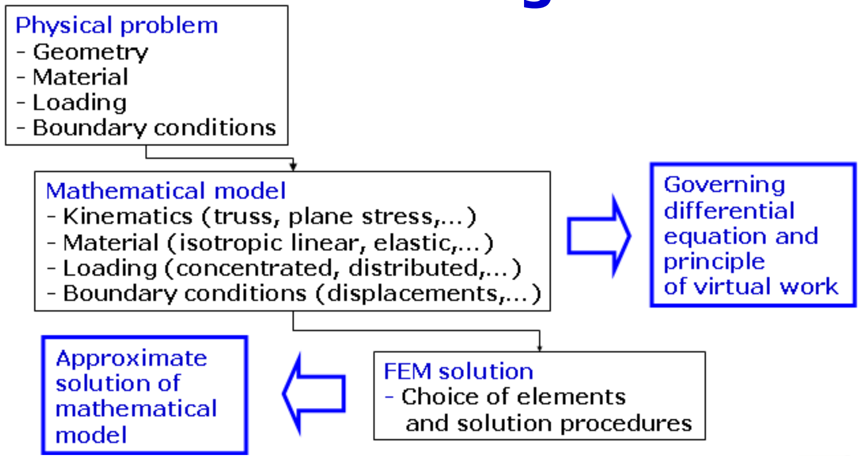

Physical problem
- Geometry
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Mathematical model
- Kinematics (truss, plane stress,...)
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Governing differential equation and principle of virtual work

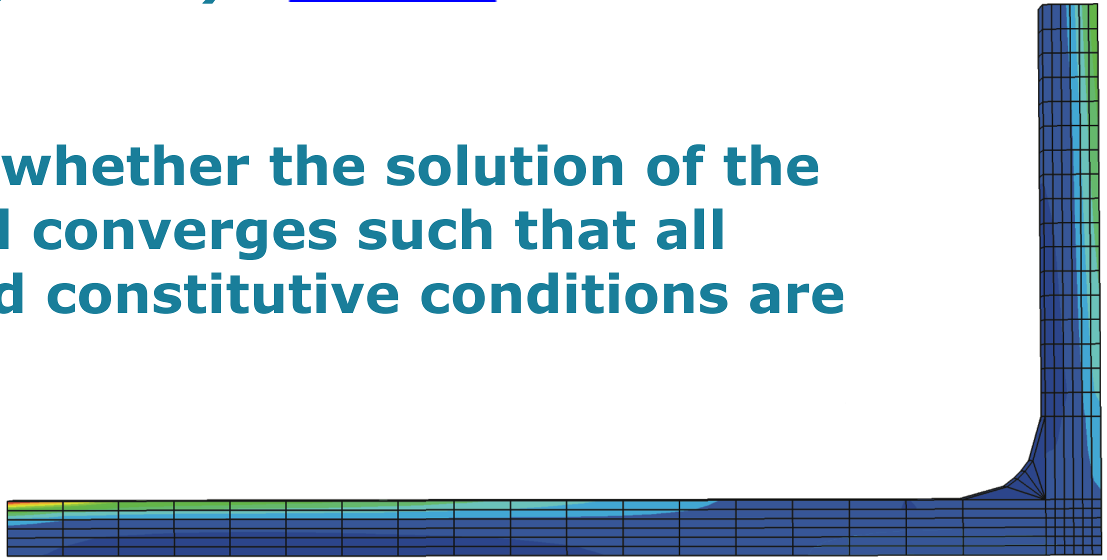
FEM solution
- Choice of elements and solution procedures

Approximate solution of mathematical model

We are interested in the exact solution to the problem!

We don’t know (in general) the exact solution!

We can only assess whether the solution of the mathematical model converges such that all kinematic, static and constitutive conditions are satisfied



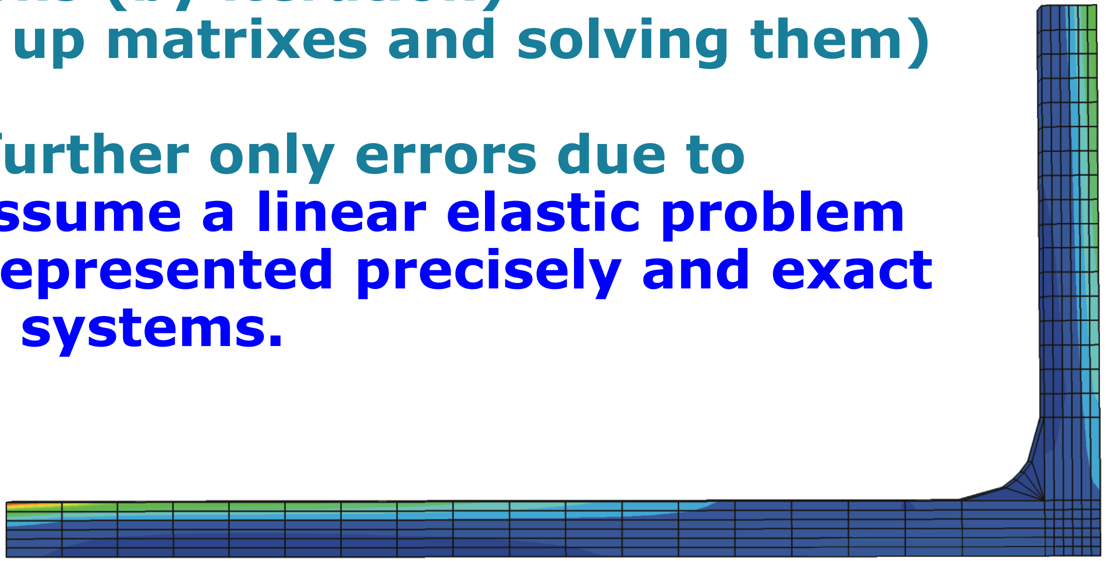
Convergence of analysis results

- **The model problem and a definition of convergence**

The solution is subject to the following possible errors:

- Discretization (interpolation functions)
- Numerical integration (finite element matrixes)
- Evaluation of constitutive relations (non-linear)
- Solution of equations (by iteration)
- Round off (setting up matrixes and solving them)

We consider in the further only errors due to discretization; we assume a linear elastic problem with the geometry represented precisely and exact solution of equation systems.



Convergence of analysis results

- **The model problem and a definition of convergence**

To proceed we consider the principle of virtual work:

$$\int_V \overline{\boldsymbol{\varepsilon}}^T \boldsymbol{\tau} dV = \int_{S_f} \overline{\mathbf{u}}^{S_f T} \mathbf{f}^{S_f} dS + \int_V \overline{\mathbf{u}}^T \mathbf{f}^B dV$$

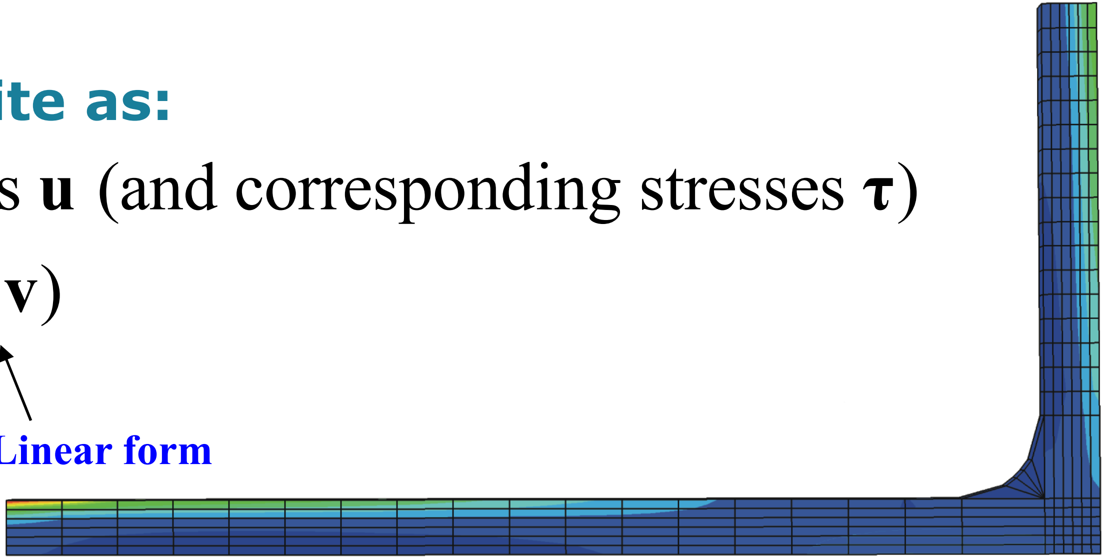
which we now rewrite as:

Find the displacements $\mathbf{u}$ (and corresponding stresses $\boldsymbol{\tau}$)

such that $a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v})$

$$\underset{\text{Bi-linear form}}{\uparrow} \qquad \underset{\text{Linear form}}{\uparrow}$$

Bi-linear form **Linear form**



Convergence of analysis results

- **The model problem and a definition of convergence**

$$\int_V \overline{\boldsymbol{\varepsilon}}^T \boldsymbol{\tau} dV = \int_{S_f} \overline{\mathbf{u}}^{S_f^T} \mathbf{f}^{S_f} dS + \int_V \overline{\mathbf{u}}^T \mathbf{f}^B dV$$

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

$$\frac{1}{2} a(\mathbf{u}, \mathbf{u}) = \text{strain energy}$$

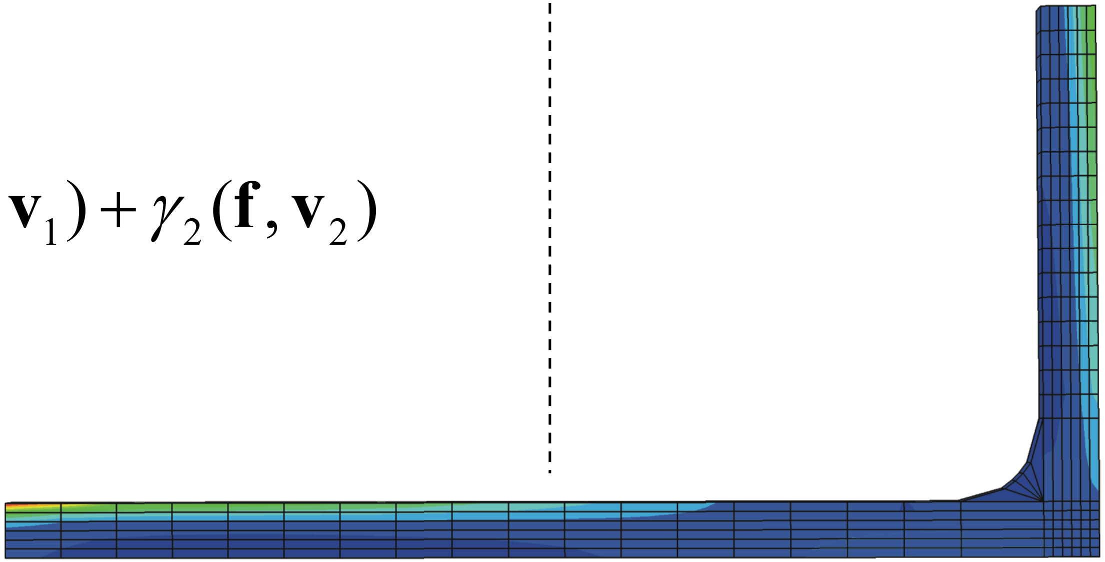
Bi-linearity refers to:

$$a(\gamma_1 \mathbf{u}_1 + \gamma_2 \mathbf{u}_2, \mathbf{v}) = \gamma_1 a(\mathbf{u}_1, \mathbf{v}) + \gamma_2 a(\mathbf{u}_2, \mathbf{v})$$

$$a(\mathbf{u}, \gamma_1 \mathbf{v}_1 + \gamma_2 \mathbf{v}_2) = \gamma_1 a(\mathbf{u}, \mathbf{v}_1) + \gamma_2 a(\mathbf{u}, \mathbf{v}_2)$$

Linearity refers to:

$$(\mathbf{f}, \gamma_1 \mathbf{v}_1 + \gamma_2 \mathbf{v}_2) = \gamma_1 (\mathbf{f}, \mathbf{v}_1) + \gamma_2 (\mathbf{f}, \mathbf{v}_2)$$



Convergence of analysis results

- **The model problem and a definition of convergence**

Assuming that the FEM solution is: $\mathbf{u}_h$
and the exact solution is: $\mathbf{u}$

then convergence may be defined as:

$$\boxed{a(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h) \rightarrow 0, \quad \text{as } h \rightarrow 0}$$

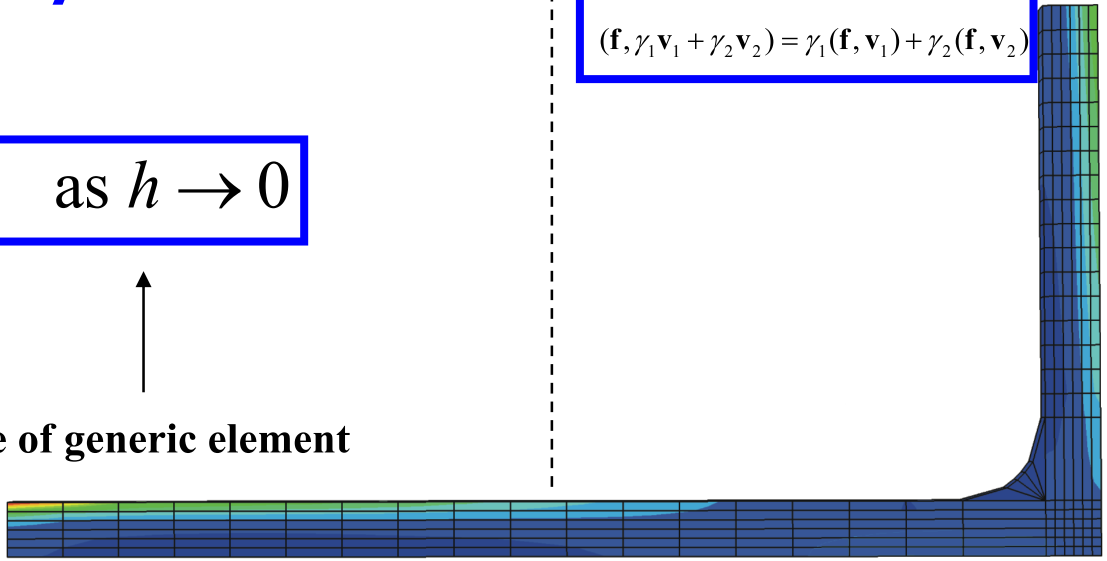
Size of generic element

$$\int_V \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\tau} dV = \int_{S_f} \bar{\mathbf{u}}^{S_f^T} \mathbf{f}^{S_f} dS + \int_V \bar{\mathbf{u}}^T \mathbf{f}^B dV$$

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

$$a(\gamma_1 \mathbf{u}_1 + \gamma_2 \mathbf{u}_2, \mathbf{v}) = \gamma_1 a(\mathbf{u}_1, \mathbf{v}) + \gamma_2 a(\mathbf{u}_2, \mathbf{v})$$
$$a(\mathbf{u}, \gamma_1 \mathbf{v}_1 + \gamma_2 \mathbf{v}_2) = \gamma_1 a(\mathbf{u}, \mathbf{v}_1) + \gamma_2 a(\mathbf{u}, \mathbf{v}_2)$$

$$(\mathbf{f}, \gamma_1 \mathbf{v}_1 + \gamma_2 \mathbf{v}_2) = \gamma_1 (\mathbf{f}, \mathbf{v}_1) + \gamma_2 (\mathbf{f}, \mathbf{v}_2)$$



