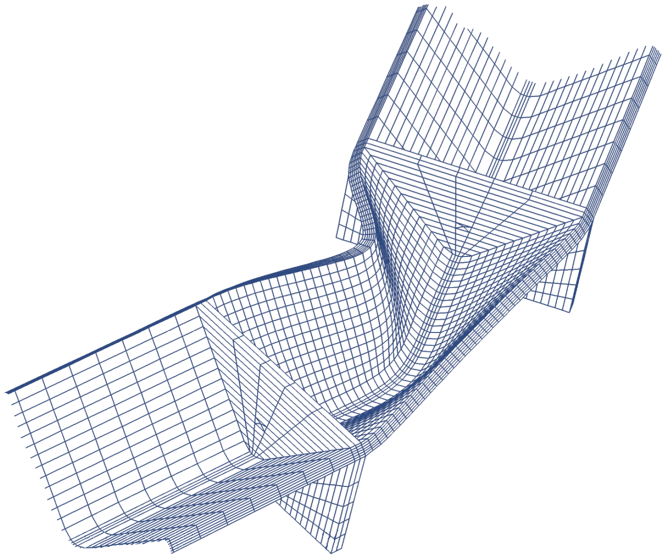
The Finite Element Method for the Analysis of Linear Systems



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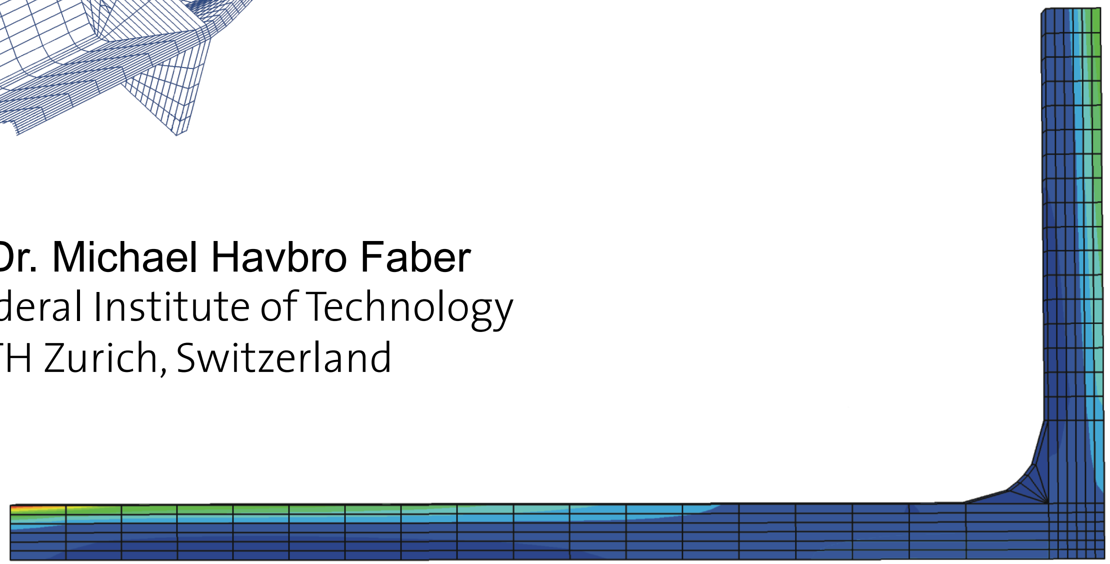
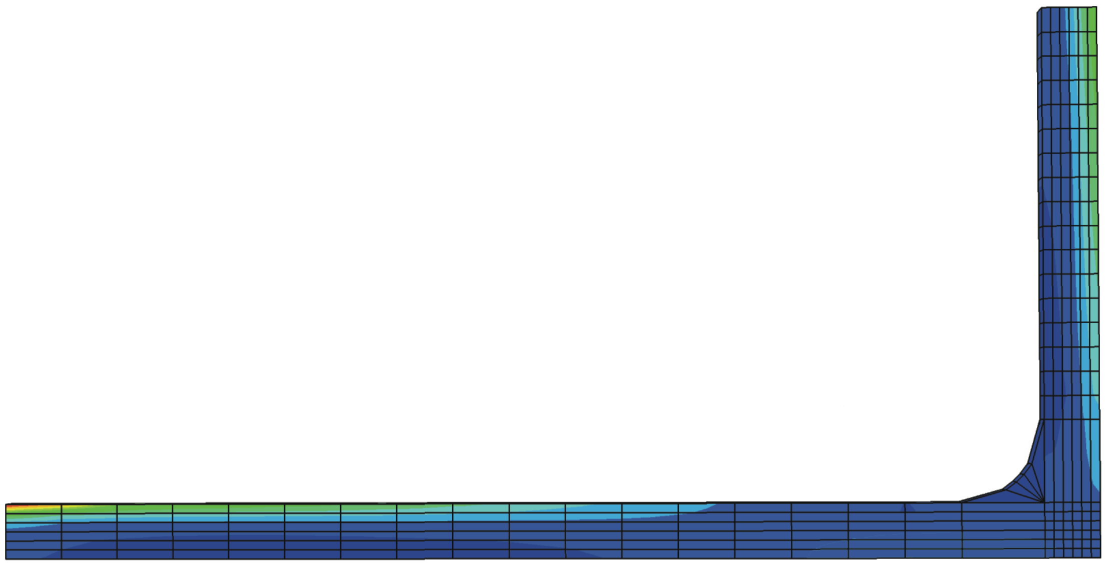


Contents of Today's Lecture

- Gauss elimination
- LDL^T solution
- Cholesky factorization and other related methods
- Solution errors



Gauss elimination

In general:

Solve Ax=b for x

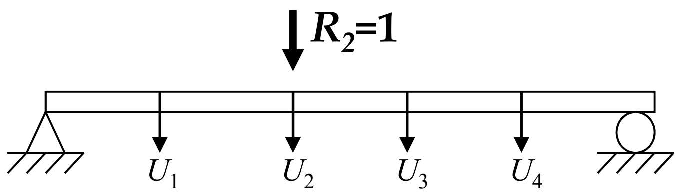
A: a matrix of coefficients,
x: the vector of unknowns,
b: the right-hand side vector

In the context of finite element problems:

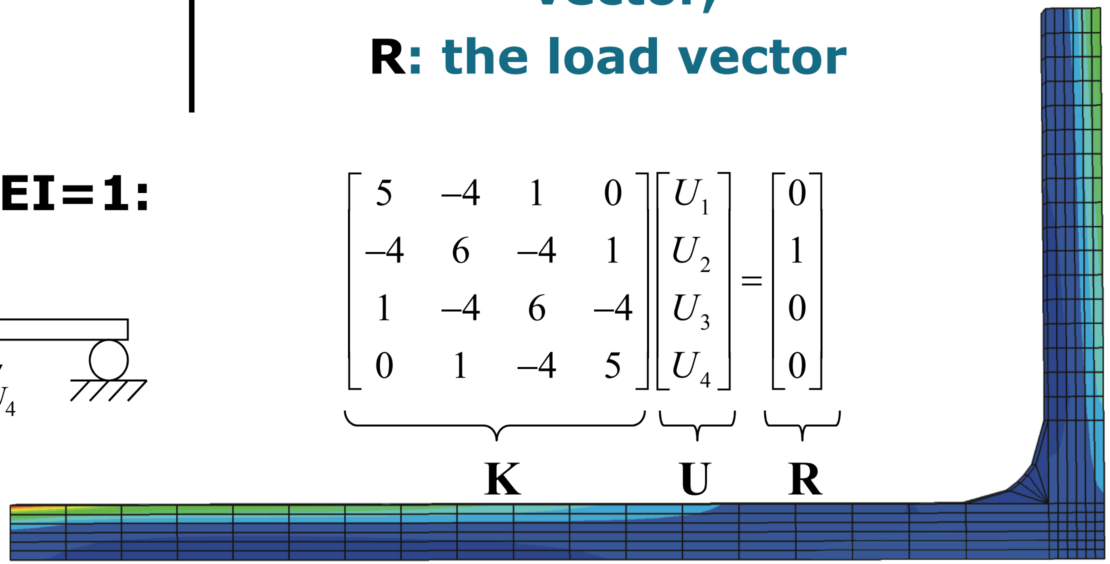
Solve KU=R for U

K: the stiffness matrix,
U: the displacement vector,
R: the load vector

Example 3.27 with L=5, EI=1:



