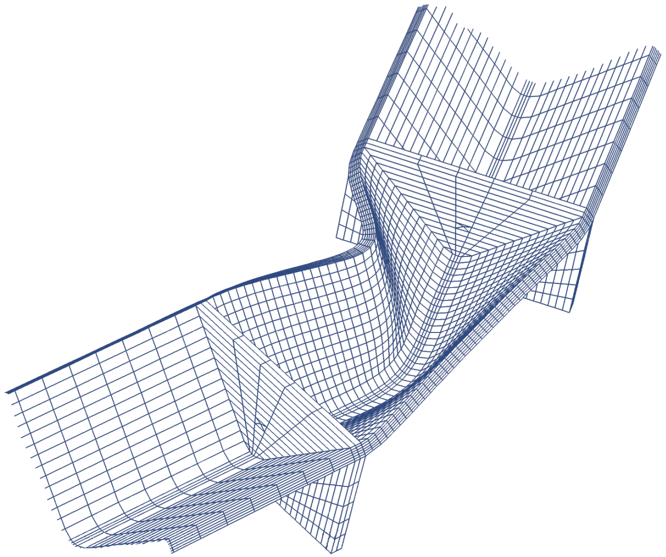
The Finite Element Method for the Analysis of Linear Systems



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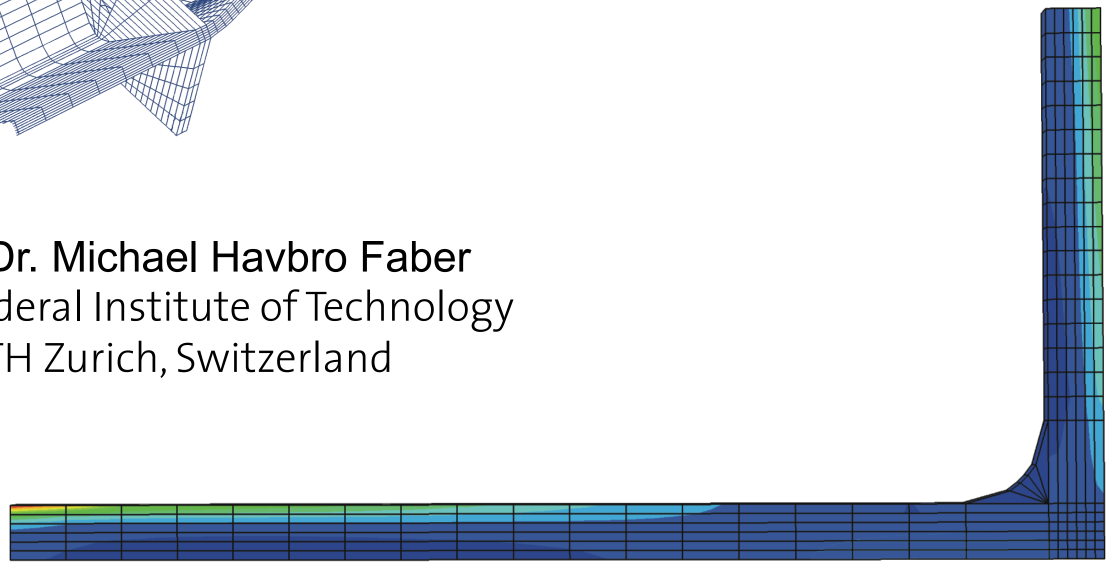


Contents of Today's Lecture

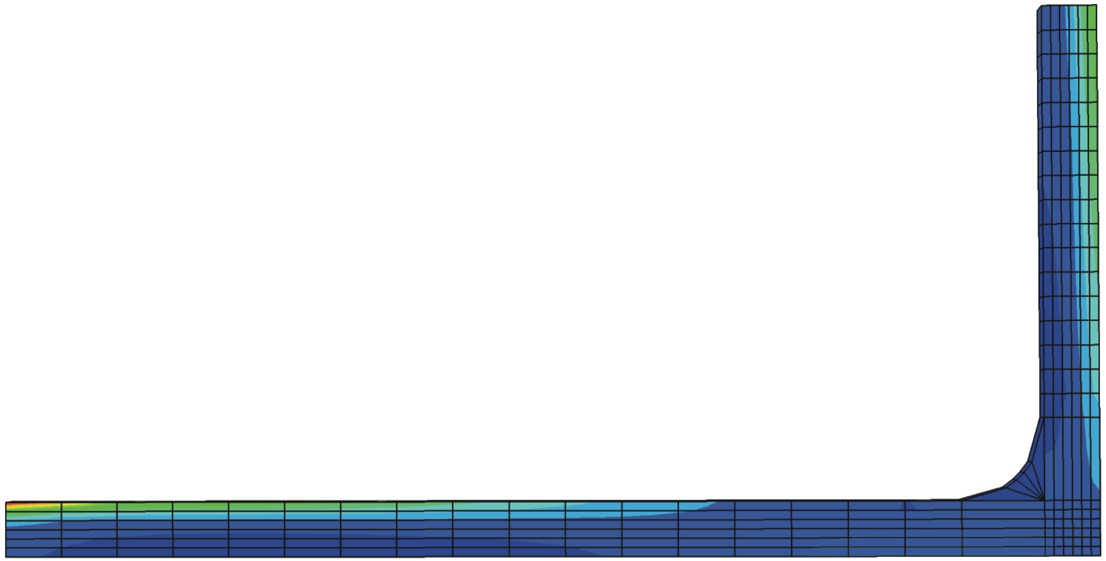
- General Shell Elements

  - Pure displacement based formulation

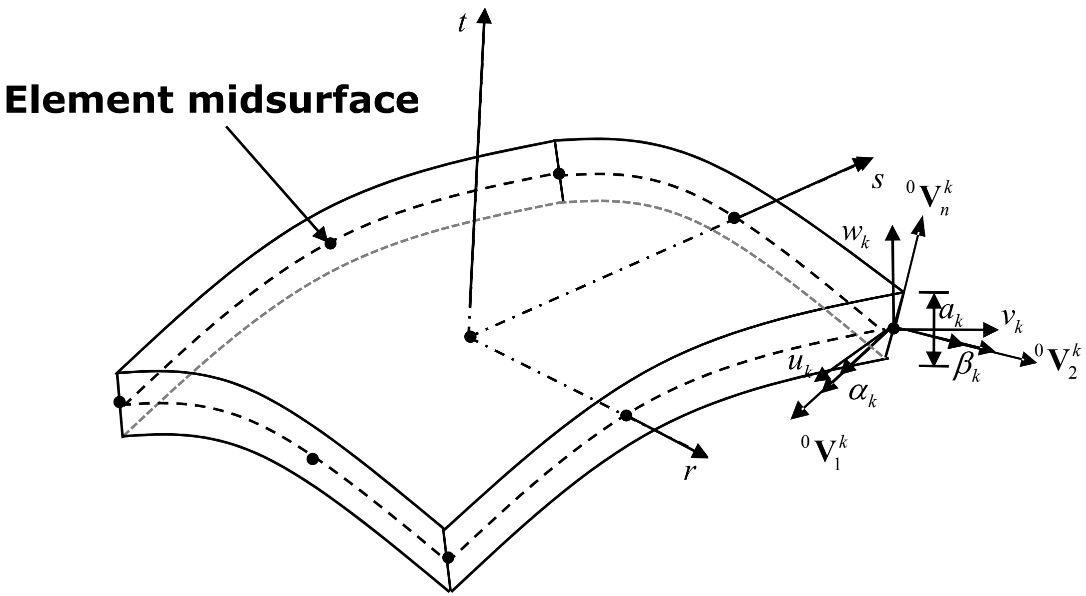
  - Mixed interpolation elements (MITC*n*)

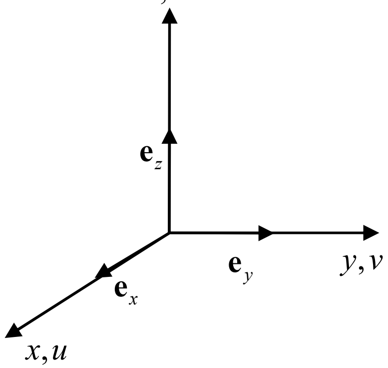
- Boundary conditions

- Assignment 5

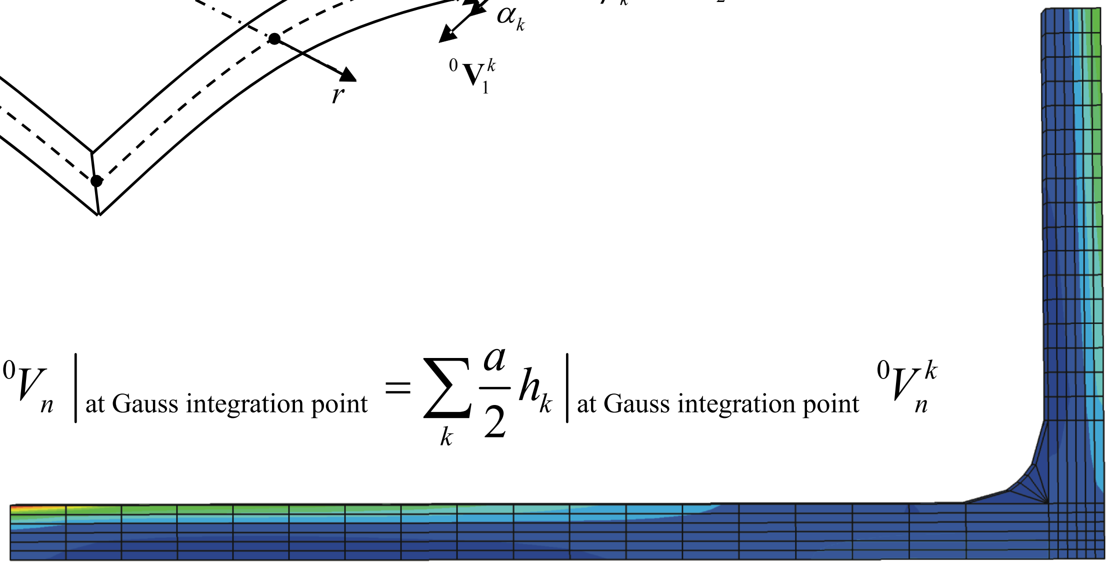


General Shell Elements

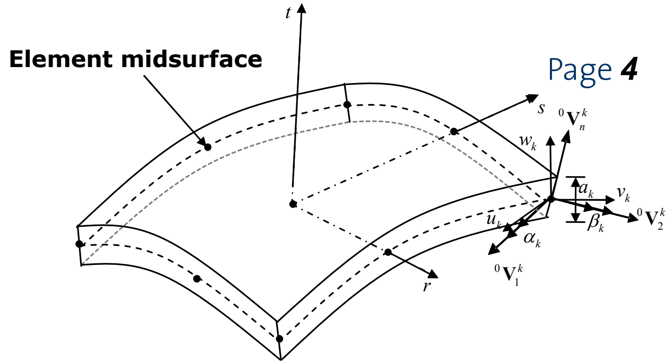


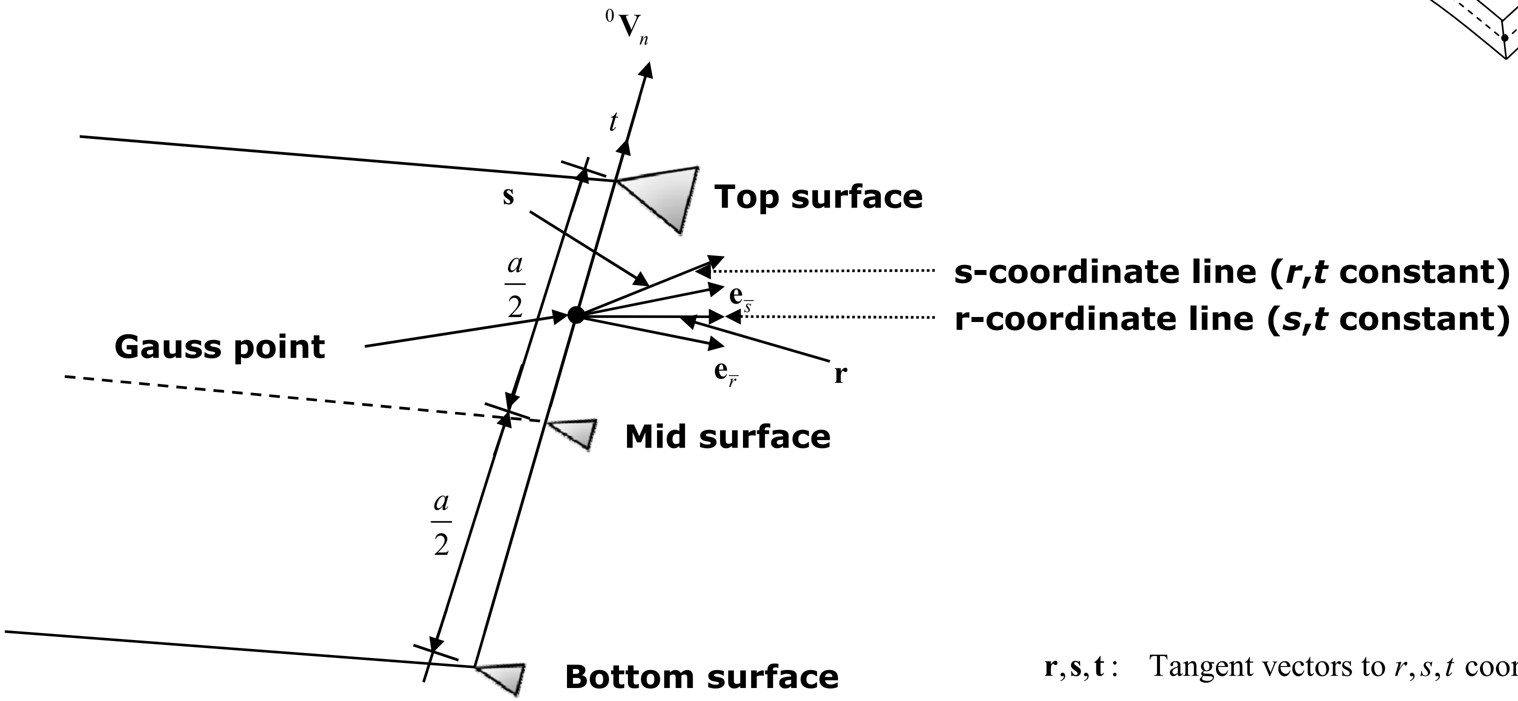


$$\frac{a}{2}\,{}^0V_n\Big|_{\text{at Gauss integration point}} = \sum_k \frac{a}{2} h_k\Big|_{\text{at Gauss integration point}}\ {}^0V_n^k$$