Convergence of analysis results

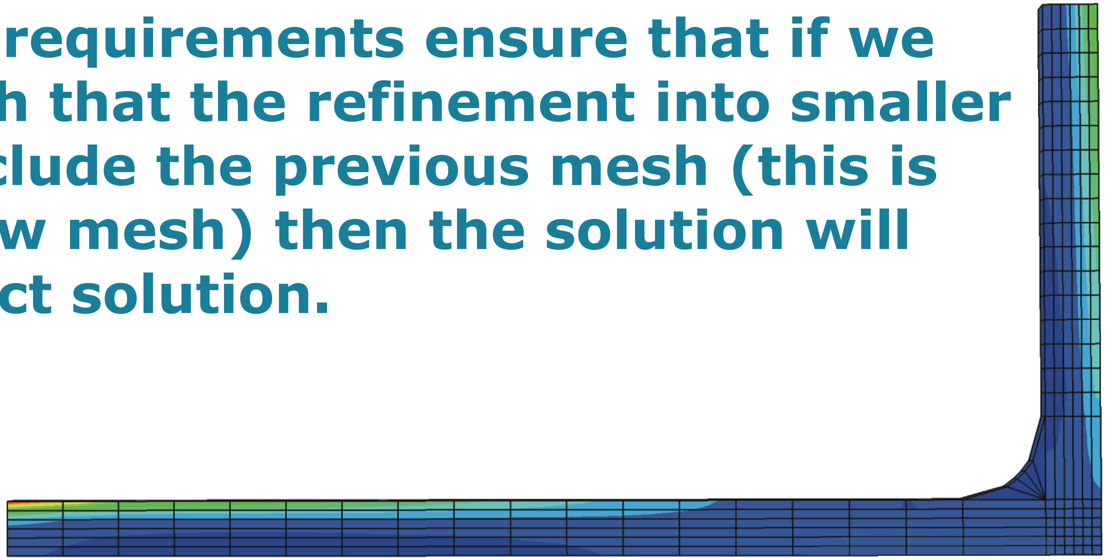
- **Criteria for monotonic convergence**

For monotonic convergence, the elements must be:

Complete

Compatible

Fulfillment of these requirements ensure that if we refine the mesh such that the refinement into smaller elements always include the previous mesh (this is embedded in the new mesh) then the solution will converge to the exact solution.

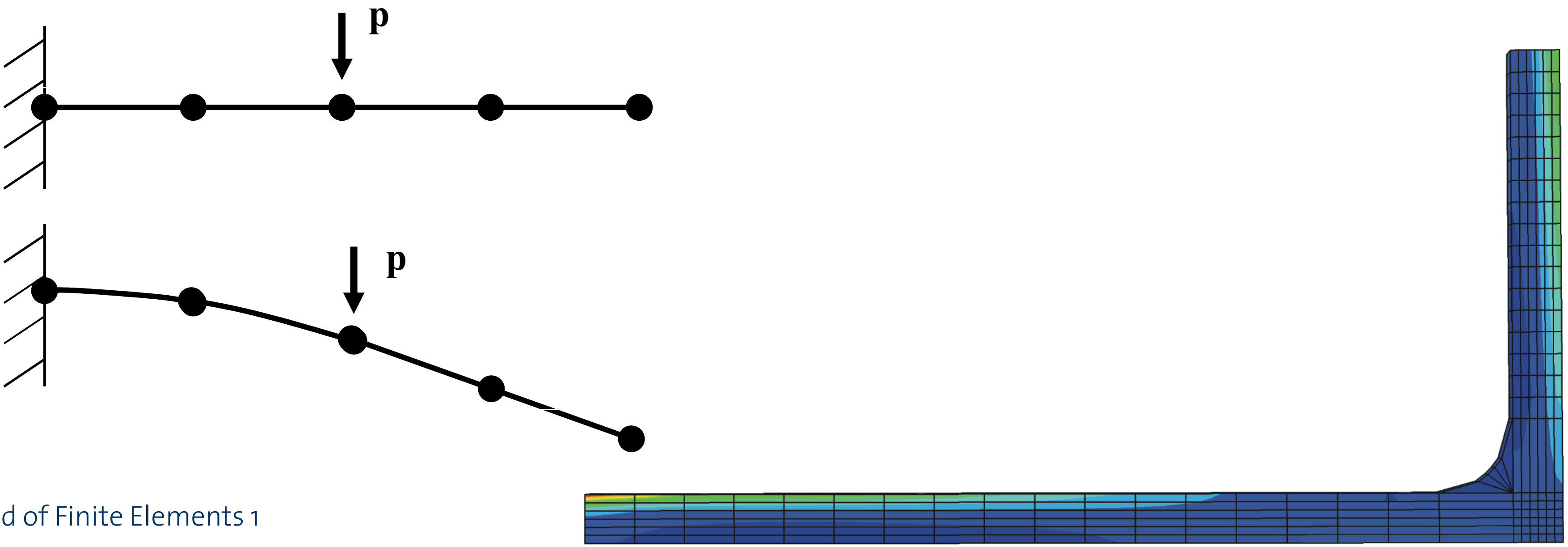


Convergence of analysis results

- **Criteria for monotonic convergence**

Completeness:

The elements must be able to represent all rigid body displacements and also constant strain state



Convergence of analysis results

- **Criteria for monotonic convergence**

Completeness:

The displacement modes which may be represented by a given element can be identified by solving the eigenvalue problem