$$\underbrace{\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{R}}$$



Gauss elimination

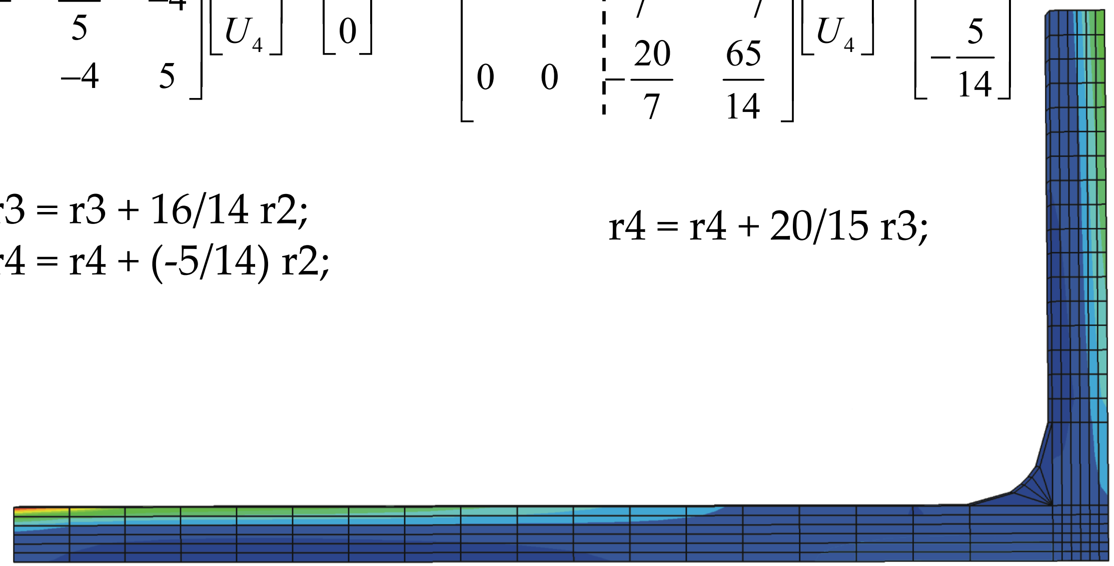
In a Gauss elimination, we reduce the matrix of coefficients to an upper triangular form, by a successive addition of multiples of the i^{th} row $(i = 1,\ldots,n-1)$ to the remaining $n-i$ rows j $(j = i+1,\ldots,n)$.

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ -\frac{5}{14} \end{bmatrix}$$

r2 = r2 + 4/5 r1;
r3 = r3 + (-1/5) r1;
r4 = r4;

r3 = r3 + 16/14 r2;
r4 = r4 + (-5/14) r2;

r4 = r4 + 20/15 r3;



Gauss elimination

The result is an upper-triangular matrix which we can solve for the unknowns U_i in the order $U_n, U_{n-1}, \ldots, U_1$.

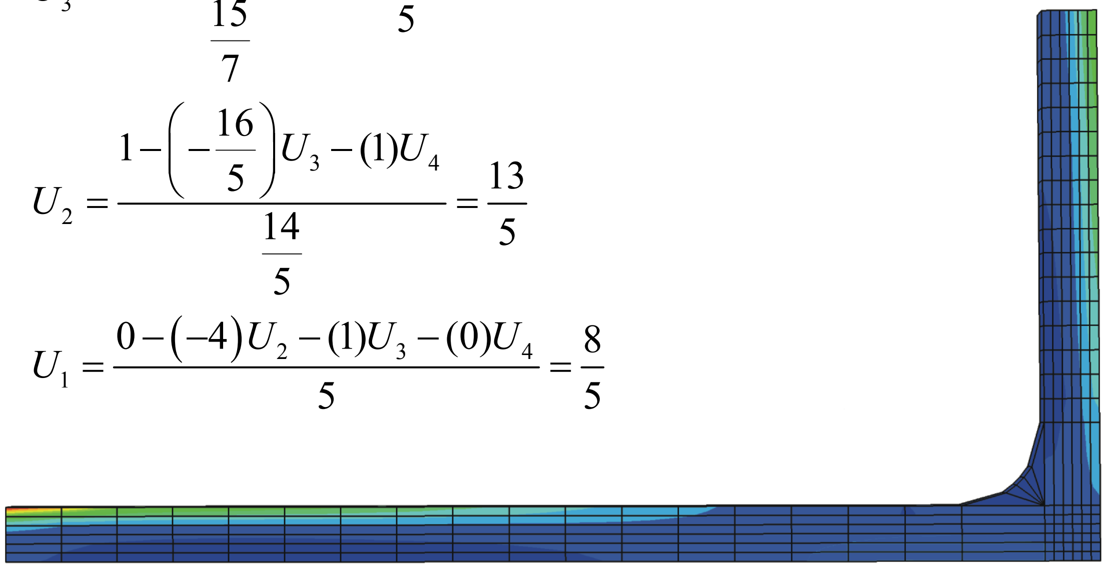
$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ \frac{7}{6} \end{bmatrix}$$

$$U_4 = \frac{\frac{7}{6}}{\frac{5}{6}} = \frac{7}{5}$$

$$U_3 = \frac{\frac{8}{7} - \left(-\frac{20}{7}\right) U_4}{\frac{15}{7}} = \frac{12}{5}$$

$$U_2 = \frac{1 - \left(-\frac{16}{5}\right) U_3 - (1) U_4}{\frac{14}{5}} = \frac{13}{5}$$

$$U_1 = \frac{0 - (-4) U_2 - (1) U_3 - (0) U_4}{5} = \frac{8}{5}$$



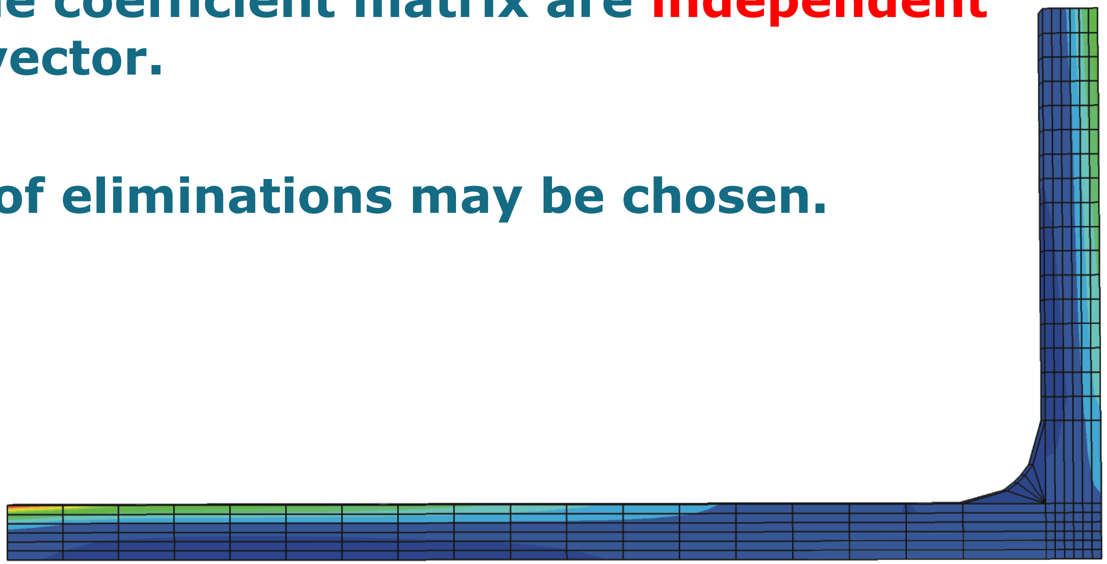
Gauss elimination

- After step i (i.e. after the full addition procedure involving multiples of row i), the lower right $(n-i) \times (n-i)$ submatrix is symmetric $\rightarrow$ storage implications

- Solution based on non-vanishing i^{th} diagonal element of coefficient matrix in step i.

- The operations on the coefficient matrix are independent of the right-hand side vector.

- Any desirable order of eliminations may be chosen.



Gauss elimination

In order to identify the physical process corresponding to the mathematical operations in Gauss elimination, we note first that the operations on the coefficient matrix K are independent of the elements in the load vector R.