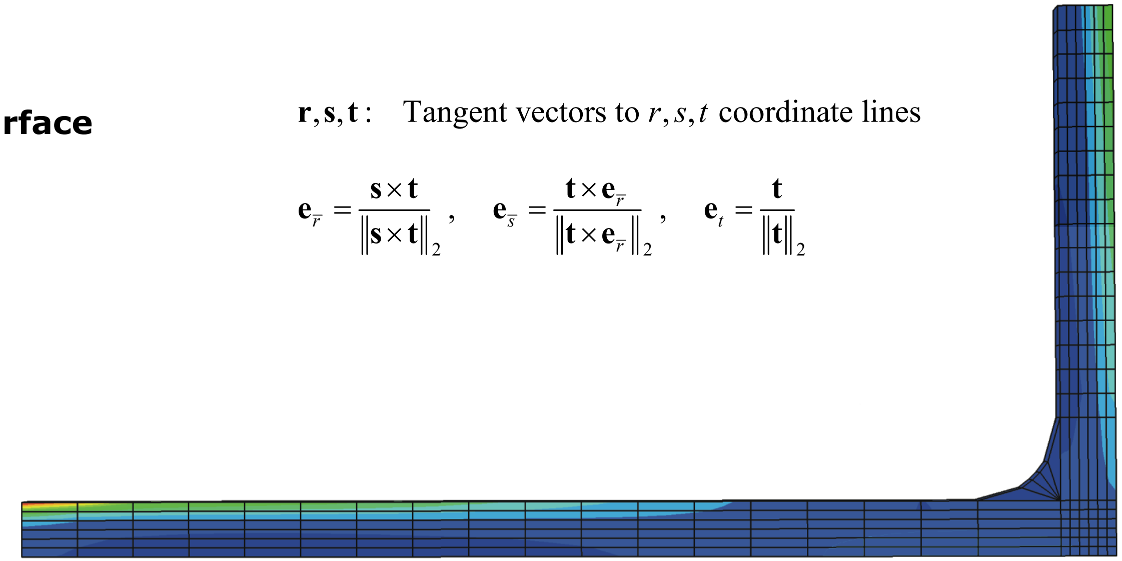
General Shell Elements

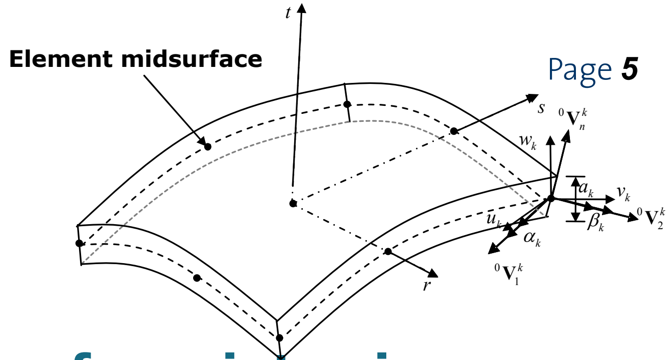




$\mathbf{r},\mathbf{s},\mathbf{t}$: Tangent vectors to r,s,t coordinate lines

$$\mathbf{e}_{\bar{r}} = \frac{\mathbf{s}\times\mathbf{t}}{\|\mathbf{s}\times\mathbf{t}\|_2}, \quad \mathbf{e}_{\bar{s}} = \frac{\mathbf{t}\times\mathbf{e}_{\bar{r}}}{\|\mathbf{t}\times\mathbf{e}_{\bar{r}}\|_2}, \quad \mathbf{e}_t = \frac{\mathbf{t}}{\|\mathbf{t}\|_2}$$





General Shell Elements

- **We may write the Cartesian coordinates of a point using natural coordinates before and after deformation as:**

$$ {}^{l}x(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{l}x_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{nx}^{k} $$

$$ {}^{l}y(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{l}y_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{ny}^{k} $$

$$ {}^{l}z(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{l}z_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{nz}^{k} $$

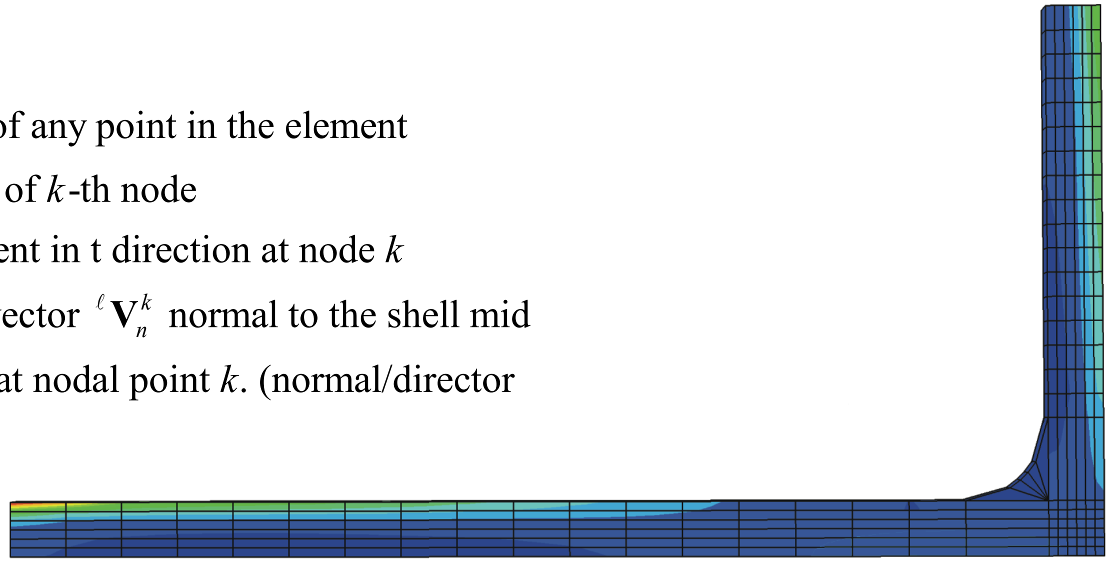
where:

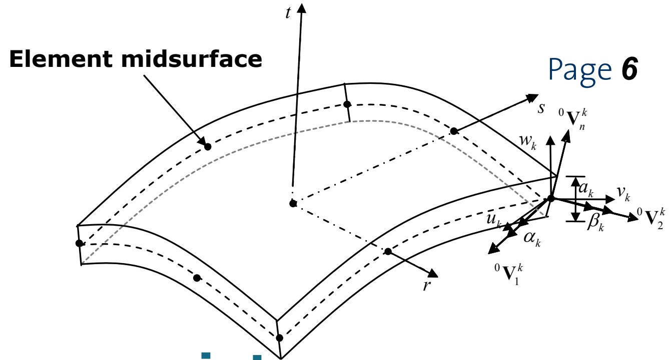
${}^{l}x, {}^{l}y, {}^{l}z$: Cartesian coordinates of any point in the element

${}^{l}x_k, {}^{l}y_k, {}^{l}z_k$: Cartesian coordinates of k-th node

a_k : Thickness of the element in t direction at node k

${}^{\ell}V_{nx}^{k}, {}^{\ell}V_{ny}^{k}, {}^{\ell}V_{nz}^{k}$: Components of unit vector ${}^{\ell}\mathbf{V}_n^k$ normal to the shell mid surface in direction t at nodal point k. (normal/director vector at node k)





General Shell Elements

- **Now we can write the displacements at any point as:**

$$u(r,s,t)=\sum_{k=1}^{q}h_k u_{x_k}+\frac{t}{2}\sum_{k=1}^{q}a_k h_k V_{nx}^k$$

$$v(r,s,t)=\sum_{k=1}^{q}h_k v_{y_k}+\frac{t}{2}\sum_{k=1}^{q}a_k h_k V_{ny}^k$$

$$w(r,s,t)=\sum_{k=1}^{q}h_k w_{z_k}+\frac{t}{2}\sum_{k=1}^{q}a_k h_k V_{nz}^k$$

where

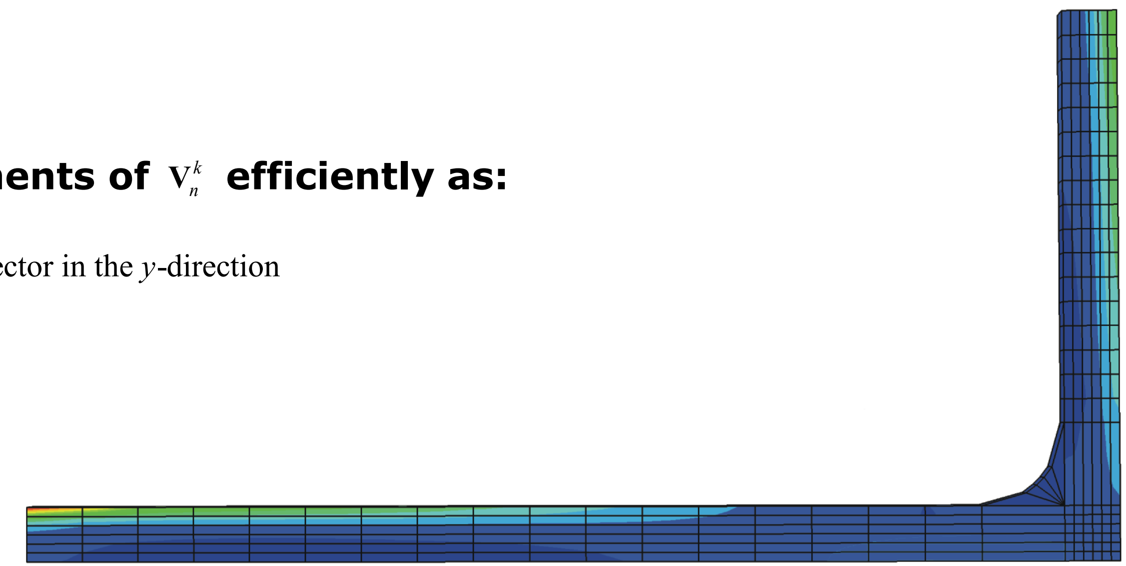
$$\mathbf{V}_n^k={}^1\mathbf{V}_n^k-{}^0\mathbf{V}_n^k$$

We can express the components of V_n^k efficiently as:

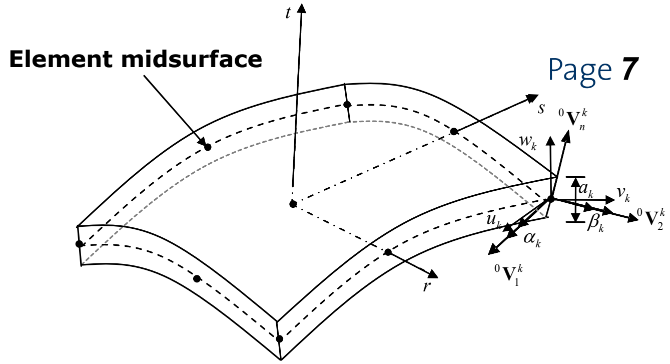
$${}^0\mathbf{V}_1^k=\frac{\mathbf{e}_y\times{}^0\mathbf{V}_n^k}{\left\|\mathbf{e}_y\times{}^0\mathbf{V}_n^k\right\|_2},\quad \text{with } \mathbf{e}_y \text{ being the unit vector in the } y\text{-direction}$$

$\Downarrow$

$${}^0\mathbf{V}_2^k={}^0\mathbf{V}_n^k\times{}^0\mathbf{V}_1^k$$



General Shell Elements



Letting α_k and β_k be the rotations of the director vector $\mathbf{V}_n^k$ around ${}^0\mathbf{V}_1^k$ and ${}^0\mathbf{V}_2^k$ then we can write (small rotations):

$$\mathbf{V}_n^k = -{}^0\mathbf{V}_2^k \alpha_k + {}^0\mathbf{V}_1^k \beta_k$$

Now we substitute this result into the relations for the displacements and get:

$$u(r,s,t) = \sum_{k=1}^{q} h_k u_{x_k} + \frac{t}{2} \sum_{k=1}^{q} a_k h_k (-{}^0V_{2x}^k \alpha_k + {}^0V_{1x}^k \beta_k)$$