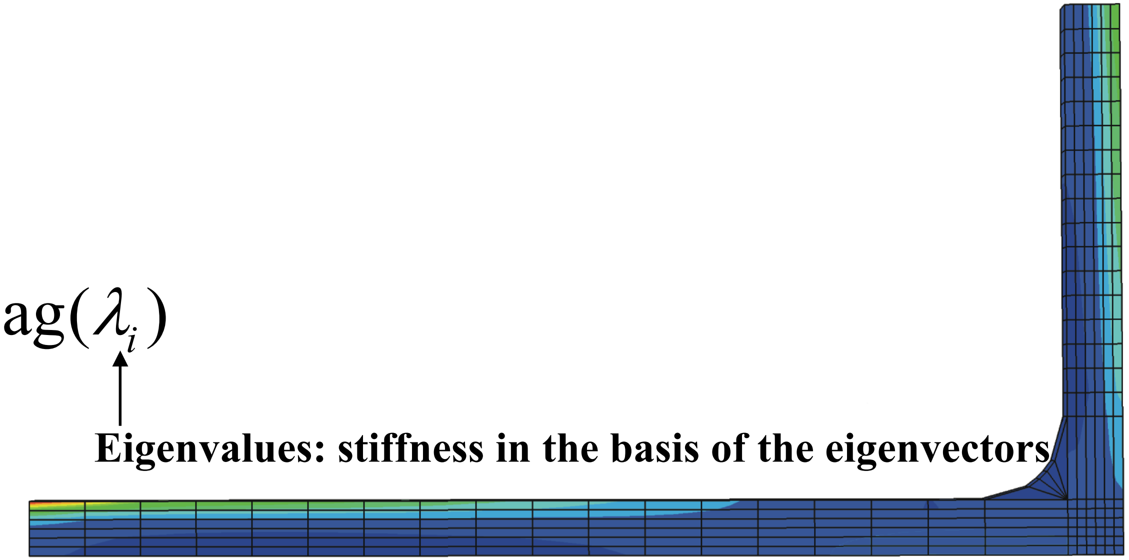
$$\mathbf{K}\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}$$

Eigen vectors

$$\mathbf{K}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\Lambda}, \qquad \boldsymbol{\Lambda} = \mathrm{diag}(\lambda_i)$$

$$\boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi} = \boldsymbol{\Lambda}$$

Eigenvalues: stiffness in the basis of the eigenvectors

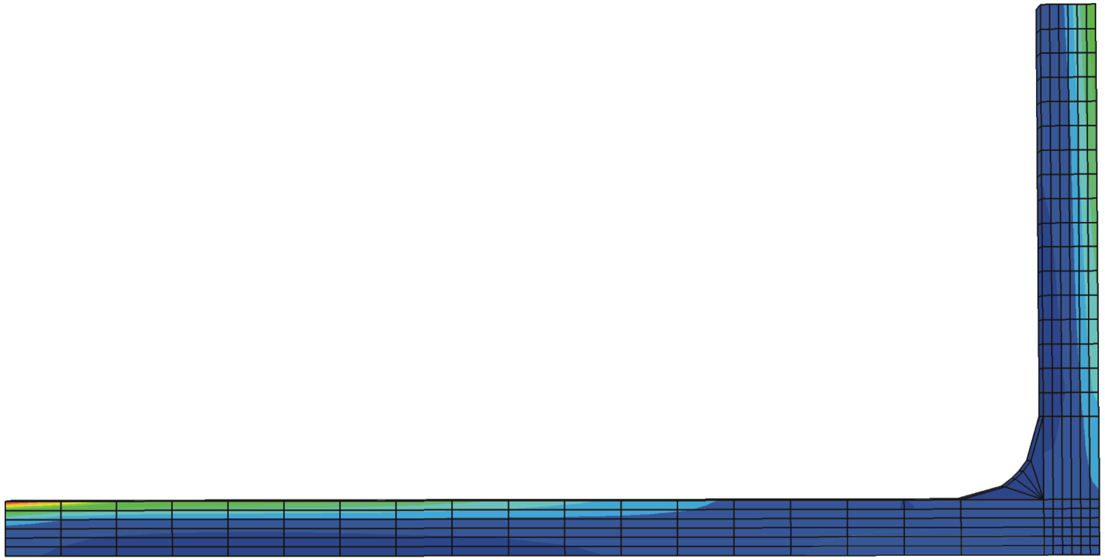


Convergence of analysis results

- **Criteria for monotonic convergence**

Completeness:

The constant strain state is required as when we reduce the size of the generic element *h* – then in the limit as *h* approaches zero the strain must approach a constant stress state