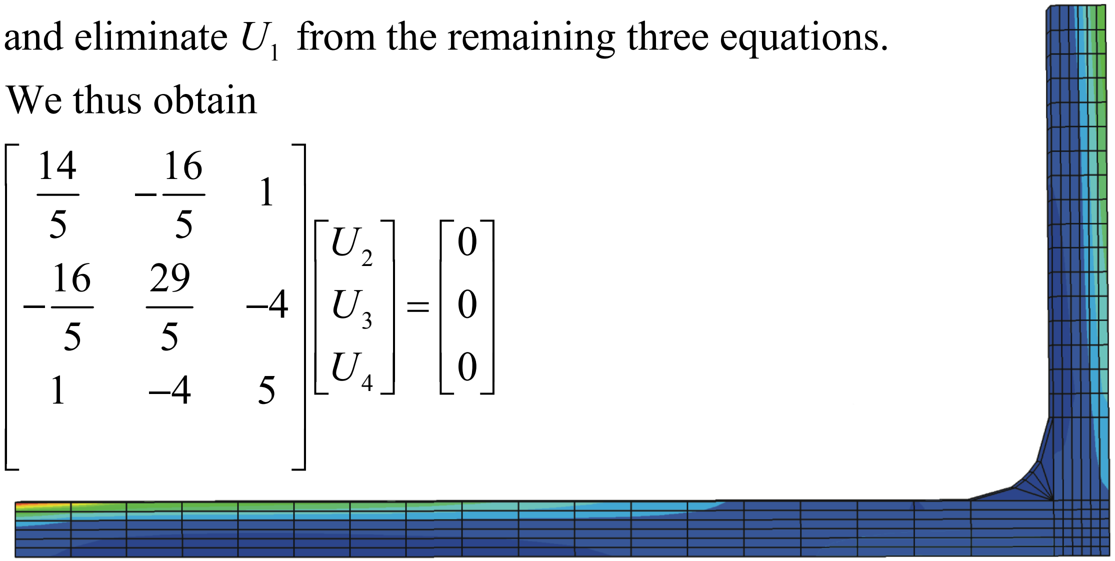
We consider that no loads are applied and hence have

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

First equation: $5U_1 - 4U_2 + U_3 = 0 \Rightarrow U_1 = \frac{4}{5}U_2 - \frac{1}{5}U_3$

and eliminate U_1 from the remaining three equations. We thus obtain

$$\begin{bmatrix} \frac{14}{5} & -\frac{16}{5} & 1 \\ -\frac{16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



Gauss elimination

We consider that no loads are applied and hence have

$$\begin{bmatrix} \frac{14}{5} & -\frac{16}{5} & 1 \\ -\frac{16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

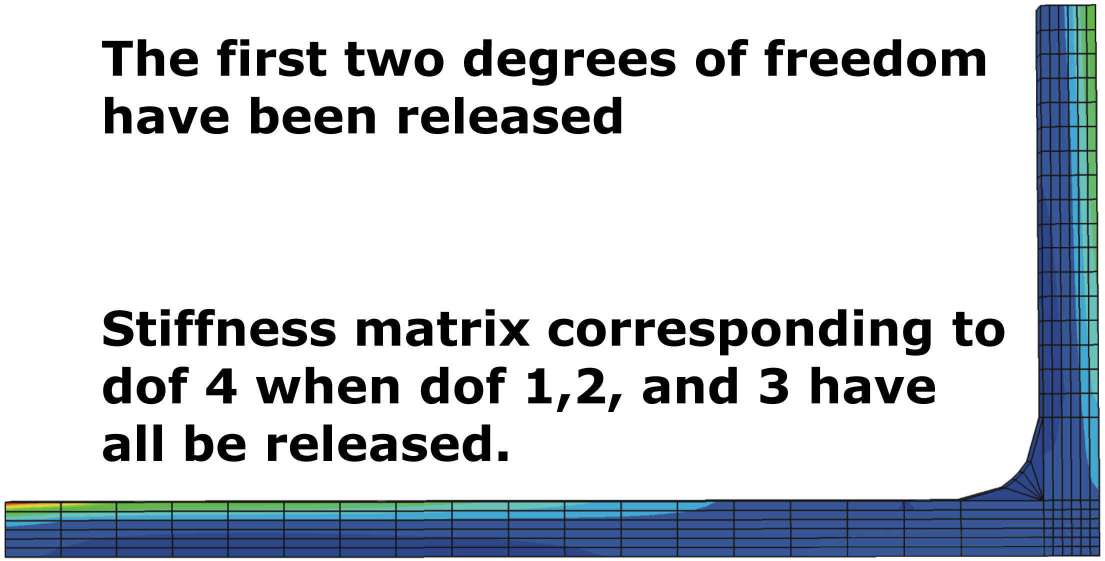
Stiffness matrix corresponding to beam after release of dof 1. (dof 1"statically condensed out")

$$\begin{bmatrix} \frac{15}{7} & -\frac{20}{7} \\ -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

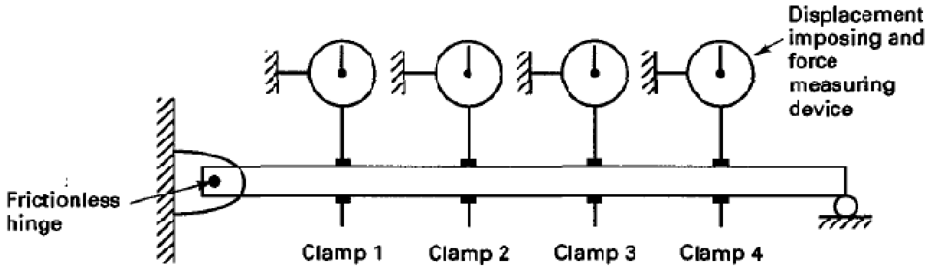
The first two degrees of freedom have been released

$$\begin{bmatrix} \frac{5}{6} \end{bmatrix} [U_4] = [0]$$

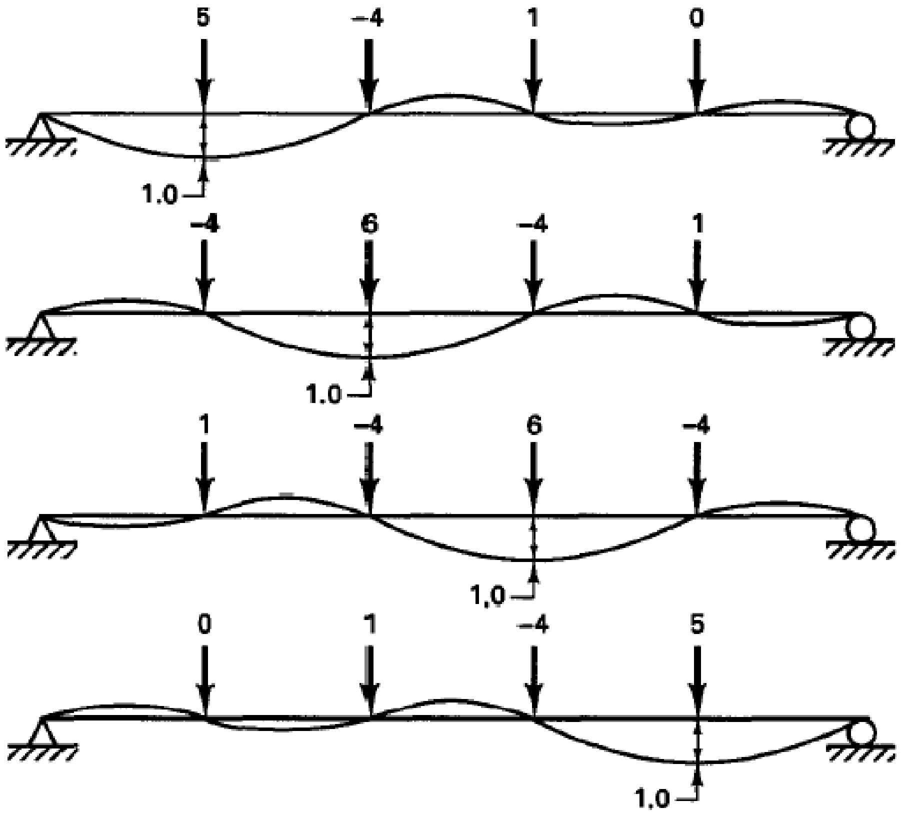
Stiffness matrix corresponding to dof 4 when dof 1,2, and 3 have all be released.