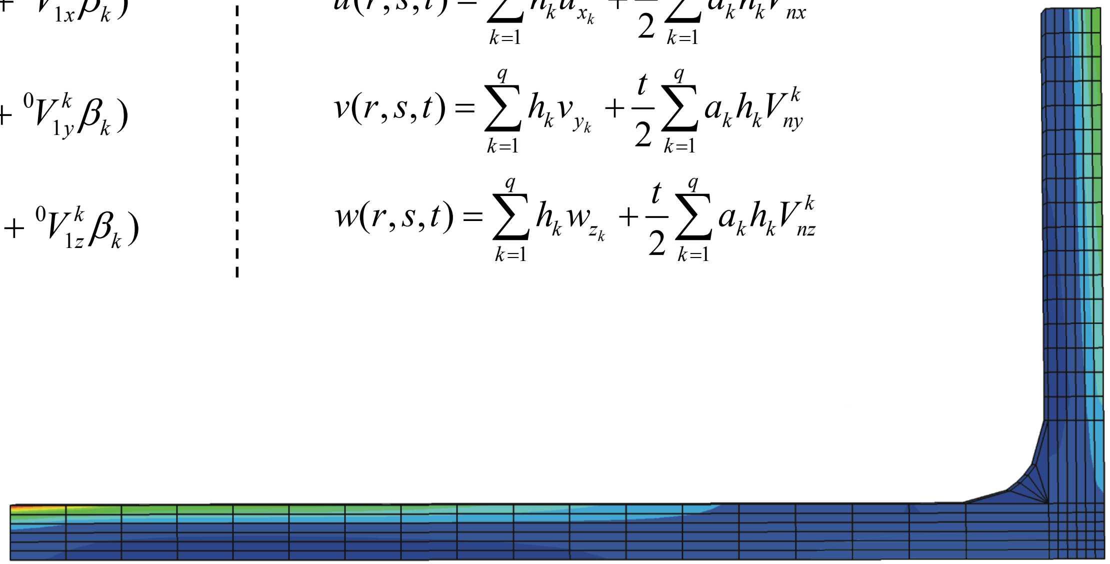
$$v(r,s,t) = \sum_{k=1}^{q} h_k v_{y_k} + \frac{t}{2} \sum_{k=1}^{q} a_k h_k (-{}^0V_{2y}^k \alpha_k + {}^0V_{1y}^k \beta_k)$$

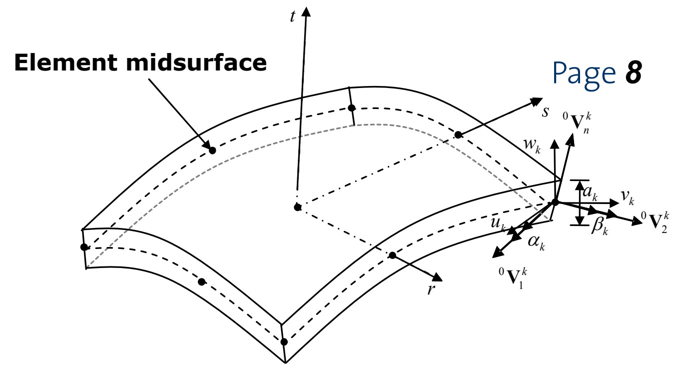
$$w(r,s,t) = \sum_{k=1}^{q} h_k w_{z_k} + \frac{t}{2} \sum_{k=1}^{q} a_k h_k (-{}^0V_{2z}^k \alpha_k + {}^0V_{1z}^k \beta_k)$$

$$u(r,s,t) = \sum_{k=1}^{q} h_k u_{x_k} + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{nx}^k$$

$$v(r,s,t) = \sum_{k=1}^{q} h_k v_{y_k} + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{ny}^k$$

$$w(r,s,t) = \sum_{k=1}^{q} h_k w_{z_k} + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{nz}^k$$





General Shell Elements

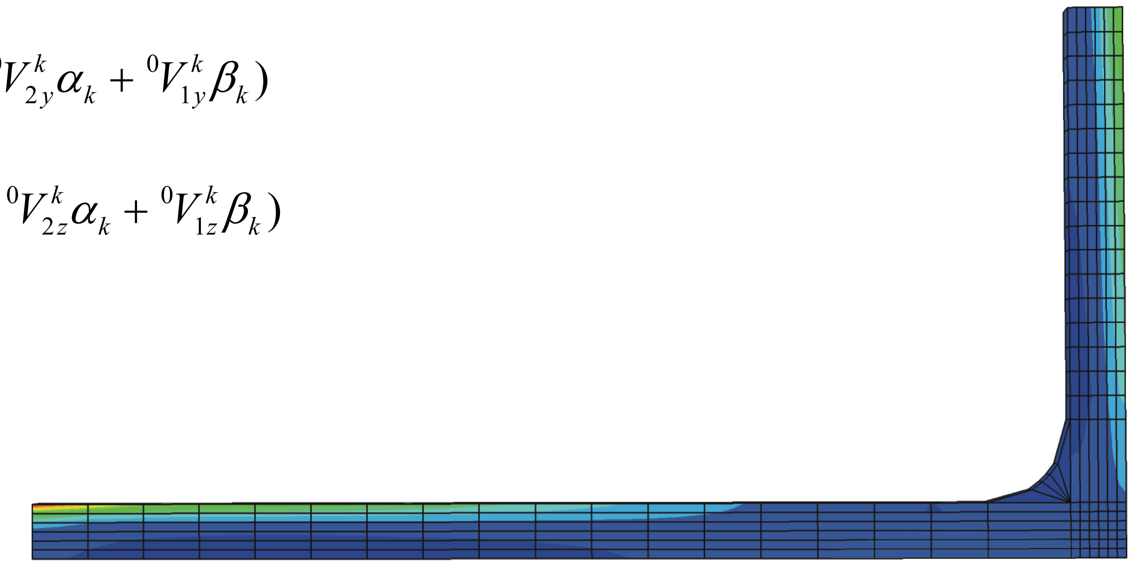
Now we can follow the same procedure as for the beam element in developing the element matrixes

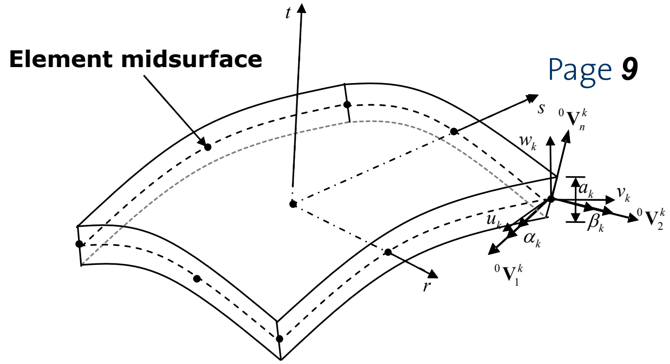
The components of the displacement interpolation matrix H are given in the relation:

$$u(r,s,t) = \sum_{k=1}^{q} h_k u_{x_k} + \frac{t}{2}\sum_{k=1}^{q} a_k h_k (-{}^0V_{2x}^k \alpha_k + {}^0V_{1x}^k \beta_k)$$

$$v(r,s,t) = \sum_{k=1}^{q} h_k v_{y_k} + \frac{t}{2}\sum_{k=1}^{q} a_k h_k (-{}^0V_{2y}^k \alpha_k + {}^0V_{1y}^k \beta_k)$$

$$w(r,s,t) = \sum_{k=1}^{q} h_k w_{z_k} + \frac{t}{2}\sum_{k=1}^{q} a_k h_k (-{}^0V_{2z}^k \alpha_k + {}^0V_{1z}^k \beta_k)$$





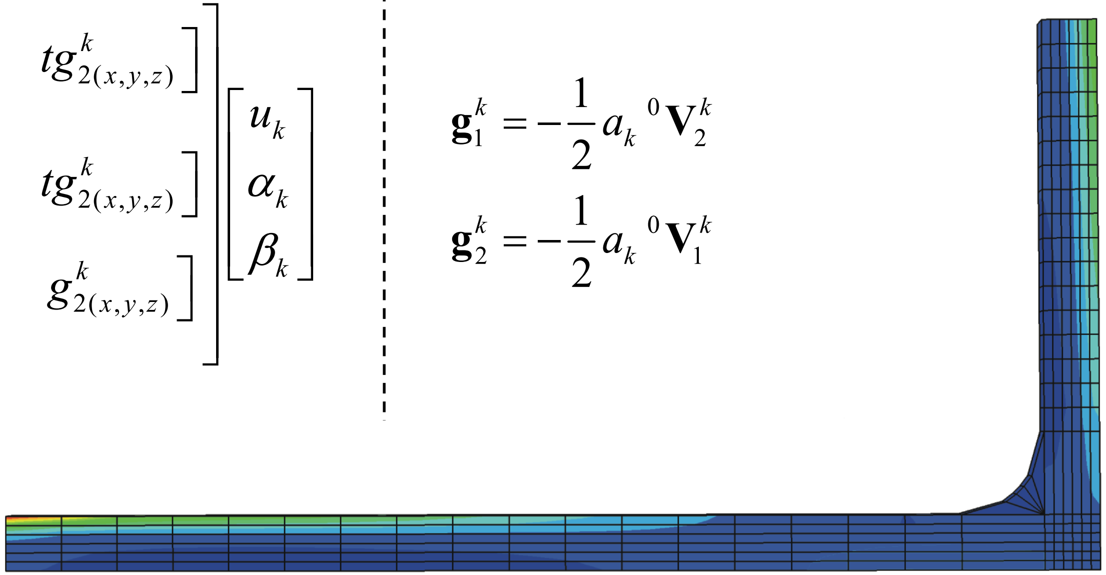
General Shell Elements

In evaluating the strain-displacement matrix B we first need the derivatives of the displacements in regard to the natural coordinates

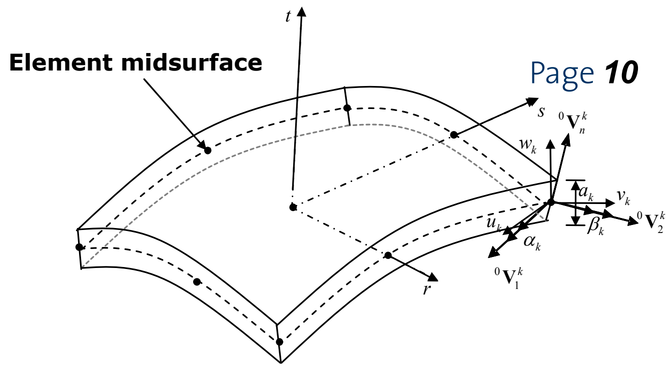
$$\begin{bmatrix} \dfrac{\partial(u,v,w)}{\partial r} \\ \dfrac{\partial(u,v,w)}{\partial s} \\ \dfrac{\partial(u,v,w)}{\partial t} \end{bmatrix} = \sum_{k=1}^{q} \begin{bmatrix} \dfrac{\partial h_k}{\partial r}\begin{bmatrix}1 & t g^k_{1(x,y,z)} & t g^k_{2(x,y,z)}\end{bmatrix} \\ \dfrac{\partial h_k}{\partial s}\begin{bmatrix}1 & t g^k_{1(x,y,z)} & t g^k_{2(x,y,z)}\end{bmatrix} \\ h_k\begin{bmatrix}0 & g^k_{1(x,y,z)} & g^k_{2(x,y,z)}\end{bmatrix} \end{bmatrix} \begin{bmatrix} u_k \\ \alpha_k \\ \beta_k \end{bmatrix}$$

$$\mathbf{g}_1^k = -\frac{1}{2} a_k \, {}^0\mathbf{V}_2^k$$

$$\mathbf{g}_2^k = -\frac{1}{2} a_k \, {}^0\mathbf{V}_1^k$$



General Shell Elements



As usual we must now transform the derivatives to achieve their Cartesian components (global coordinates):

$$\frac{\partial}{\partial \mathbf{x}} = \mathbf{J}^{-1} \frac{\partial}{\partial \mathbf{r}}$$