Convergence of analysis results

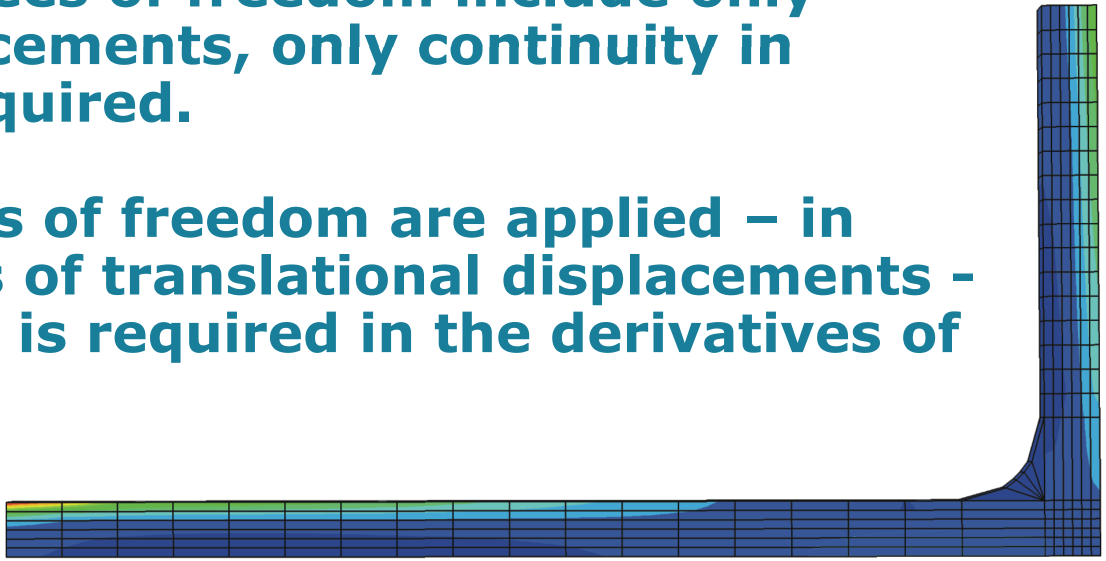
- **Criteria for monotonic convergence**

Compatibility:

The displacements within and between elements must be continuous;
Avoiding gaps between elements in a loaded situation

If the element degrees of freedom include only translational displacements, only continuity in displacements is required.

If rotational degrees of freedom are applied – in terms of derivatives of translational displacements - then also continuity is required in the derivatives of the displacements.



Convergence of analysis results

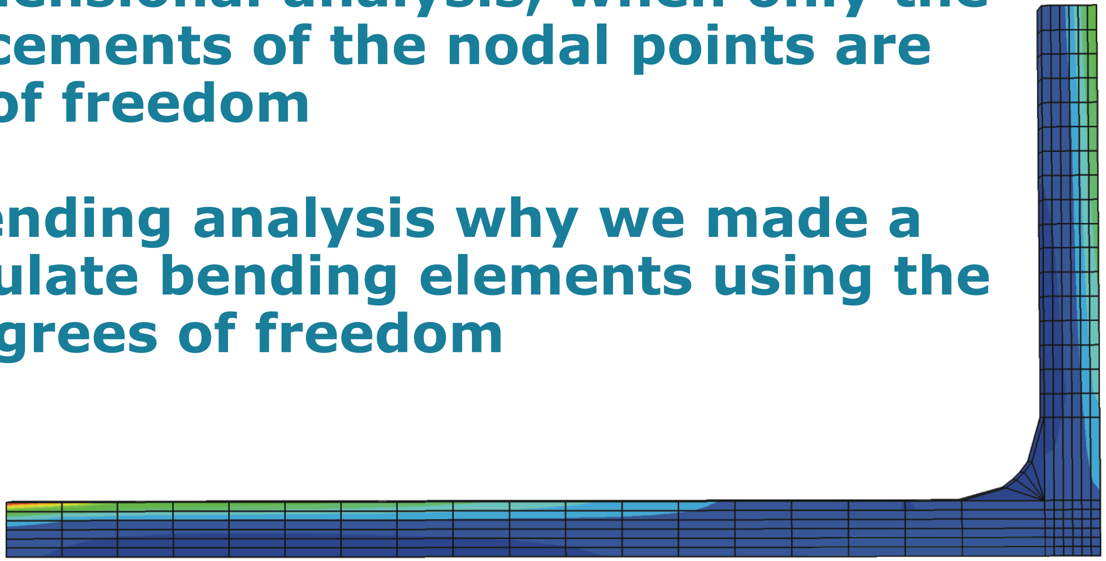
- **Criteria for monotonic convergence**

Compatibility:

Automatically ensured between truss and beam elements as they only join in the nodal points

As we have seen also, compatibility is relatively easy to ensure in 2-3 dimensional analysis, when only the translational displacements of the nodal points are applied as degrees of freedom

Difficult for plate bending analysis why we made a great effort to formulate bending elements using the rotations also as degrees of freedom