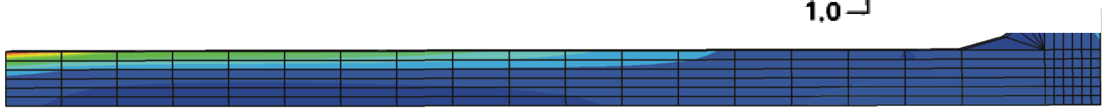


Gauss elimination

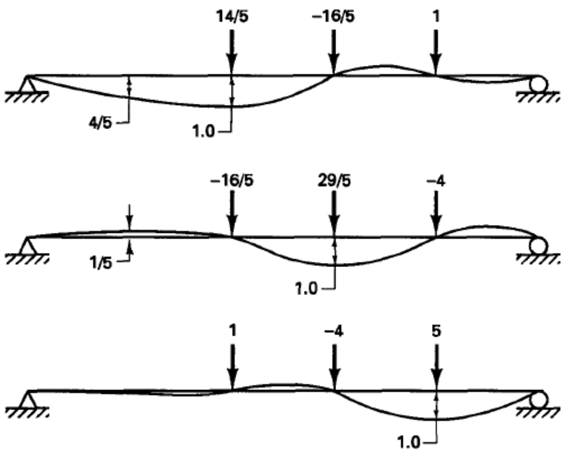


$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

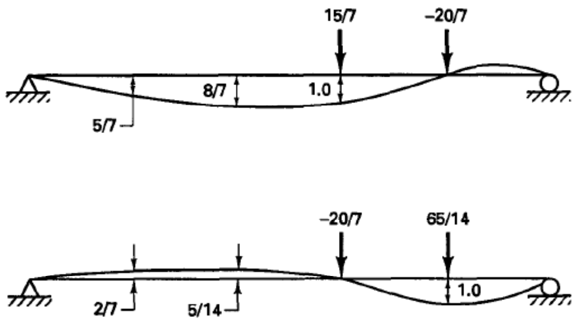




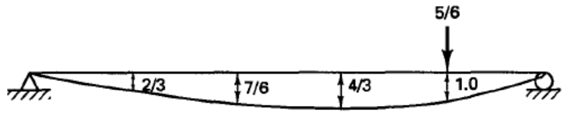
Gauss elimination



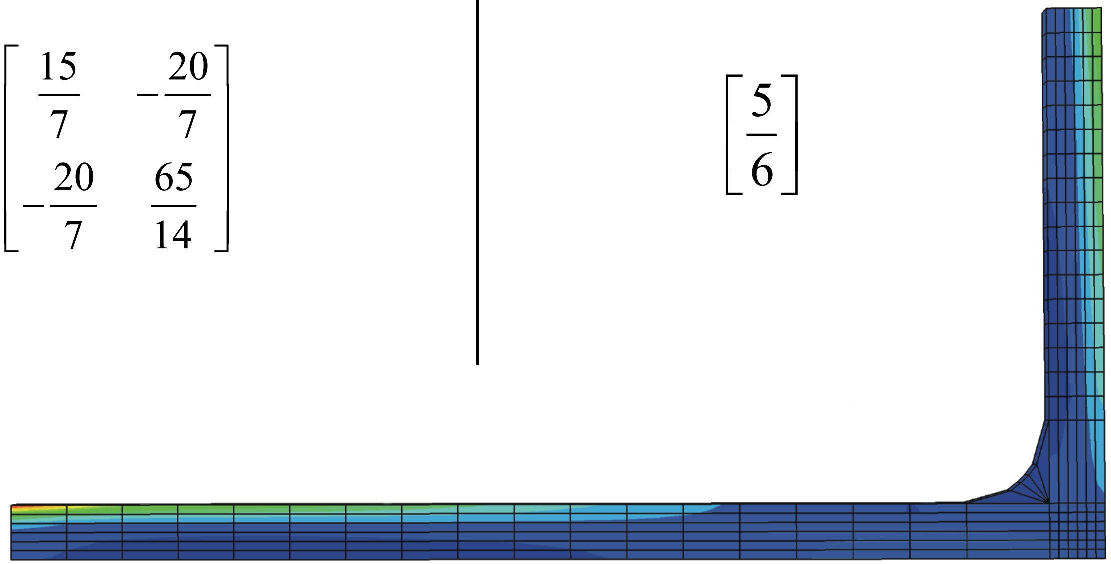
$$\begin{bmatrix} \frac{14}{5} & -\frac{16}{5} & 1 \\ -\frac{16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{bmatrix}$$



$$\begin{bmatrix} \frac{15}{7} & -\frac{20}{7} \\ -\frac{20}{7} & \frac{65}{14} \end{bmatrix}$$



$$\begin{bmatrix} \frac{5}{6} \end{bmatrix}$$

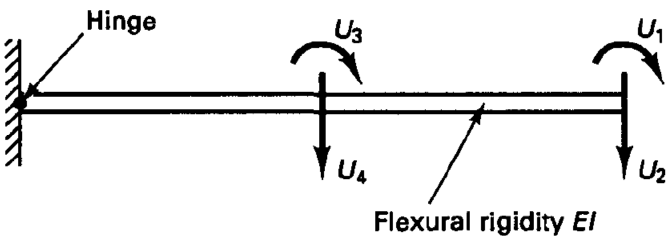


Gauss elimination

- We get a total of n stiffness matrices of decreasing order (n, n-1,…, 2, 1), each describing a set of n-i degrees of freedom (i = 0, 1,…,n-1) of the same physical system.

- If **R≠0**, then we also establish the load vectors pertaining to these stiffness matrices.

- The physical picture suggests that the diagonal elements remain positive during the Gauss elimination: Stiffness should be positive; a non-positive diagonal element implies an unstable structure.



If a zero (or negative) diagonal element occurs, the structure is not stable.
Here, after release of dofs U_1, U_2 and U_3 the last diagonal element (i.e. the stiffness at dof U_4) is zero.

LDL^T solution

The successive matrix operations during a Gauss elimination can be cast into a general form, which leads, likewise, to the reduction of K to an upper triangular form, S,

$$\mathbf{L}_{n-1}^{-1}\ldots\ldots\mathbf{L}_2^{-1}\mathbf{L}_1^{-1}\mathbf{K} = \mathbf{S}$$

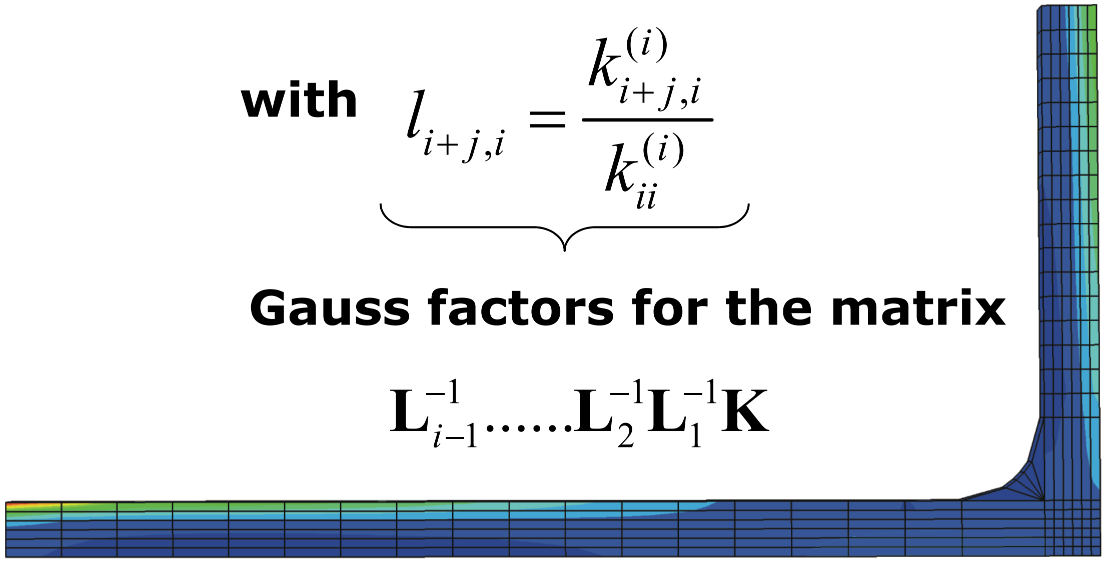
where

$$\mathbf{L}_i^{-1} = \begin{bmatrix} 1 & & & & & & & & \\ & \bullet & & & & & & & \\ & & \bullet & & & & & & \\ & & & \bullet & & & & & \\ & & & & 1 & & & & \\ & & & & -l_{i+1,i} & \bullet & & & \\ & & & & -l_{i+2,i} & & \bullet & & \\ & & & & \vdots & & & \bullet & \\ & & & & -l_{ni} & & & & 1 \end{bmatrix}$$

with $$l_{i+j,i} = \frac{k_{i+j,i}^{(i)}}{k_{ii}^{(i)}}$$

Gauss factors for the matrix

$$\mathbf{L}_{i-1}^{-1}\ldots\ldots\mathbf{L}_2^{-1}\mathbf{L}_1^{-1}\mathbf{K}$$



LDL^T solution

$$\mathbf{L}_{n-1}^{-1} \ldots\ldots \mathbf{L}_2^{-1}\mathbf{L}_1^{-1}\mathbf{K} = \mathbf{S}$$