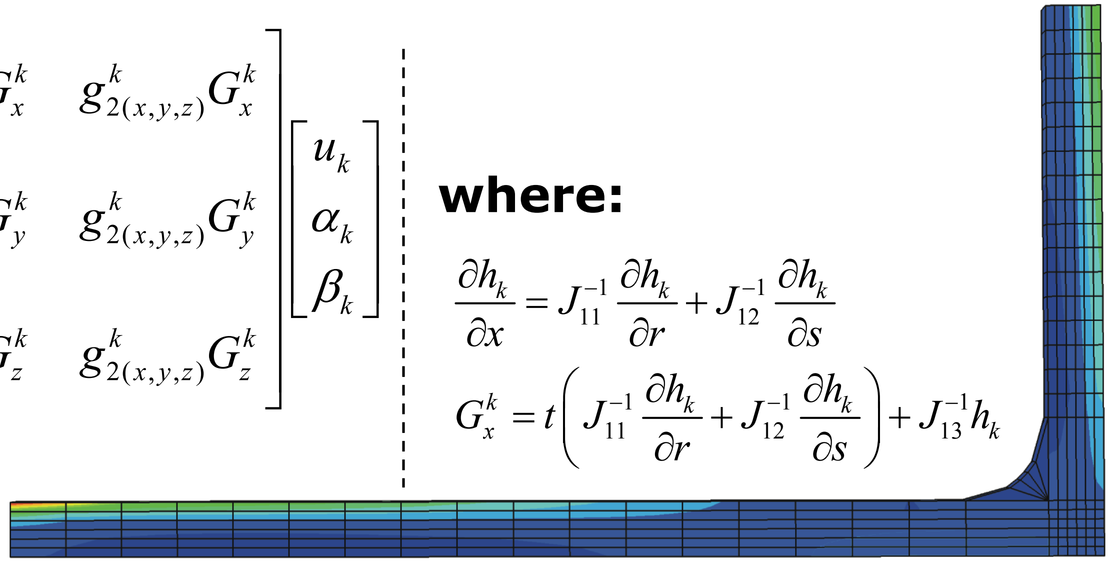
then we may write:

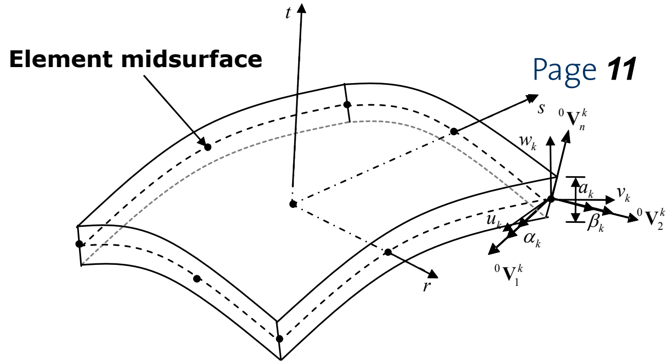
$$\begin{bmatrix} \dfrac{\partial(u,v,w)}{\partial r} \\ \dfrac{\partial(u,v,w)}{\partial s} \\ \dfrac{\partial(u,v,w)}{\partial t} \end{bmatrix} = \sum_{k=1}^{q} \begin{bmatrix} \dfrac{\partial h_k}{\partial x} & g^k_{1(x,y,z)} G^k_x & g^k_{2(x,y,z)} G^k_x \\ \dfrac{\partial h_k}{\partial y} & g^k_{1(x,y,z)} G^k_y & g^k_{2(x,y,z)} G^k_y \\ \dfrac{\partial h_k}{\partial z} & g^k_{1(x,y,z)} G^k_z & g^k_{2(x,y,z)} G^k_z \end{bmatrix} \begin{bmatrix} u_k \\ \alpha_k \\ \beta_k \end{bmatrix}$$

where:

$$\frac{\partial h_k}{\partial x} = J^{-1}_{11} \frac{\partial h_k}{\partial r} + J^{-1}_{12} \frac{\partial h_k}{\partial s}$$

$$G^k_x = t\left(J^{-1}_{11} \frac{\partial h_k}{\partial r} + J^{-1}_{12} \frac{\partial h_k}{\partial s} \right) + J^{-1}_{13} h_k$$





General Shell Elements

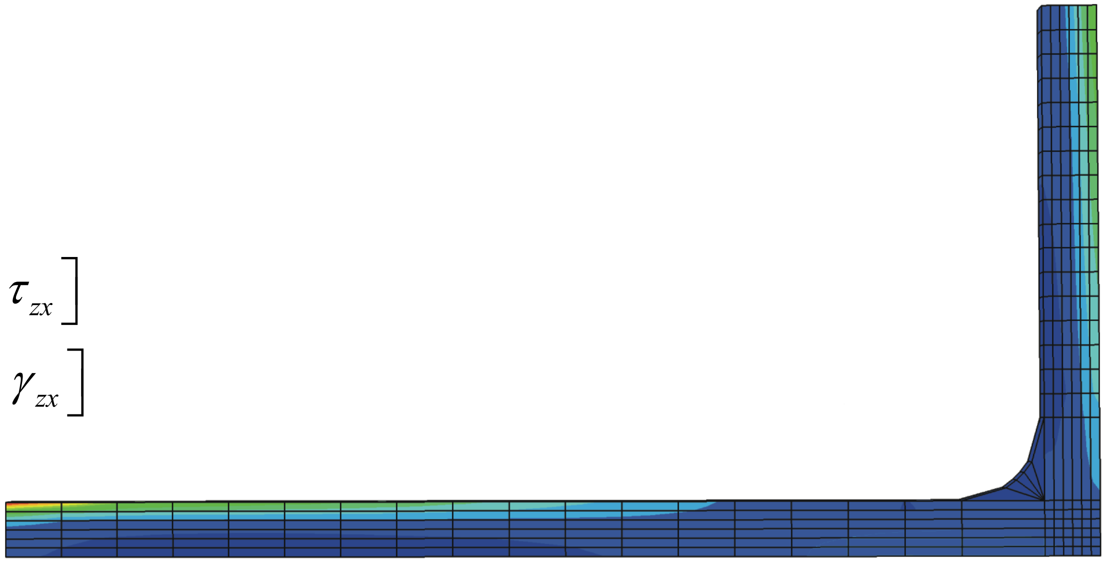
Having established the displacement derivatives we can now assemble the strain-displacement matrix B for the shell element.

In order to establish the strain-stress matrix (constitutive law) we must impose the shell assumption that the stress in the direction of the normal to the shell surface is zero

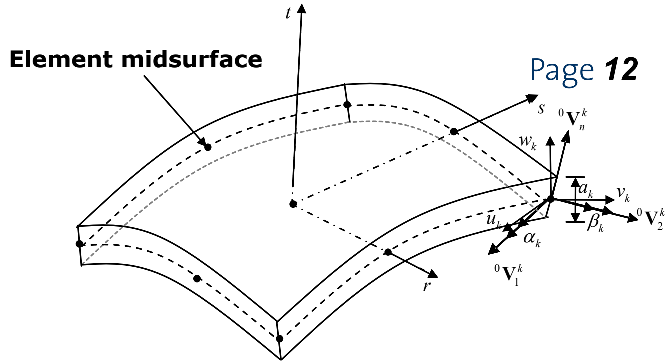
$$\boldsymbol{\tau} = \mathbf{C}_{sh}\boldsymbol{\varepsilon}$$ **with:**

$$\boldsymbol{\tau}^T = \begin{bmatrix} \tau_{xx} & \tau_{yy} & \tau_{zz} & \tau_{xy} & \tau_{yz} & \tau_{zx} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}^T = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \boldsymbol{\varepsilon}_{yy} & \boldsymbol{\varepsilon}_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}$$



General Shell Elements



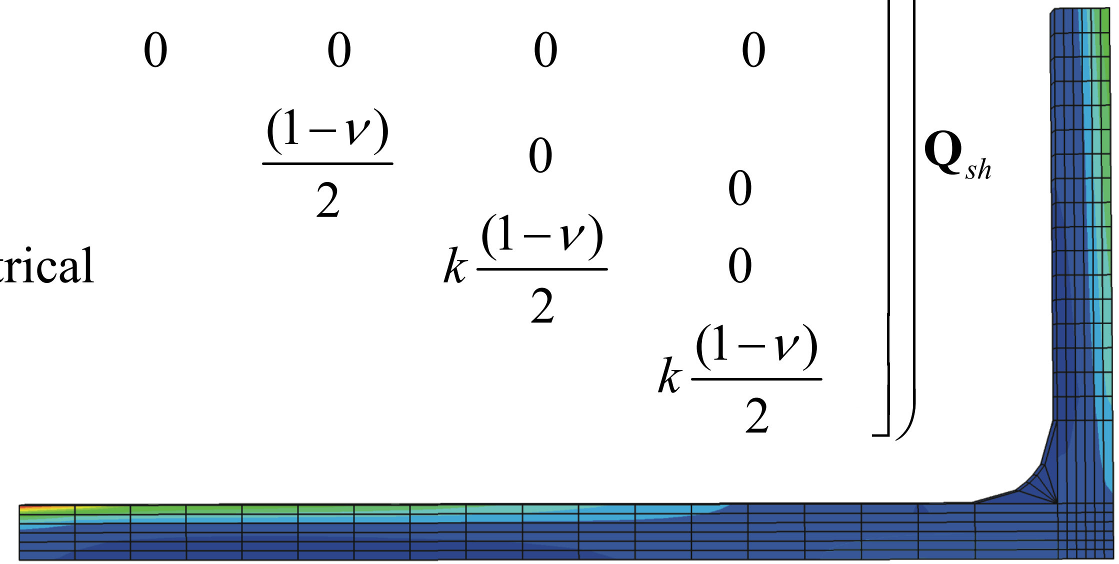
Where

$$\boldsymbol{\tau} = \mathbf{C}_{sh}\boldsymbol{\varepsilon}$$

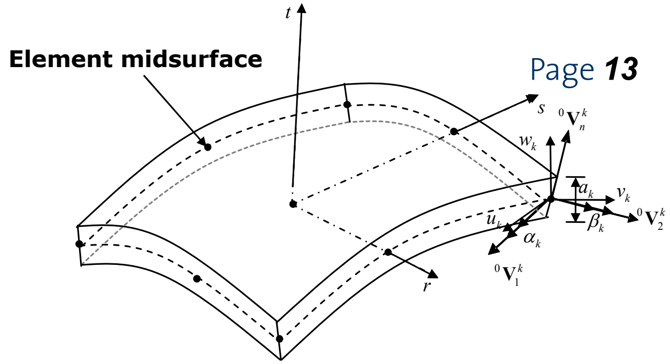
$$\boldsymbol{\tau}^T = \begin{bmatrix} \tau_{xx} & \tau_{yy} & \tau_{zz} & \tau_{xy} & \tau_{yz} & \tau_{zx} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}^T = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \boldsymbol{\varepsilon}_{yy} & \boldsymbol{\varepsilon}_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}$$

$$\mathbf{C}_{sh} = \mathbf{Q}_{sh}^T \left(\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & \frac{(1-\nu)}{2} & 0 & 0 \\ & \text{symmetrical} & & & k\frac{(1-\nu)}{2} & 0 \\ & & & & & k\frac{(1-\nu)}{2} \end{bmatrix} \right) \mathbf{Q}_{sh}$$



General Shell Elements



The transformation from the $\bar{r}, \bar{s}, t$ Cartesian shell aligned coordinate system to the global coordinate system is made through the 4th order tensor transformation

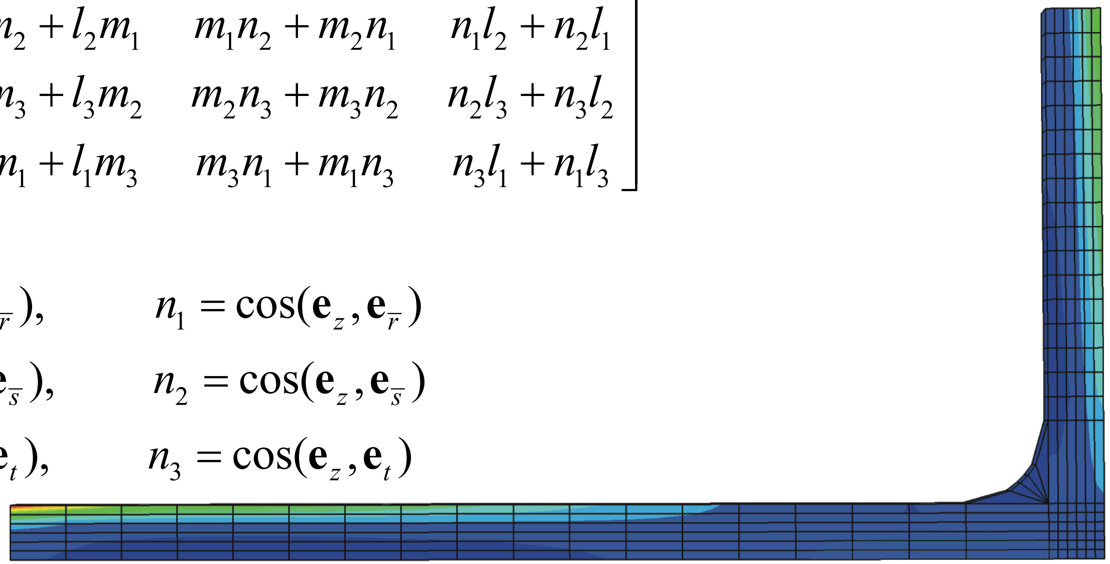
$$\mathbf{Q}_{sh} = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & l_1 m_1 & m_1 n_1 & n_1 l_1 \\ l_2^2 & m_2^2 & n_2^2 & l_2 m_2 & m_2 n_2 & n_2 l_2 \\ l_3^2 & m_3^2 & n_3^2 & l_3 m_3 & m_3 n_3 & n_3 l_3 \\ 2 l_1 l_2 & 2 m_1 m_2 & 2 n_1 n_2 & l_1 m_2 + l_2 m_1 & m_1 n_2 + m_2 n_1 & n_1 l_2 + n_2 l_1 \\ 2 l_2 l_3 & 2 m_2 m_3 & 2 n_2 n_3 & l_2 m_3 + l_3 m_2 & m_2 n_3 + m_3 n_2 & n_2 l_3 + n_3 l_2 \\ 2 l_3 l_1 & 2 m_3 m_1 & 2 n_3 n_1 & l_3 m_1 + l_1 m_3 & m_3 n_1 + m_1 n_3 & n_3 l_1 + n_1 l_3 \end{bmatrix}$$

where

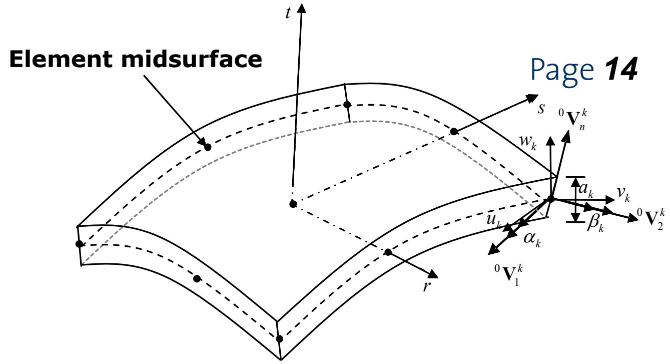
$$l_1 = \cos(\mathbf{e}_x, \mathbf{e}_{\bar{r}}), \qquad m_1 = \cos(\mathbf{e}_y, \mathbf{e}_{\bar{r}}), \qquad n_1 = \cos(\mathbf{e}_z, \mathbf{e}_{\bar{r}})$$

$$l_2 = \cos(\mathbf{e}_x, \mathbf{e}_{\bar{s}}), \qquad m_2 = \cos(\mathbf{e}_y, \mathbf{e}_{\bar{s}}), \qquad n_2 = \cos(\mathbf{e}_z, \mathbf{e}_{\bar{s}})$$

$$l_3 = \cos(\mathbf{e}_x, \mathbf{e}_t), \qquad m_3 = \cos(\mathbf{e}_y, \mathbf{e}_t), \qquad n_3 = \cos(\mathbf{e}_z, \mathbf{e}_t)$$