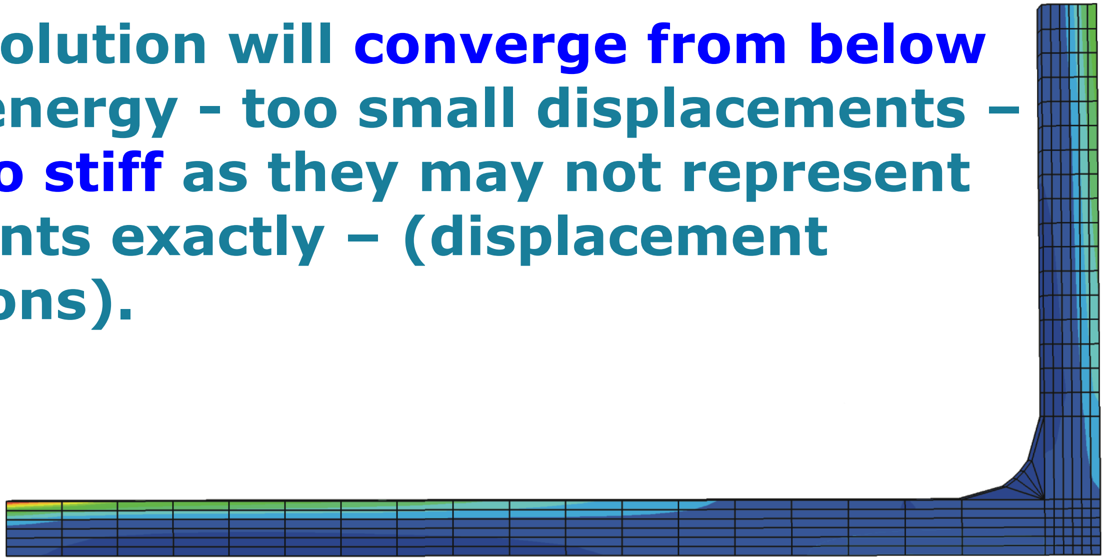


Convergence of analysis results

- **Properties of the Finite Element Solution**

  Uniqueness:
  The exact solution to our elasticity problem is unique meaning that **there are no two different exact solutions.**

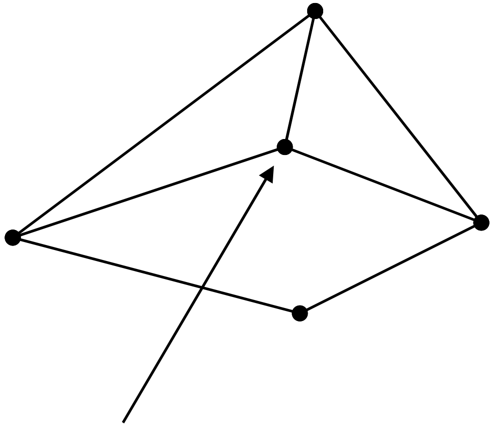
  Convergence:
  The finite element solution will **converge from below** to the exact strain energy - too small displacements – **the elements are too stiff** as they may not represent the true displacements exactly – (displacement interpolation functions).



Convergence of analysis results

- **The “Patch Test“**

  The idea in this test is to consider an arbitrary patch of elements:

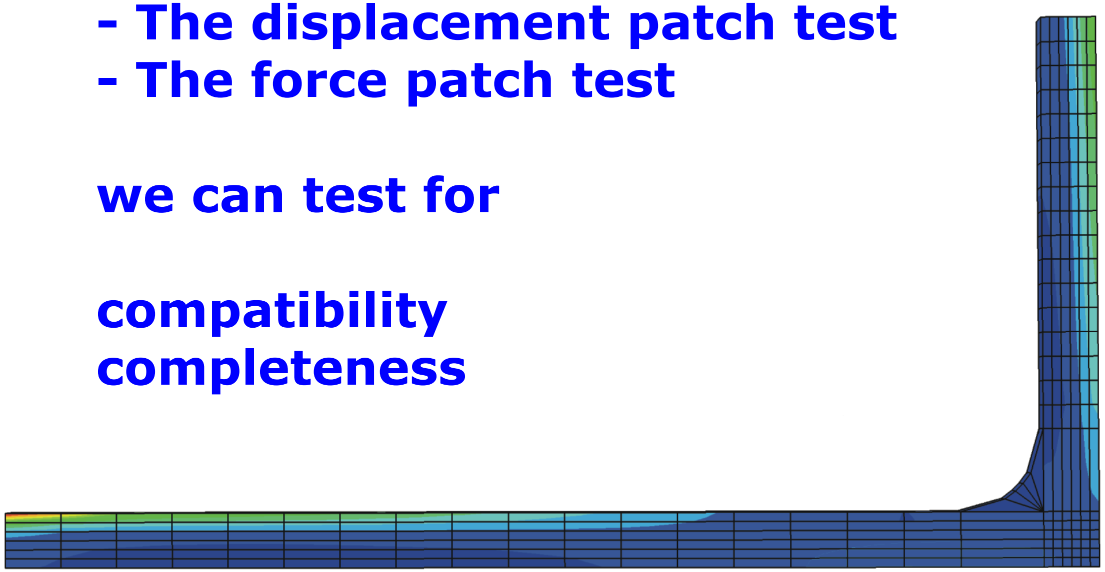


There are two dual tests !

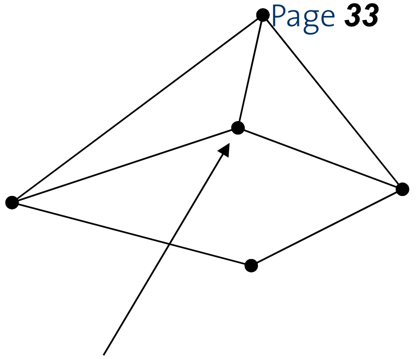
- The displacement patch test
- The force patch test