Solve for K:

$$\mathbf{K} = \mathbf{L}_1\mathbf{L}_2 \ldots\ldots \mathbf{L}_{n-1}\mathbf{S}$$

where

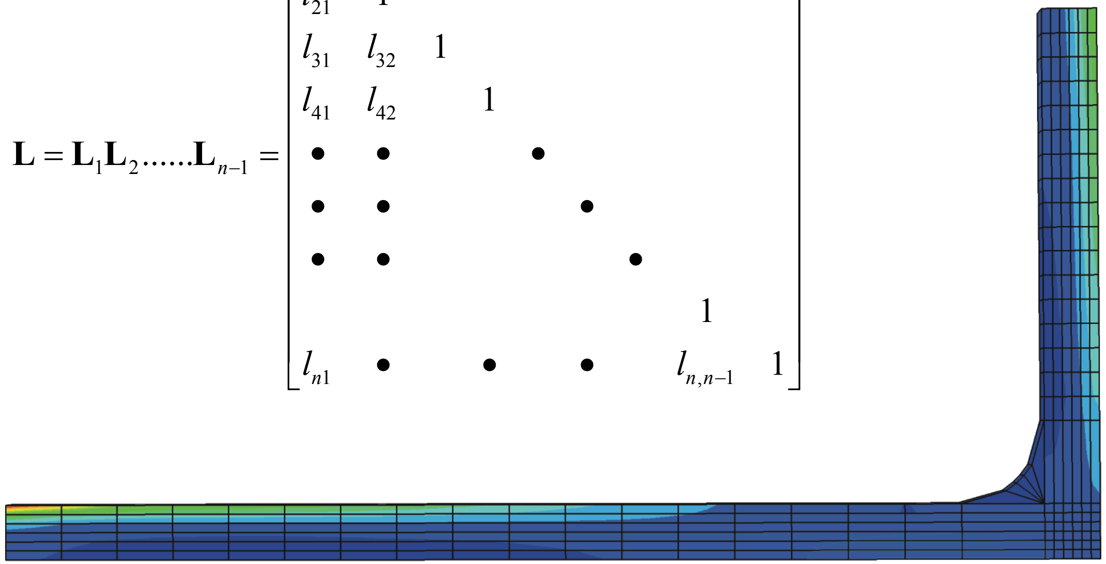
$$\mathbf{L}_i = \begin{bmatrix} 1 & & & & & & & & \\ & \bullet & & & & & & & \\ & & \bullet & & & & & & \\ & & & \bullet & & & & & \\ & & & & 1 & & & & \\ & & & & l_{i+1,i} & \bullet & & & \\ & & & & l_{i+2,i} & & \bullet & & \\ & & & & \vdots & & & \bullet & \\ & & & & l_{ni} & & & & 1 \end{bmatrix}$$

Hence,

$$\mathbf{K} = \mathbf{L}\mathbf{S}$$

with

$$\mathbf{L} = \mathbf{L}_1\mathbf{L}_2 \ldots\ldots \mathbf{L}_{n-1} = \begin{bmatrix} 1 & & & & & & & & \\ l_{21} & 1 & & & & & & & \\ l_{31} & l_{32} & 1 & & & & & & \\ l_{41} & l_{42} & & 1 & & & & & \\ \bullet & \bullet & & & \bullet & & & & \\ \bullet & \bullet & & & & \bullet & & & \\ \bullet & \bullet & & & & & \bullet & & \\ & & & & & & & 1 & \\ l_{n1} & \bullet & & \bullet & & \bullet & & l_{n,n-1} & 1 \end{bmatrix}$$



LDL^T solution

$$\mathbf{K} = \mathbf{LS}$$

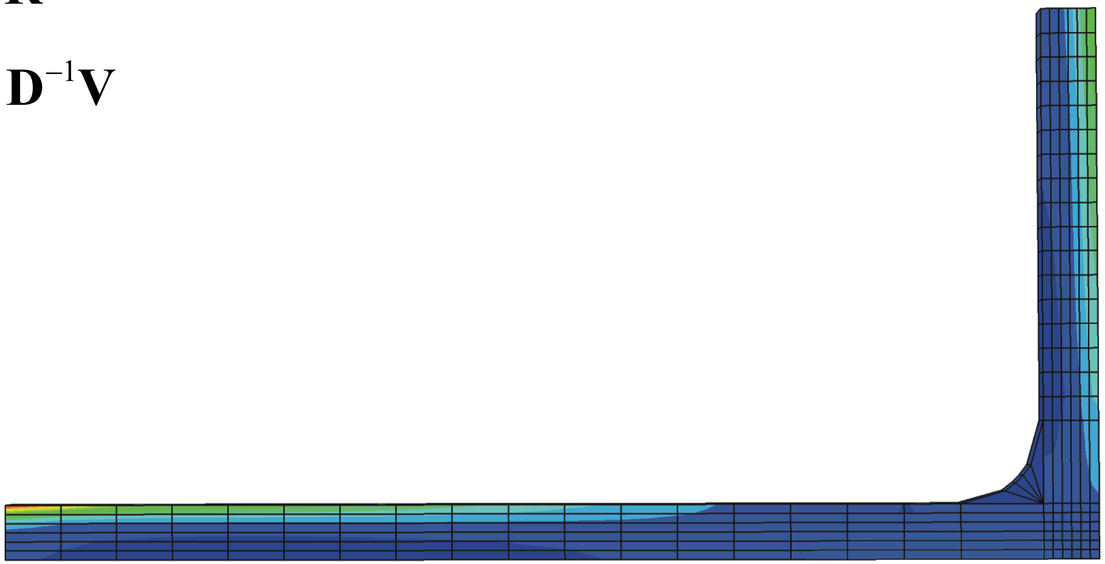
Now $\mathbf{S} = \mathbf{D}\tilde{\mathbf{S}}$ **where** $d_{ij}=\delta_{ij}s_{ij}$, **hence** $\mathbf{K} = \mathbf{L}\mathbf{D}\tilde{\mathbf{S}}$ **and since** $k_{ij}=k_{ji}$, $\tilde{\mathbf{S}} = \mathbf{L}^T$

$$\boxed{\mathbf{K} = \mathbf{LDL}^T}$$

In practice:

$$\mathbf{LV} = \mathbf{R} \quad \Rightarrow \quad \mathbf{V} = \mathbf{L}^{-1}\mathbf{R}$$

$$\mathbf{DL}^T\mathbf{U} = \mathbf{V} \quad \Rightarrow \quad \mathbf{U} = \left(\mathbf{L}^T\right)^{-1}\mathbf{D}^{-1}\mathbf{V}$$



LDL^T solution

- **We look at an example:**

Compute L_i^{-1}, $i = 1, 2, 3$, L, S, D and V from

$$\mathbf{K} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \quad \textbf{and} \quad \mathbf{R} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

<u>Step 1:</u>
r2 = r2 + 4/5 r1;
r3 = r3 + (-1/5) r1;
r4 = r4;

$$\mathbf{L}_1^{-1} = \begin{bmatrix} 1 & & & \\ 4/5 & 1 & & \\ -1/5 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>Step 2:</u>
r3 = r3 + 8/7 r2;
r4 = r4 + (-5/14) r2;

$$\mathbf{L}_2^{-1} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 8/7 & 1 & \\ 0 & -5/14 & 0 & 1 \end{bmatrix}$$

<u>Step 3:</u>
r4 = r4 + 4/3 r3;

$$\mathbf{L}_3^{-1} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 4/3 & 1 \end{bmatrix}$$