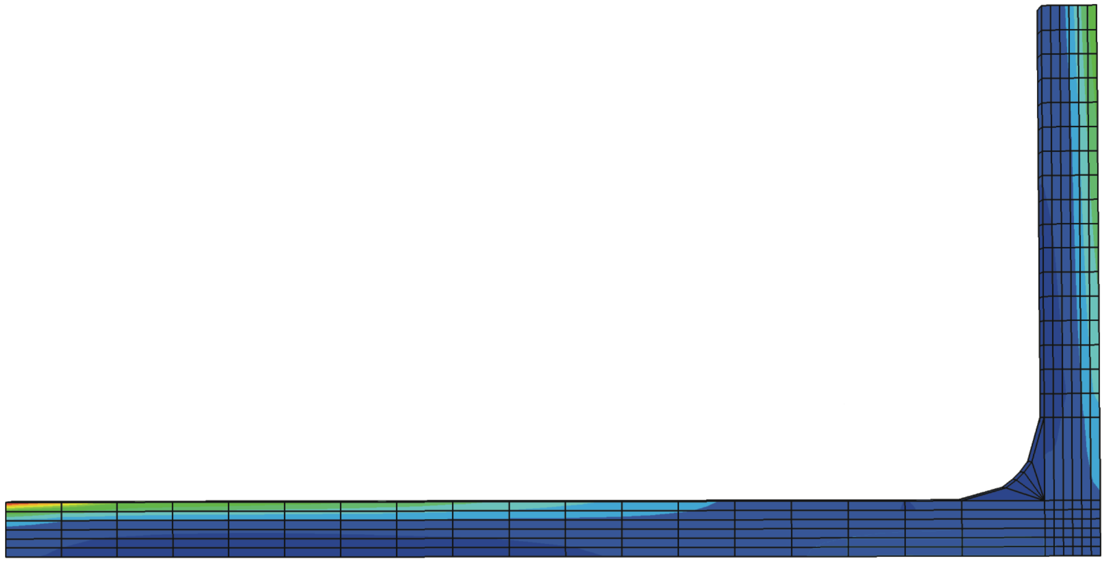


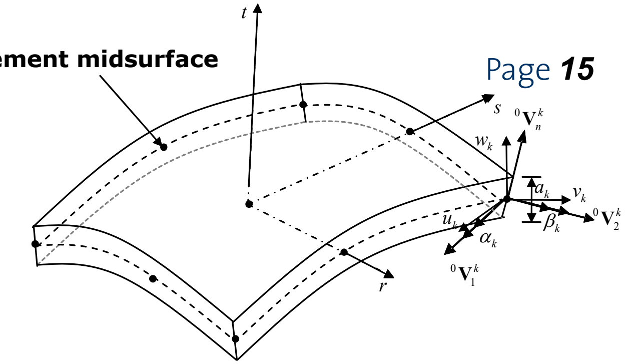
General Shell Elements



In general this transformation has to evaluated at each integration point during the integration of the stiffness matrix

– there are exceptions for e.g. flat plates





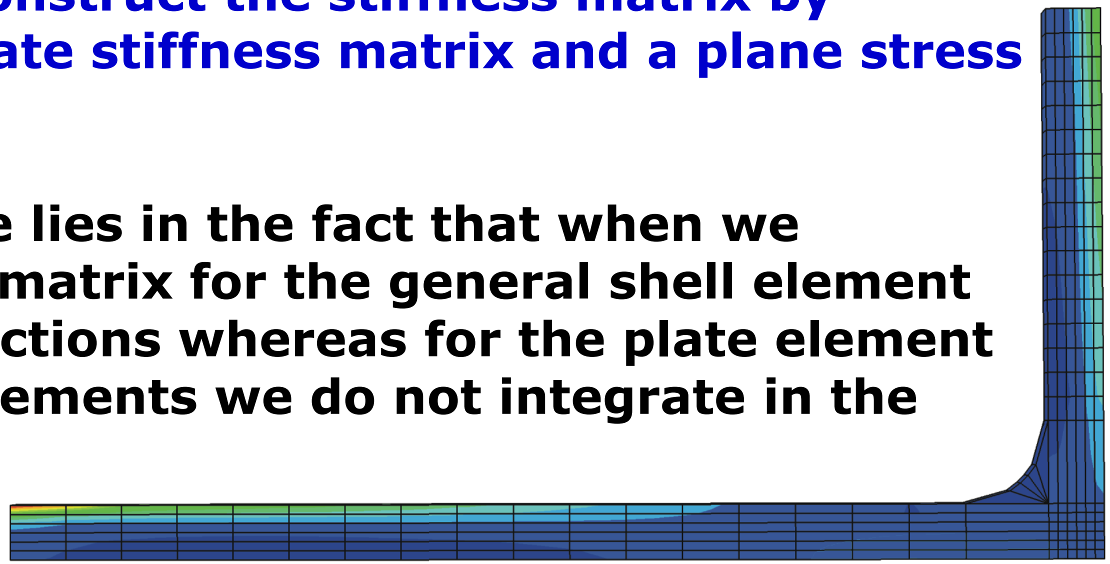
General Shell Elements

We can compare the present formulation with a shell element constructed through superposition of a plate bending and a membrane stress behavior

Let us assume that we apply the general shell element as a flat element in the modeling of a shell

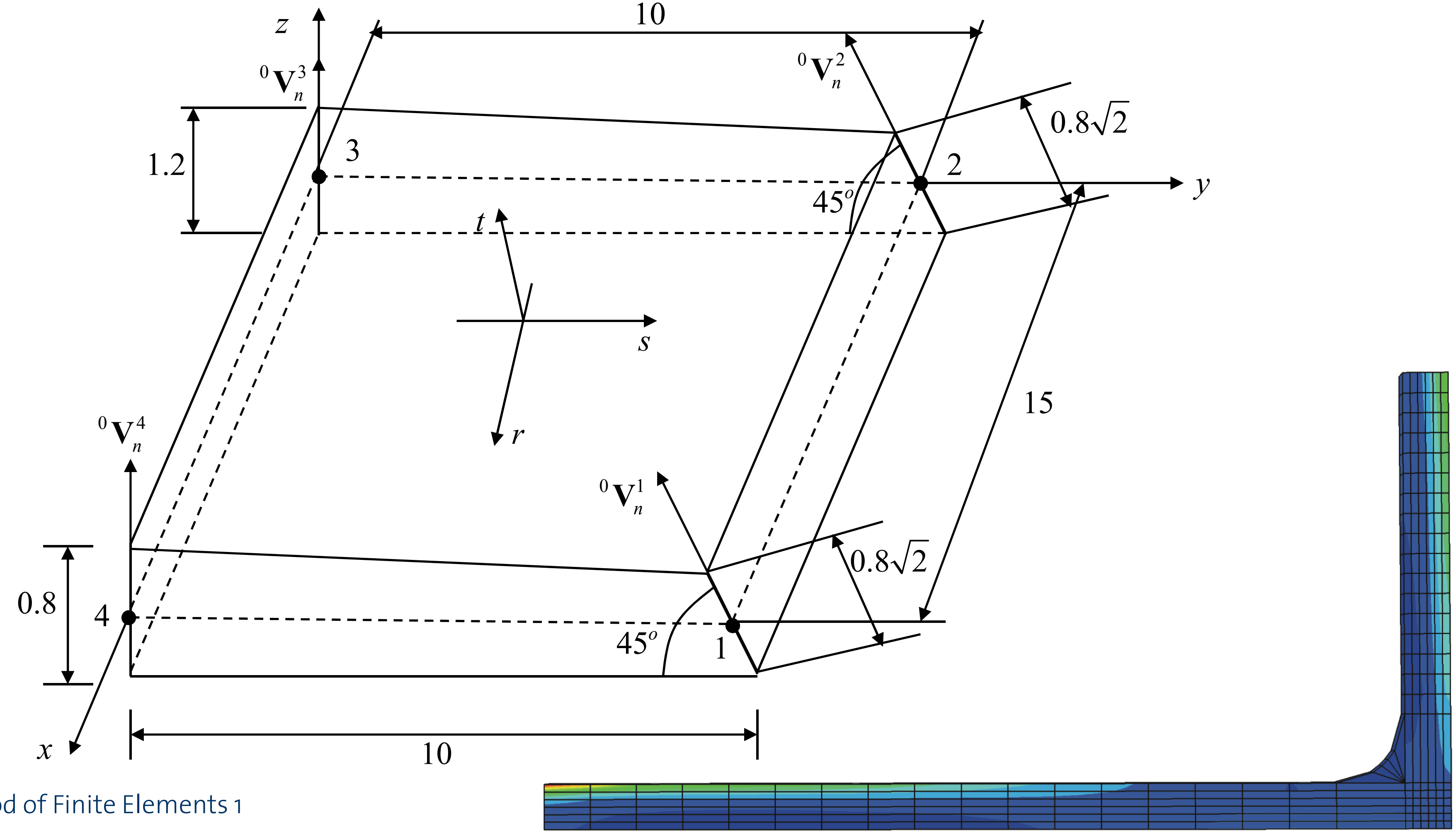
- in this case we can construct the stiffness matrix by superposition of a plate stiffness matrix and a plane stress stiffness matrix

the effective difference lies in the fact that when we establish the stiffness matrix for the general shell element we integrate in all directions whereas for the plate element and the plane stress elements we do not integrate in the *t*-direction.

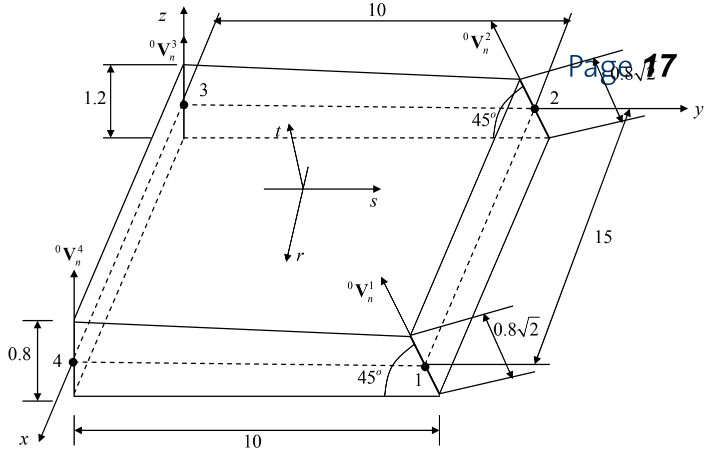


General Shell Elements

Let us consider an example



General Shell Elements



Let us consider an example

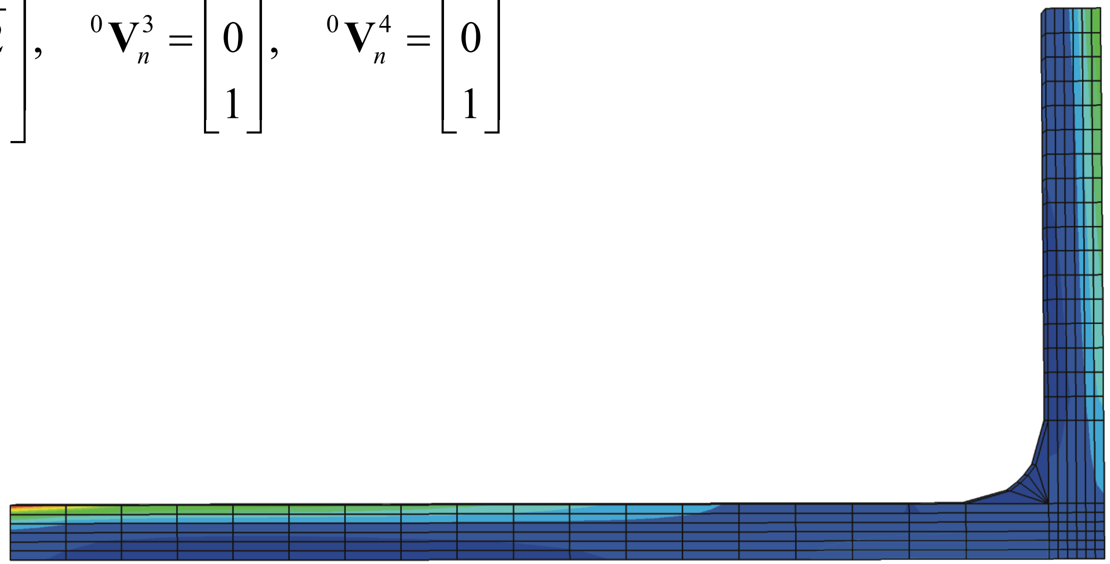
For this element the interpolation functions are those of the 4-node two dimensional element (solid element)