we can test for

compatibility
completeness



Convergence of analysis results



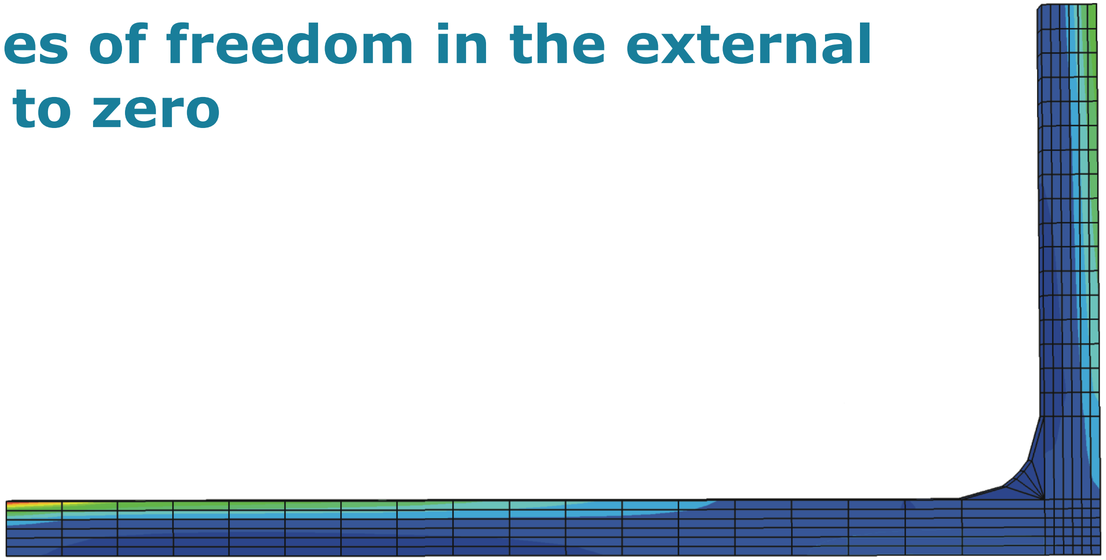
Internal node

- **The "Patch Test"**

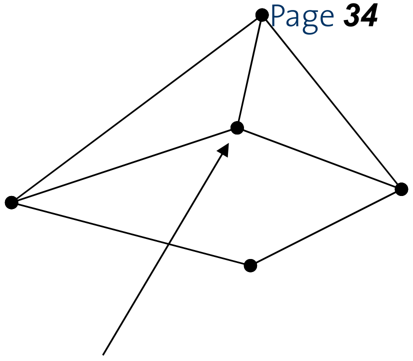
The elements (displacement interpolation functions) are **compatible** if we can prescribe:

One degree of freedom of the internal node to be equal to 1 and the other to zero

Verify that all degrees of freedom in the external nodes remain equal to zero



Convergence of analysis results

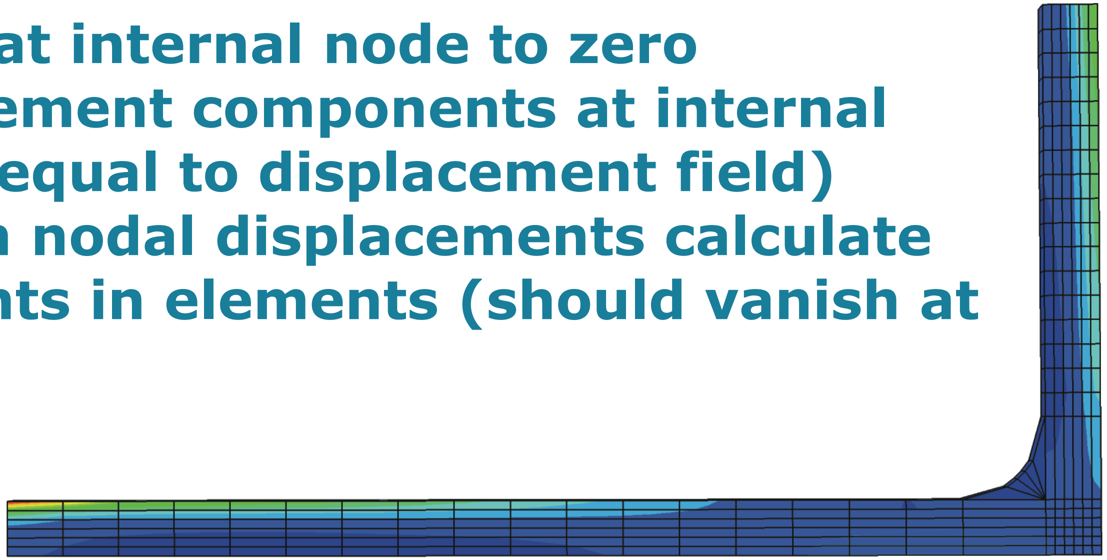


- **The "Patch Test"**

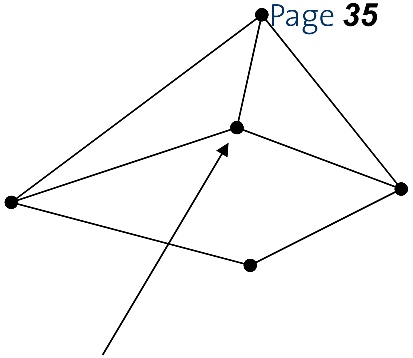
The displacement patch test

For rigid body displacement modes:

1) **Apply rigid body displacement field to external nodes**
2) **Prescribe forces at internal node to zero**
3) **Solve for displacement components at internal node (should be equal to displacement field)**
4) **Now – with given nodal displacements calculate strains at all points in elements (should vanish at all points)**



Convergence of analysis results



- **The "Patch Test"**

The displacement patch test

For constant strain displacement modes:

1) **Apply constant strain displacement field to external nodes**
2) **Prescribe forces at internal node to zero**
3) **Solve for displacement components at internal node (should be equal to displacement field)**
4) **Now – with given nodal displacements calculate strains at all points in elements (should comply with the strain corresponding to the applied displacement field at all points)**