LDL^T solution

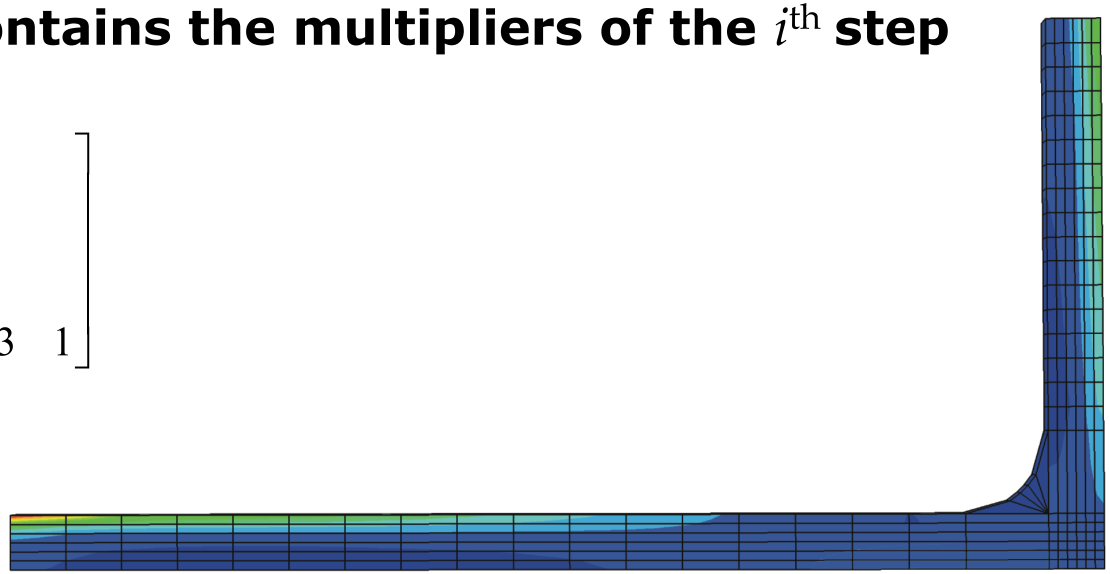
- **We look at an example:**

Compute L_i^{-1}**,** $i = 1, 2, 3$ **, L, S, D and V from**

$$\mathbf{K} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \quad \textbf{and} \quad \mathbf{R} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

the i^{th} **column of** L_i^{-1} **contains the multipliers of the** i^{th} **step**

$$\mathbf{L} = \mathbf{L}_1\mathbf{L}_2\mathbf{L}_3 = \begin{bmatrix} 1 & & & \\ -4/5 & 1 & & \\ 1/5 & -8/7 & 1 & \\ 0 & 5/14 & -4/3 & 1 \end{bmatrix}$$



LDL^T solution

- **We look at an example:**

Recall the pivots in the Gauss elimination – they enter into S (i.e. reduced K):

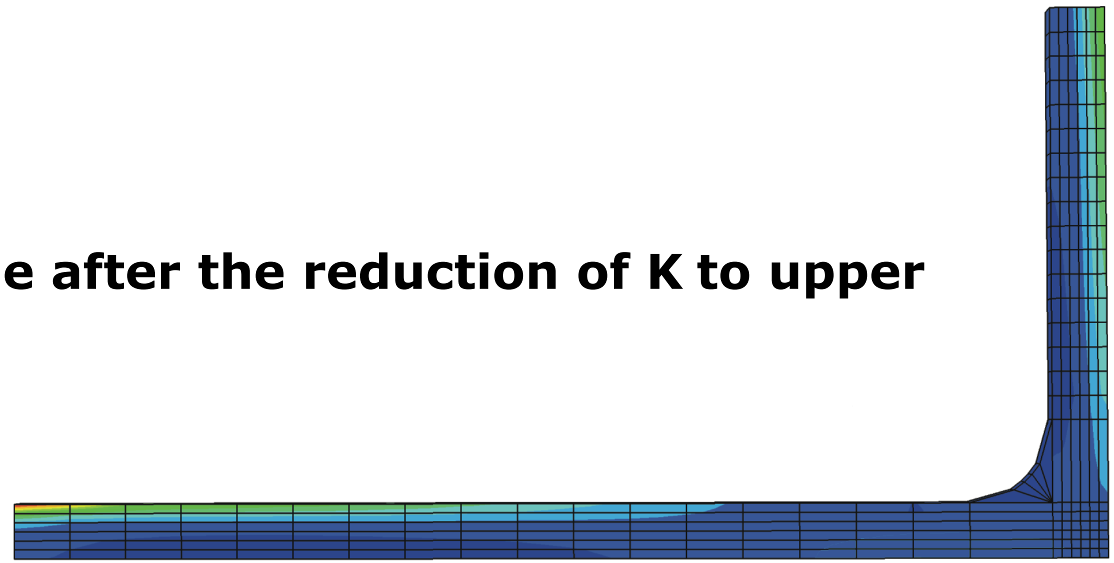
$$\mathbf{S} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ & 14/5 & -16/5 & 1 \\ & & 15/7 & -20/7 \\ & & & 5/6 \end{bmatrix} \begin{matrix} \longrightarrow \text{First row of K} \\ \longrightarrow \text{Second row of K after step 1} \\ \longrightarrow \text{Third row of K after step 2} \\ \longrightarrow \text{Fourth row of K after step 3} \end{matrix}$$

For the matrix D: $d_{ij}=\delta_{ij}s_{ij}$

$$\mathbf{D} = \begin{bmatrix} 5 & & & \\ & 14/5 & & \\ & & 15/7 & \\ & & & 5/6 \end{bmatrix}$$

V is the right-hand side after the reduction of K to upper triangular form

$$\mathbf{V} = \begin{bmatrix} 0 & 1 & 8/7 & 7/6 \end{bmatrix}^T$$



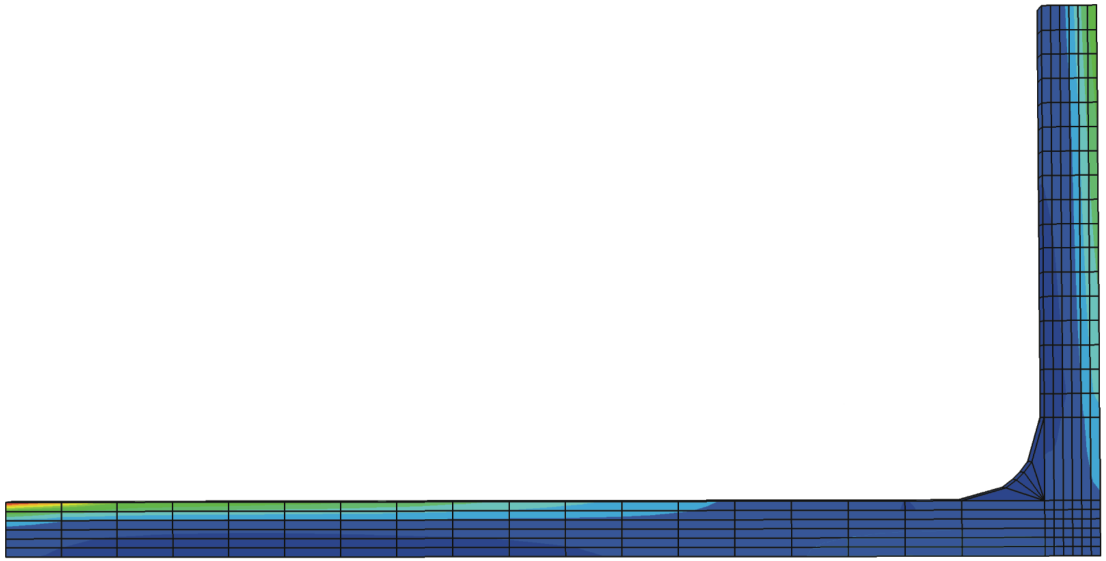
Cholesky factorization

- In addition to $\mathbf{LDL}^T$ decomposition, Cholesky factorization is used that are closely related. Both of the two methods are applications of the basic Gauss elimination procedure.

In the Cholesky factorization the stiffness matrix is decomposed as follows

$$\mathbf{K} = \tilde{\mathbf{L}}\tilde{\mathbf{L}}^T$$

where $\tilde{\mathbf{L}} = \mathbf{L}\mathbf{D}^{\frac{1}{2}}$



Cholesky factorization

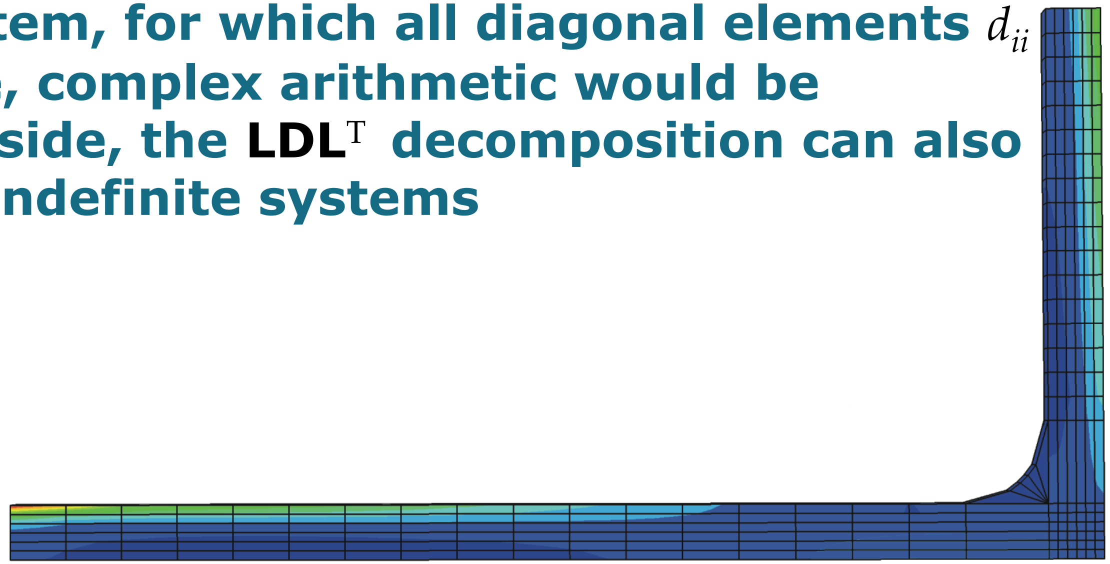
$$\mathbf{K} = \widetilde{\mathbf{L}}\widetilde{\mathbf{L}}^T$$

$$\text{where } \widetilde{\mathbf{L}} = \mathbf{L}\mathbf{D}^{\frac{1}{2}}$$

- Therefore, the Cholesky factors could be calculated from the **D** and **L** factors, but, more generally, the elements of the Cholesky factors are calculated directly.