The directional vectors are:

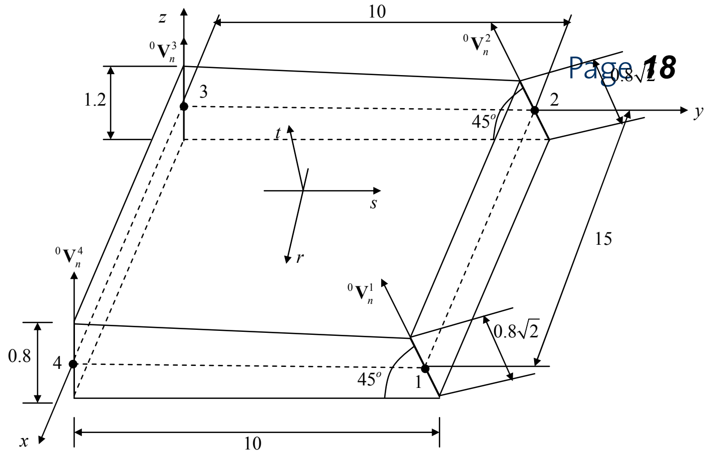
$$ {}^{0}\mathbf{V}_n^1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad {}^{0}\mathbf{V}_n^2 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad {}^{0}\mathbf{V}_n^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad {}^{0}\mathbf{V}_n^4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

hence

$$ {}^{0}\mathbf{V}_1^1 = {}^{0}\mathbf{V}_1^2 = {}^{0}\mathbf{V}_1^3 = {}^{0}\mathbf{V}_1^4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $$



General Shell Elements



Further we have

$$ {}^0\mathbf{V}_2^1 = {}^0\mathbf{V}_2^2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}; \qquad {}^0\mathbf{V}_2^3 = {}^0\mathbf{V}_2^4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} $$

and

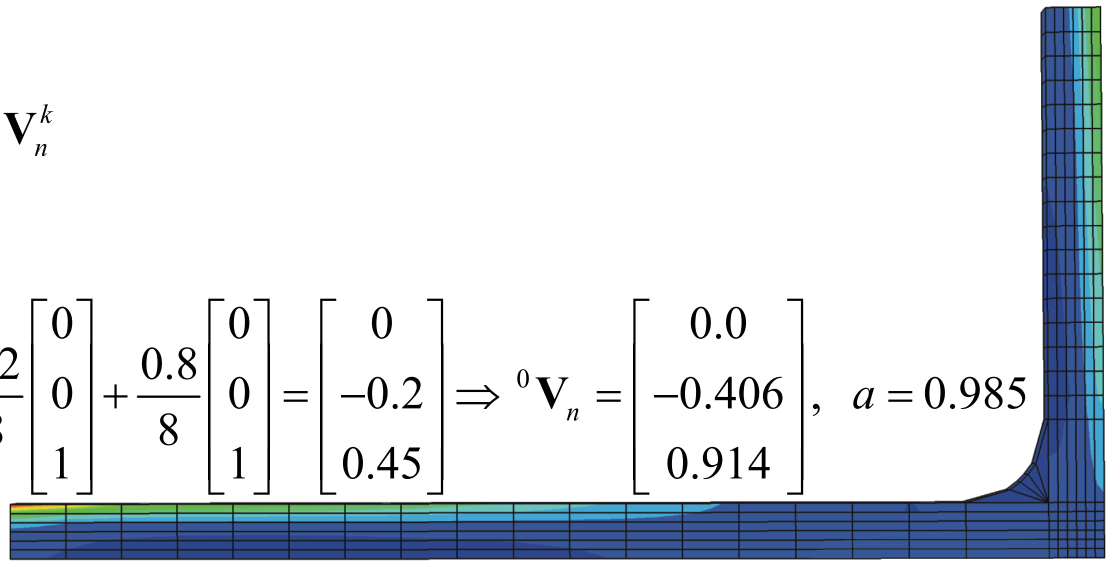
$$ a_1 = a_2 = 0.8\sqrt{2}, \qquad a_3 = 1.2, \qquad a_4 = 0.8 $$

The thickness and the director vector are determined as:

$$ \left(\frac{a}{2}\right) {}^0\mathbf{V}_n \Big|_{\text{midpoint}} = \sum_{k=1}^{4} \frac{a_k}{2} h_k \Big|_{r,s=0} {}^0\mathbf{V}_n^k $$

$$ \Downarrow $$

$$ \left(\frac{a}{2}\right) {}^0\mathbf{V}_n = \frac{0.8\sqrt{2}}{4} \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{1.2}{8} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{0.8}{8} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2 \\ 0.45 \end{bmatrix} \Rightarrow {}^0\mathbf{V}_n = \begin{bmatrix} 0.0 \\ -0.406 \\ 0.914 \end{bmatrix}, \quad a = 0.985 $$



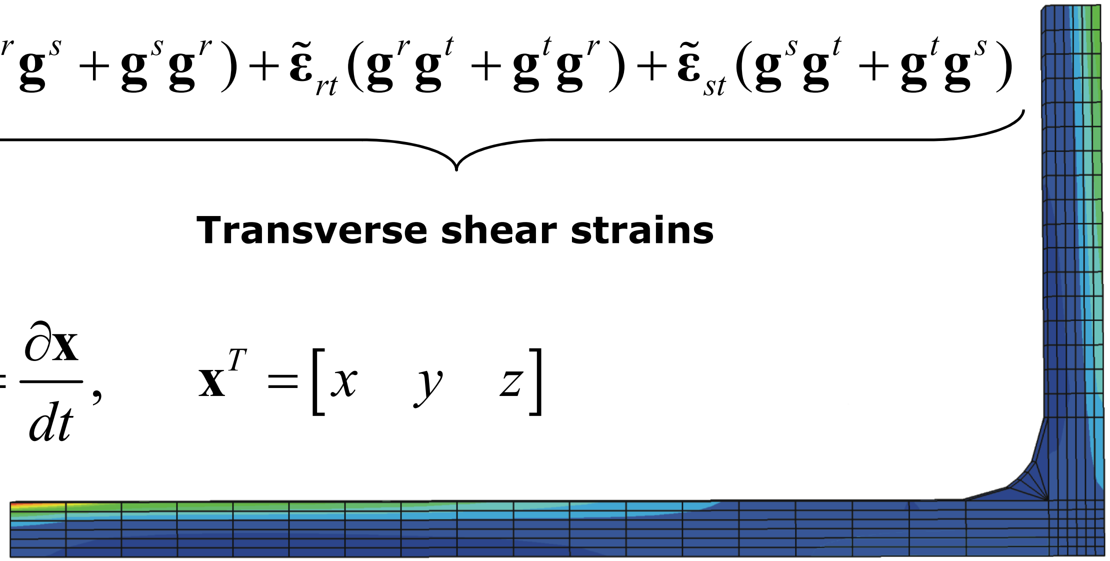
General Shell Elements

So far we have considered a pure displacement based element – however, as for the beam and the plate element shear locking (and membrane locking) is a problem (at least cubic displacement interpolation functions are required)

In order to solve this problem we use again mixed interpolation functions (Dvorkin & Bathe)

$$\boldsymbol{\varepsilon} = \underbrace{\tilde{\boldsymbol{\varepsilon}}_{rr}\mathbf{g}^r\mathbf{g}^r + \tilde{\boldsymbol{\varepsilon}}_{ss}\mathbf{g}^s\mathbf{g}^s}_{\text{In-layer strains}} + \underbrace{\tilde{\boldsymbol{\varepsilon}}_{rs}(\mathbf{g}^r\mathbf{g}^s + \mathbf{g}^s\mathbf{g}^r) + \tilde{\boldsymbol{\varepsilon}}_{rt}(\mathbf{g}^r\mathbf{g}^t + \mathbf{g}^t\mathbf{g}^r) + \tilde{\boldsymbol{\varepsilon}}_{st}(\mathbf{g}^s\mathbf{g}^t + \mathbf{g}^t\mathbf{g}^s)}_{\text{Transverse shear strains}}$$

$$\mathbf{g}_r = \frac{\partial \mathbf{x}}{dr}; \quad \mathbf{g}_s = \frac{\partial \mathbf{x}}{ds}; \quad \mathbf{g}_t = \frac{\partial \mathbf{x}}{dt}, \qquad \mathbf{x}^T = \begin{bmatrix} x & y & z \end{bmatrix}$$



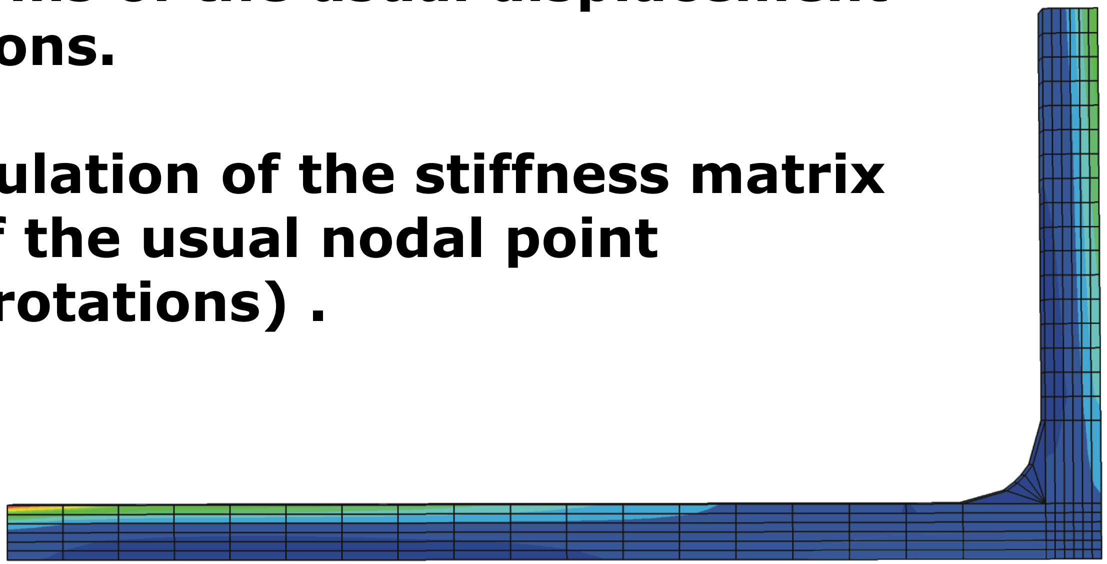
General Shell Elements

The Green-Lagrange covariant strain tensor components are determined from :

$$ {}_0^1\tilde{\varepsilon}_{ij} = \frac{1}{2}({}^1\mathbf{g}_i \bullet {}^1\mathbf{g}_j - {}^0\mathbf{g}_i \bullet {}^0\mathbf{g}_j), \quad {}^0\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial r_i}, \quad {}^1\mathbf{g}_i = \frac{\partial(\mathbf{x}+\mathbf{u})}{\partial r_i} $$

The objective is to interpolate the in-layer and transverse shear strains independently and then to express these in terms of the usual displacement interpolation functions.

We then get a formulation of the stiffness matrix as usual in terms of the usual nodal point displacement (and rotations) .