- Slightly more operations are required in the equation if the Cholesky factorization is used rather than the $\mathbf{LDL}^T$ decomposition.

- The Cholesky factorization is suitable only for the solution of **positive definite** system, for which all diagonal elements d_{ii} are positive. Otherwise, complex arithmetic would be required. On the other side, the $\mathbf{LDL}^T$ decomposition can also be used effectively on indefinite systems



Cholesky factorization

$$\mathbf{K} = \widetilde{\mathbf{L}}\widetilde{\mathbf{L}}^T$$

$$\text{where } \widetilde{\mathbf{L}} = \mathbf{L}\mathbf{D}^{\frac{1}{2}}$$

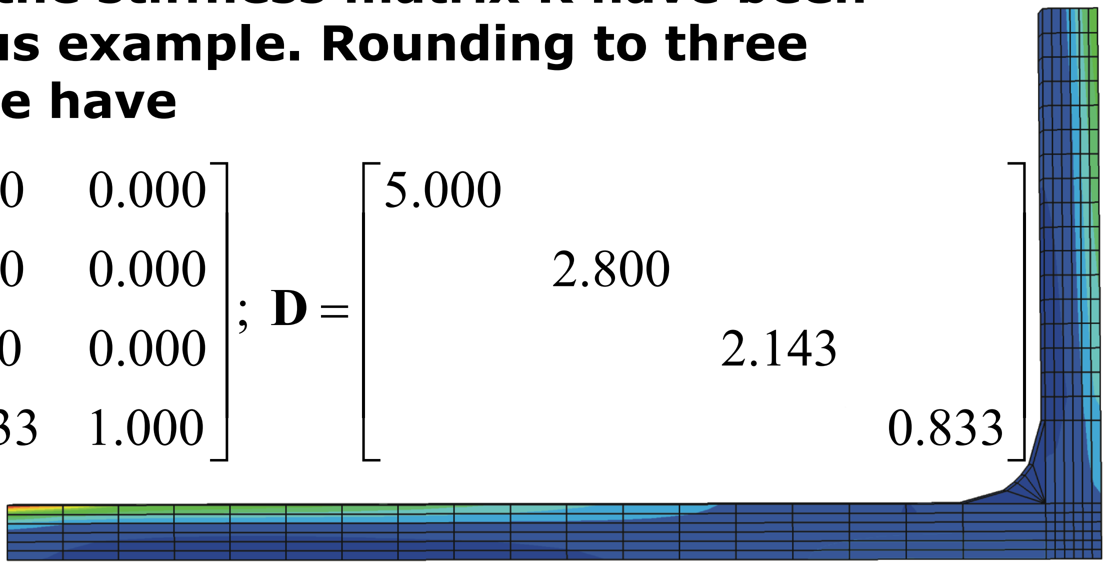
- **We look at Example 8.7:**

Calculate the Cholesky factor of the stiffness matrix K

$$\mathbf{K} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

The L and D factors of the stiffness matrix K have been obtained in the previous example. Rounding to three significant decimals, we have

$$\mathbf{L} = \begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ -0.800 & 1.000 & 0.000 & 0.000 \\ 0.200 & -1.143 & 1.000 & 0.000 \\ 0.000 & 0.357 & -1.333 & 1.000 \end{bmatrix}; \ \mathbf{D} = \begin{bmatrix} 5.000 & & & \\ & 2.800 & & \\ & & 2.143 & \\ & & & 0.833 \end{bmatrix}$$



Cholesky factorization

$$\mathbf{K} = \tilde{\mathbf{L}}\tilde{\mathbf{L}}^T$$

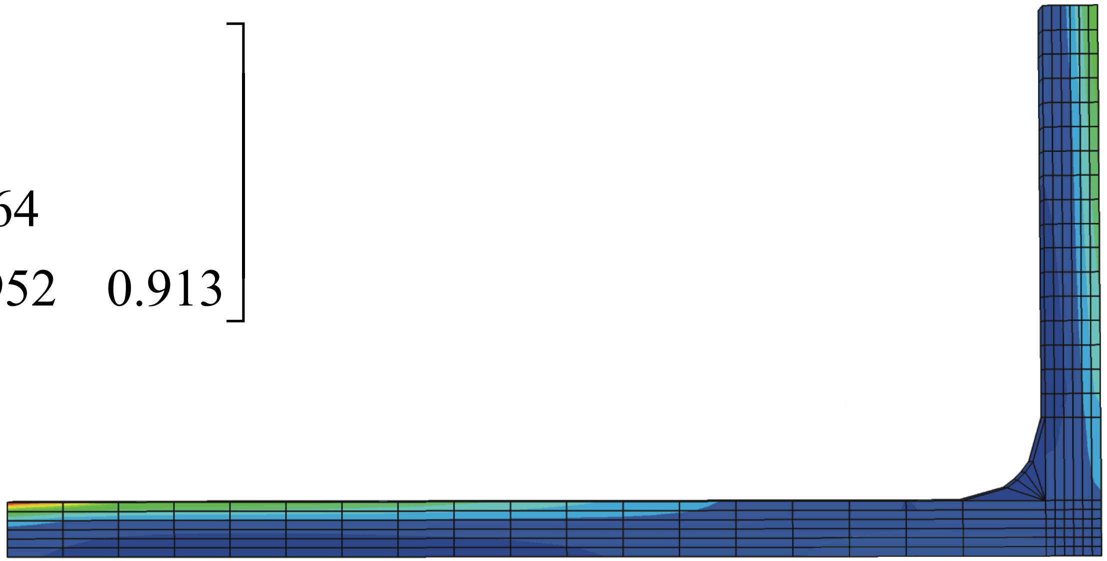
$$\text{where } \tilde{\mathbf{L}} = \mathbf{L}\mathbf{D}^{\frac{1}{2}}$$

- **We look at Example 8.7:**

Hence,

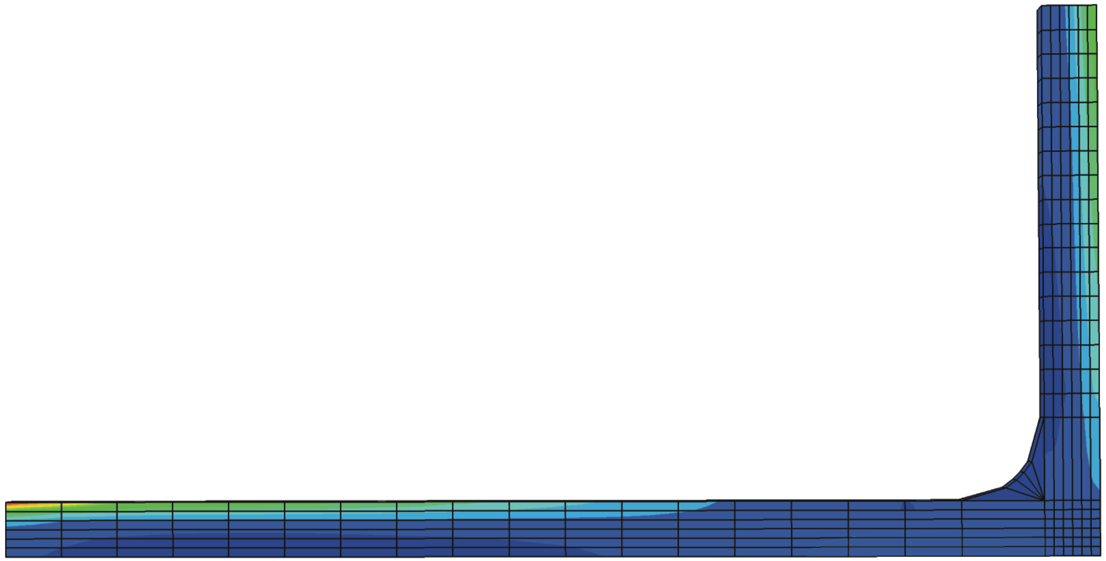
$$\tilde{\mathbf{L}} = \begin{bmatrix} 1.000 & & & \\ -0.800 & 1.000 & & \\ 0.200 & -1.143 & 1.000 & \\ 0.000 & 0.357 & -1.333 & 1.000 \end{bmatrix} \begin{bmatrix} 2.236 & & & \\ & 1.673 & & \\ & & 1.464 & \\ & & & 0.913 \end{bmatrix}$$

$$\tilde{\mathbf{L}} = \begin{bmatrix} 2.236 & & & \\ -1.789 & 1.673 & & \\ 0.447 & -1.192 & 1.464 & \\ 0.000 & 0.597 & -1.952 & 0.913 \end{bmatrix}$$



Other related methods

- Static condensation
- Substructure analysis
- Frontal solution



Solution errors

Assume that we want to solve the system of equations:

$$\begin{bmatrix} 3.42521 & -3.42521 \\ -3.42521 & 101.2431 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 1.3021 \\ 0.0 \end{bmatrix}$$

The exact solution is (to *10* digits) $U_1 = 0.3934633449;\ U_2 = 0.0133114709$

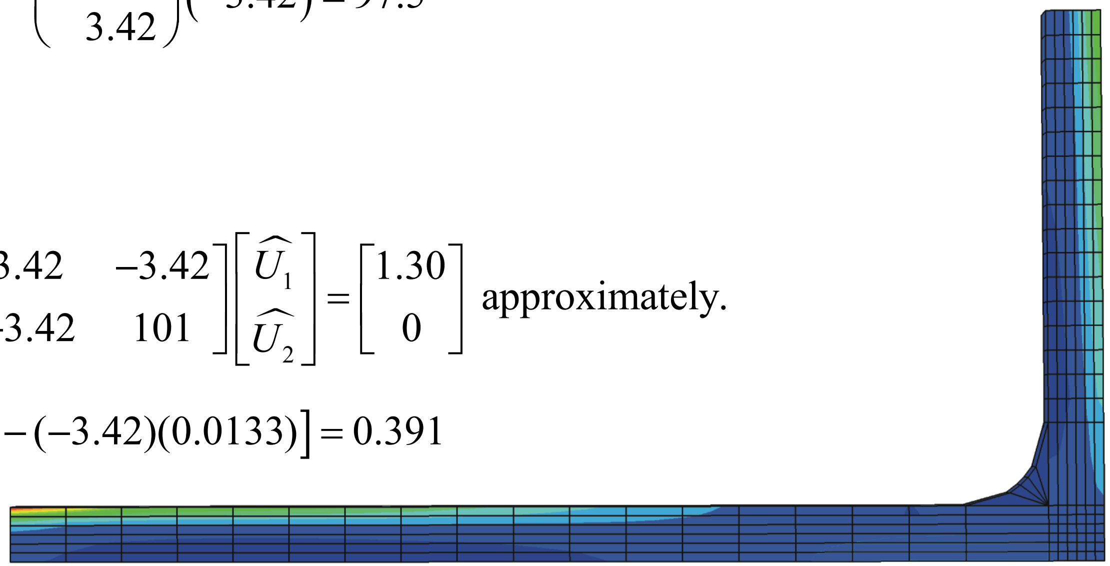
Now, we use *t=3* digits for Gauss algorithm:

$$\begin{bmatrix} 3.42 & -3.42 \\ -3.42 & 101 \end{bmatrix} \begin{bmatrix} \widehat{U}_1 \\ \widehat{U}_2 \end{bmatrix} = \begin{bmatrix} 1.30 \\ 0 \end{bmatrix},\ 101 - \left(-\frac{3.42}{3.42} \right) (-3.42) = 97.5$$

$$\begin{bmatrix} 3.42 & -3.42 \\ 0.0 & 97.5 \end{bmatrix} \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \end{bmatrix} = \begin{bmatrix} 1.30 \\ 1.30 \end{bmatrix}$$

$\overline{U}_1$ and $\overline{U}_2$ indicate that we solve $\begin{bmatrix} 3.42 & -3.42 \\ -3.42 & 101 \end{bmatrix} \begin{bmatrix} \widehat{U}_1 \\ \widehat{U}_2 \end{bmatrix} = \begin{bmatrix} 1.30 \\ 0 \end{bmatrix}$ approximately.

$$\overline{U}_2 = \frac{1.30}{97.5} = 0.0133;\ \overline{U}_1 = \frac{1}{3.42} \left[1.30 - (-3.42)(0.0133) \right] = 0.391$$



Solution errors

$$\begin{bmatrix} 3.42521 & -3.42521 \\ -3.42521 & 101.2431 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 1.3021 \\ 0.0 \end{bmatrix} \quad U_1 = 0.3934633449;\ U_2 = 0.0133114709$$

Truncation error (we solve equations exactly):

$$\begin{bmatrix} 3.42 & -3.42 \\ 0 & 97.58 \end{bmatrix}\begin{bmatrix} \widehat{U}_1 \\ \widehat{U}_2 \end{bmatrix} = \begin{bmatrix} 1.30 \\ 1.30 \end{bmatrix}$$

$$\widehat{U}_1 = 0.3934393613;\ \widehat{U}_2 = 0.0133224020$$

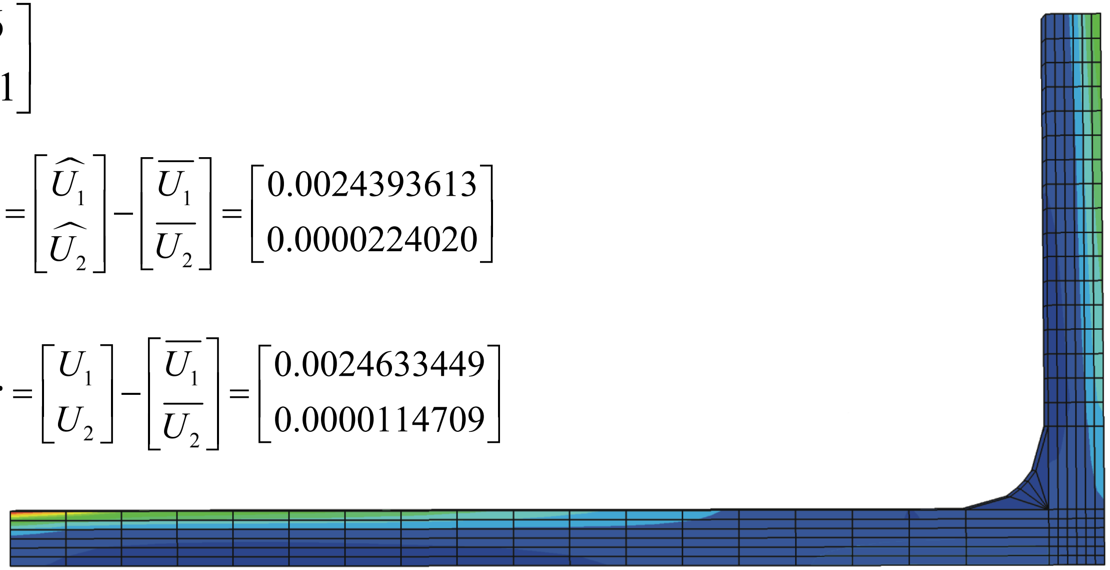
$$\hat{\mathbf{r}} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} - \begin{bmatrix} \widehat{U}_1 \\ \widehat{U}_2 \end{bmatrix} = \begin{bmatrix} 0.0000239836 \\ -0.0000109311 \end{bmatrix}$$

Round-off error:

$$\bar{\mathbf{r}} = \begin{bmatrix} \widehat{U}_1 \\ \widehat{U}_2 \end{bmatrix} - \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \end{bmatrix} = \begin{bmatrix} 0.0024393613 \\ 0.0000224020 \end{bmatrix}$$

Total error:

$$\mathbf{r} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} - \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \end{bmatrix} = \begin{bmatrix} 0.0024633449 \\ 0.0000114709 \end{bmatrix}$$



Solution errors

Summary on truncation and roundoff errors in solving KU=R (Bathe, page 739)

- **Both types of errors can be expected to be large if structures with widely varying stiffness are analyzed.**

- **Since truncation errors are most significant, to improve the solution accuracy it is necessary to evaluate both the stiffness matrix K and the solution of KU=R in double precision.**