General Shell Elements

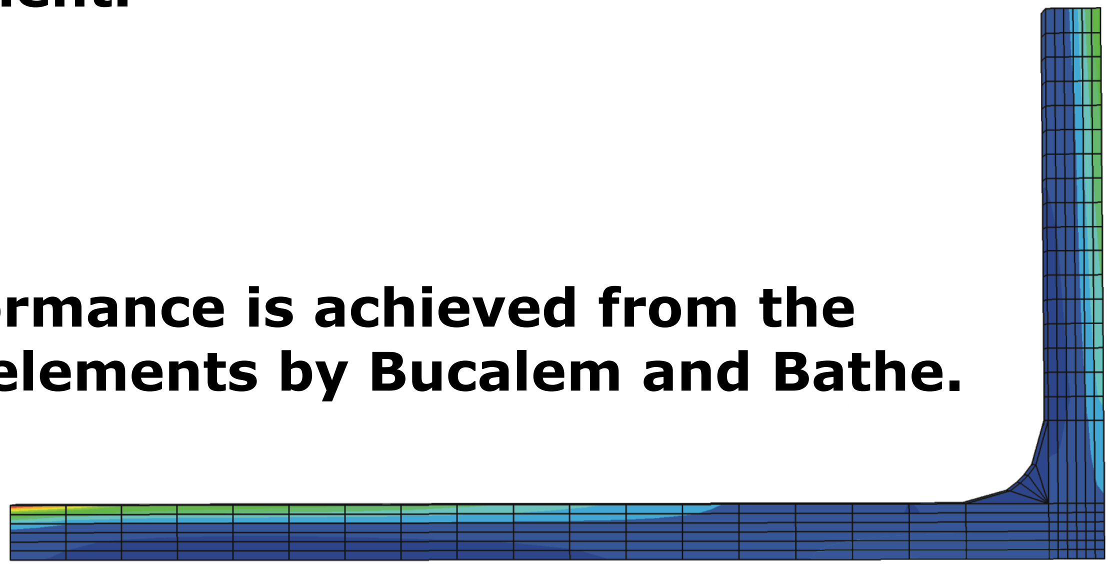
The four-node shell element (MITC4) proposed by Dvorkin and Bathe is attractive –

The in-layer strains are computed from the displacement interpolations

The covariant shear strains are computed from the displacement interpolation functions at discrete locations as for the plate element.

$$\tilde{\varepsilon}_{ij} = \sum_{k=1}^{n_{ij}} h_k^{ij} \mathbf{B}_{ij}^{DI} \Big|_k \hat{\mathbf{u}}$$

Better general performance is achieved from the MITC9 and MITC16 elements by Bucalem and Bathe.



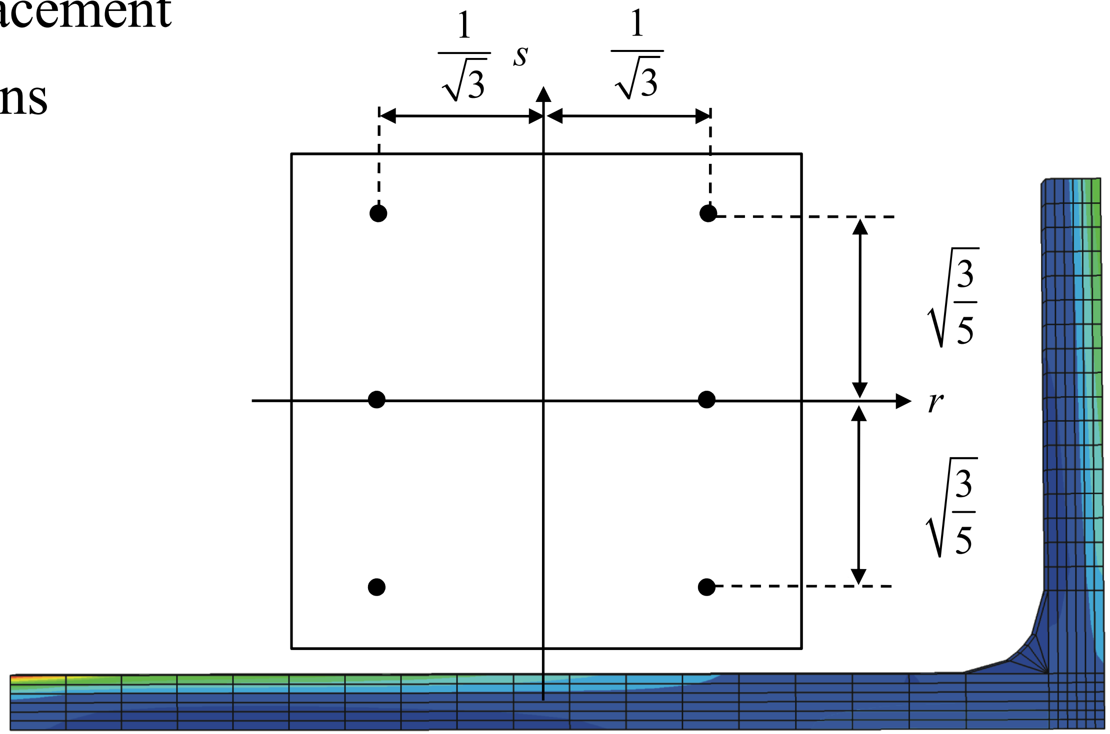
General Shell Elements

In the MITC9 element the discrete locations in which the mixed interpolations are fixed with the displacement interpolation functions are:

For the $\tilde{\varepsilon}_{rr}, \tilde{\varepsilon}_{rt}$ **strain components**

$$\tilde{\varepsilon}_{ij} = \sum_{k=1}^{n_{ij}} h_k^{ij} \mathbf{B}_{ij}^{DI} \Big|_k \hat{\mathbf{u}}$$

h_k^{ij} : are the 6-node displacement interpolation functions



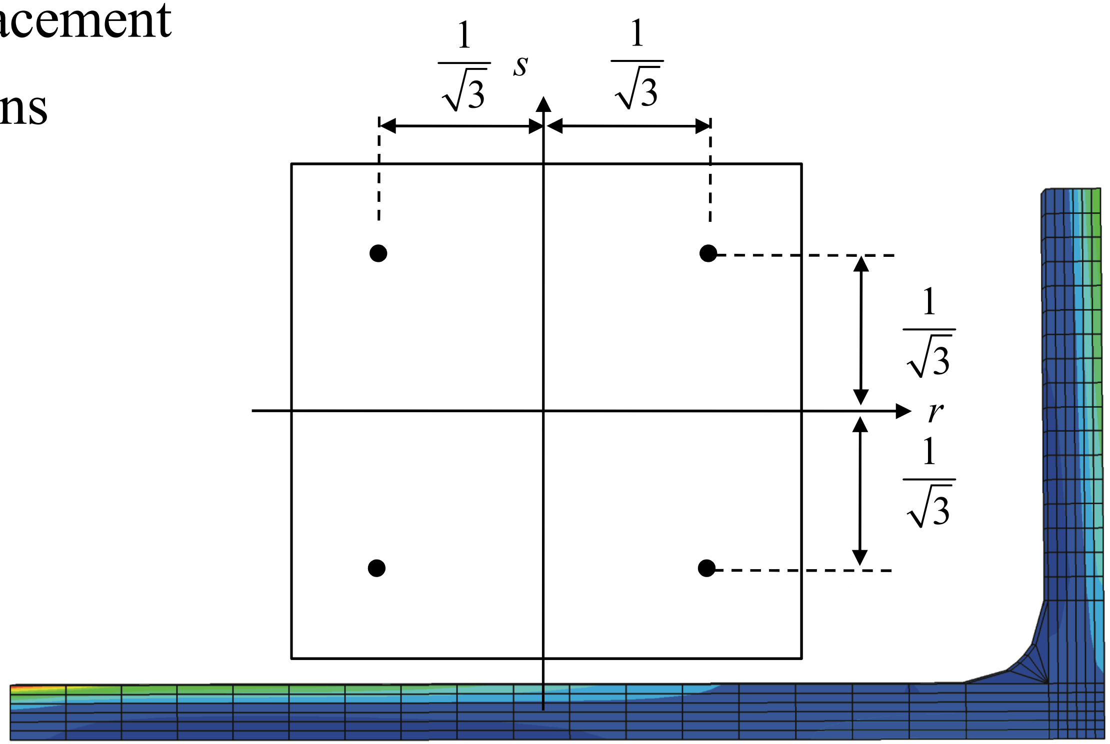
General Shell Elements

In the MITC9 element the discrete locations in which the mixed interpolations are fixed with the displacement interpolation functions are:

For the $\tilde{\varepsilon}_{rs}$ **strain component**

$$\tilde{\varepsilon}_{ij} = \sum_{k=1}^{n_{ij}} h_k^{ij} \mathbf{B}_{ij}^{DI} \Big|_k \hat{\mathbf{u}}$$

h_k^{ij} : are the 4-node displacement interpolation functions



Boundary Conditions

The plate elements we considered in the previous lecture were based on the Reissner-Midlin plate theory

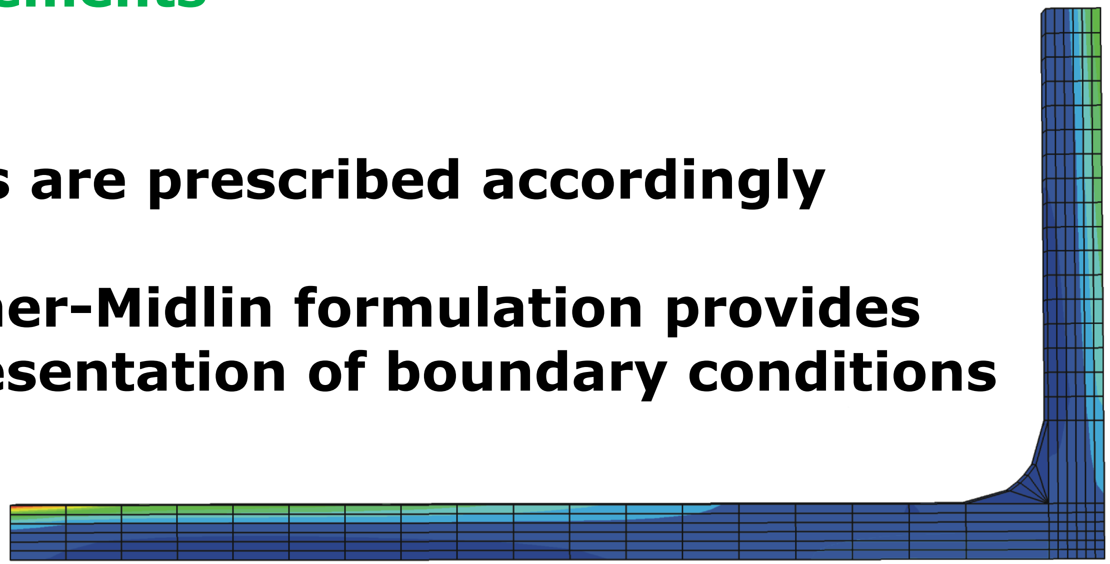
- transverse displacements and section rotations

The Kirchhoff plate theory

- transverse displacements

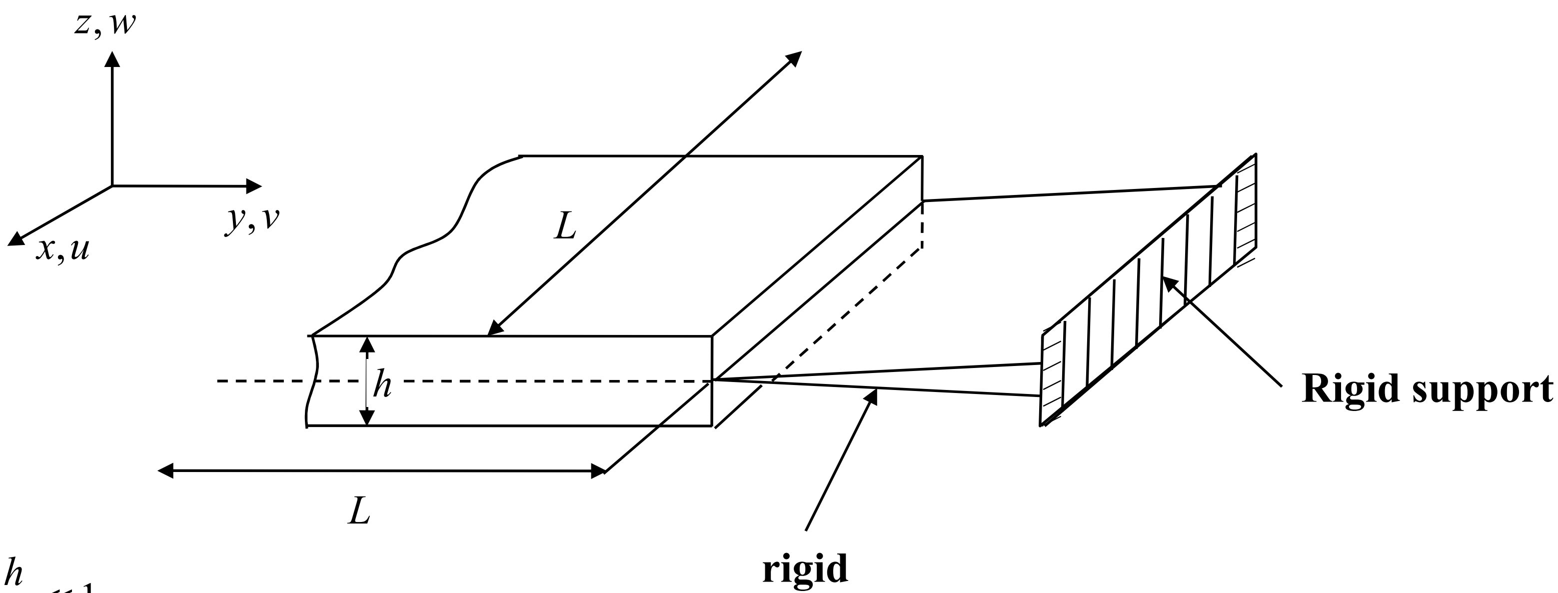
Boundary conditions are prescribed accordingly

therefore the Reissner-Midlin formulation provides more accurate representation of boundary conditions

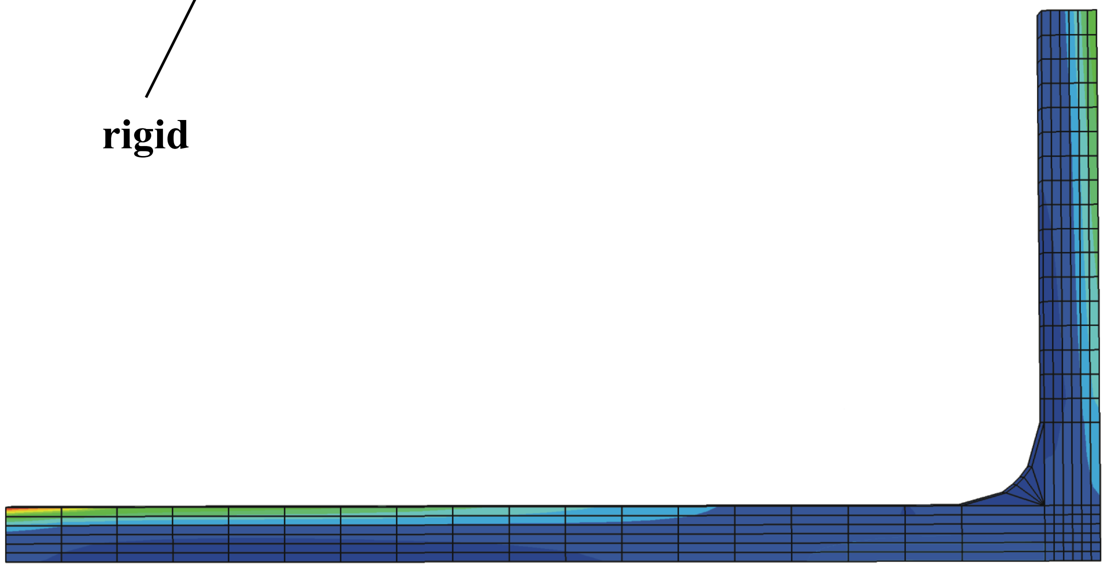


Boundary Conditions

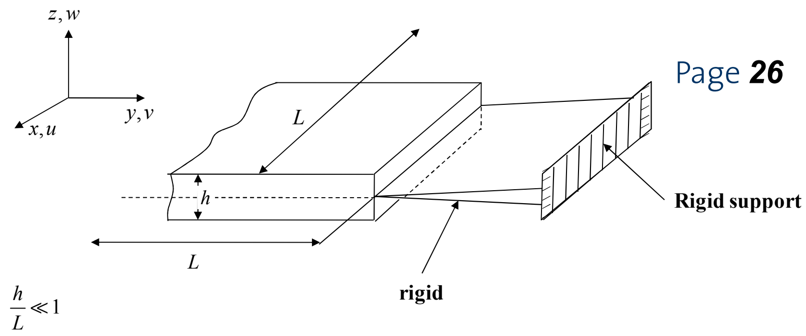
Let us consider an example:



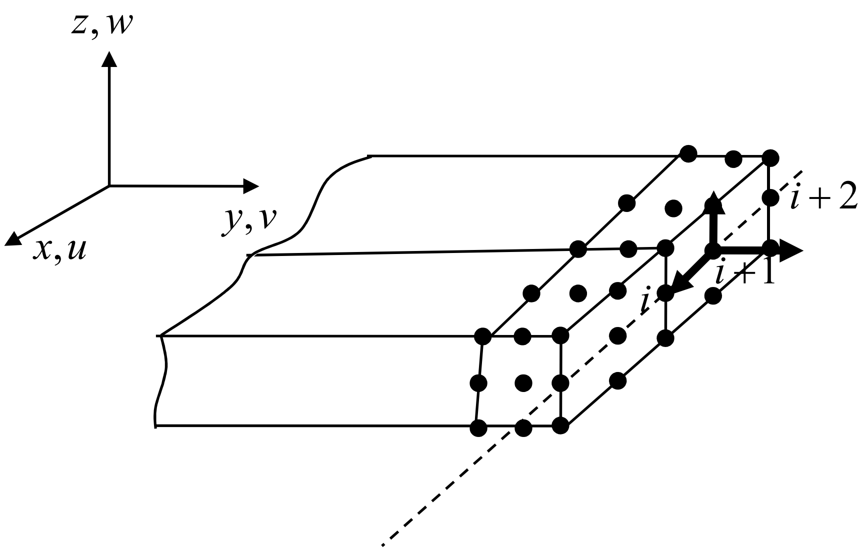
$$\frac{h}{L} \ll 1$$



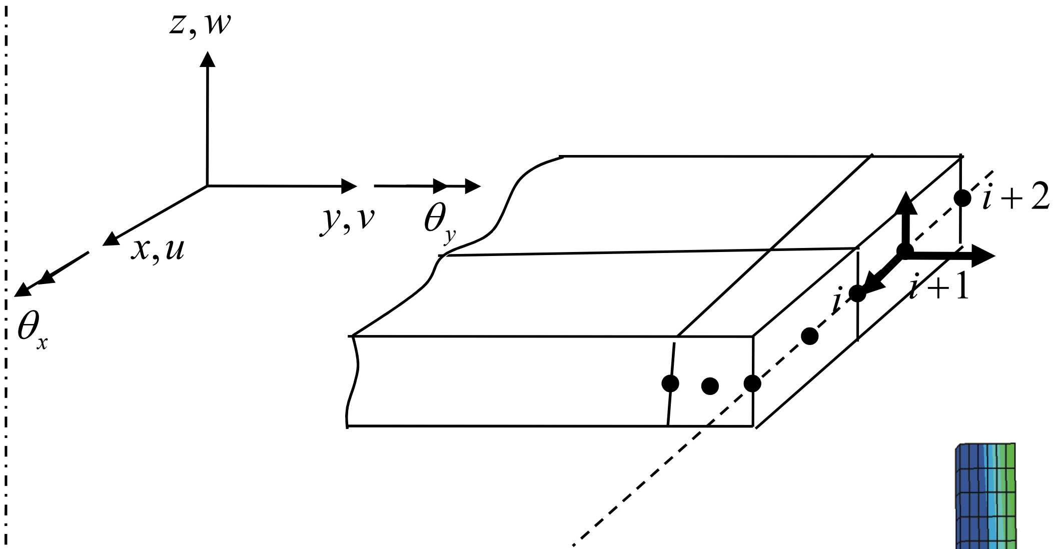
Boundary Conditions



Let us consider an example:



$$u_k = v_k = w_k = 0, \quad k = i, i+1, i+2, ..$$



$$w_k = 0, \quad k = i, i+1, i+2, ..$$

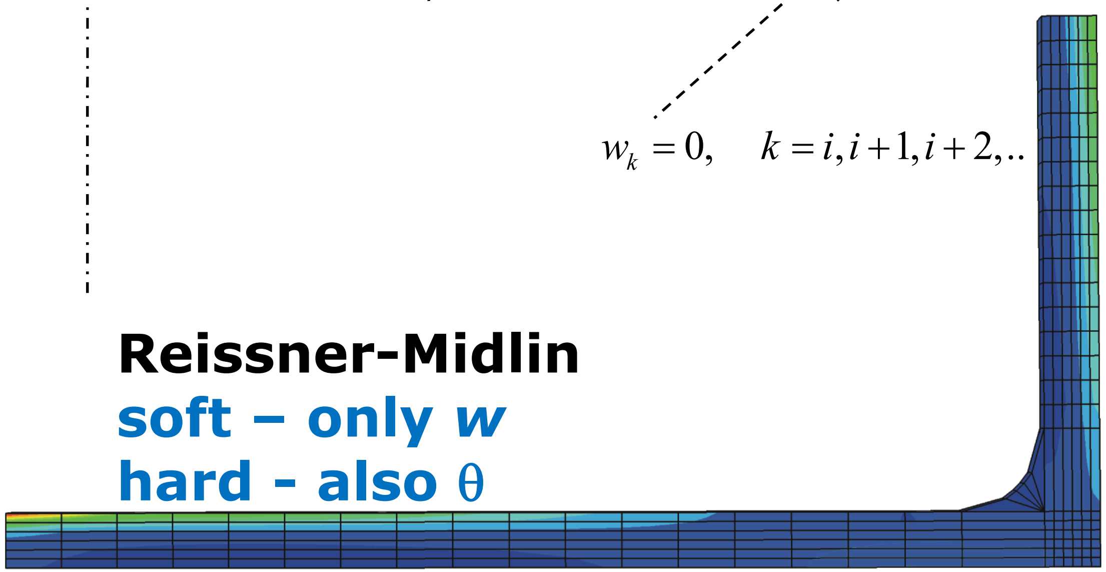
3-D solid

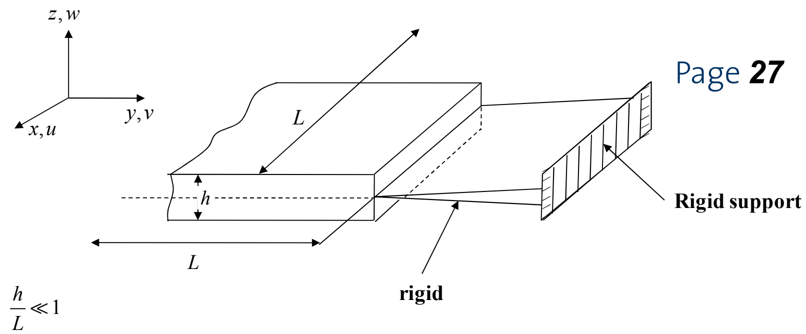
Displacement in all directions

Reissner-Midlin

soft – only *w*

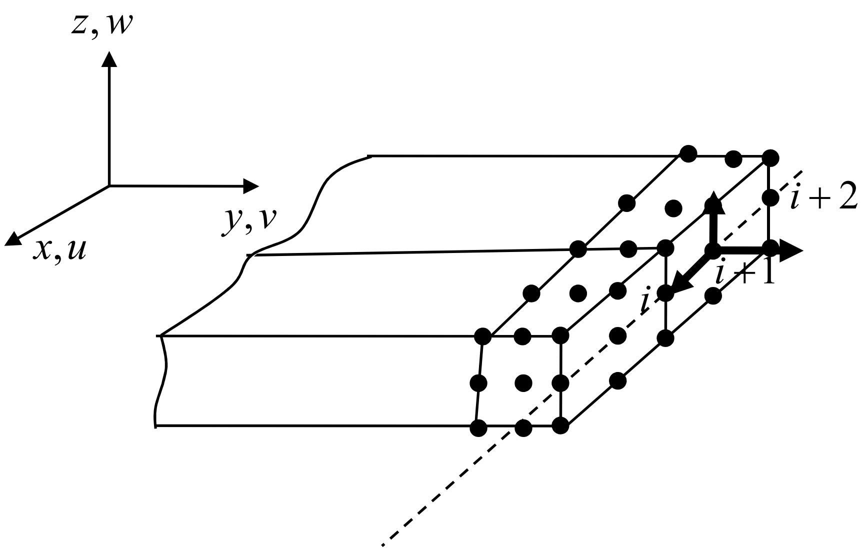
hard - also θ



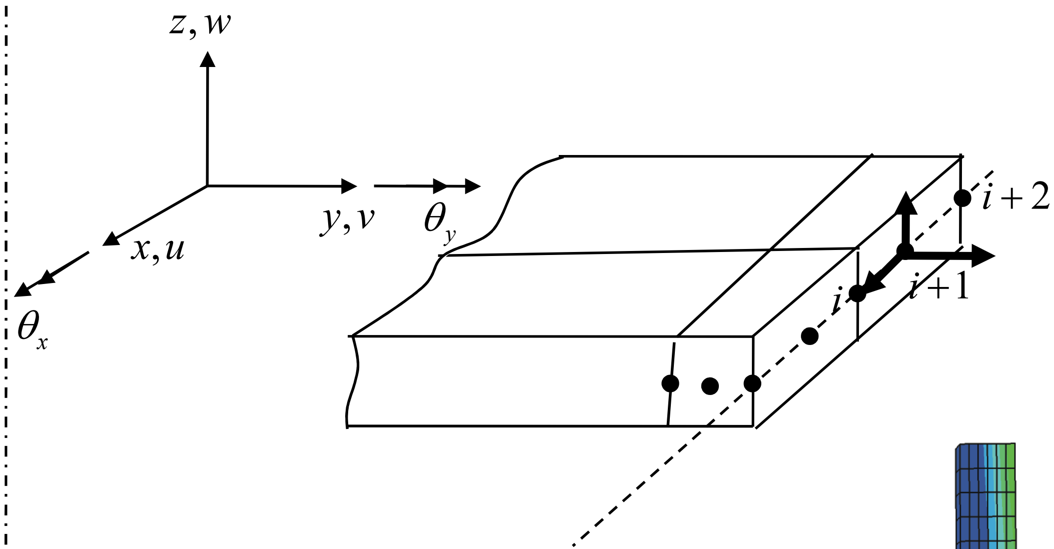


Boundary Conditions

Let us consider an example:



$$u_k = v_k = w_k = 0, \quad k = i, i+1, i+2, ..$$



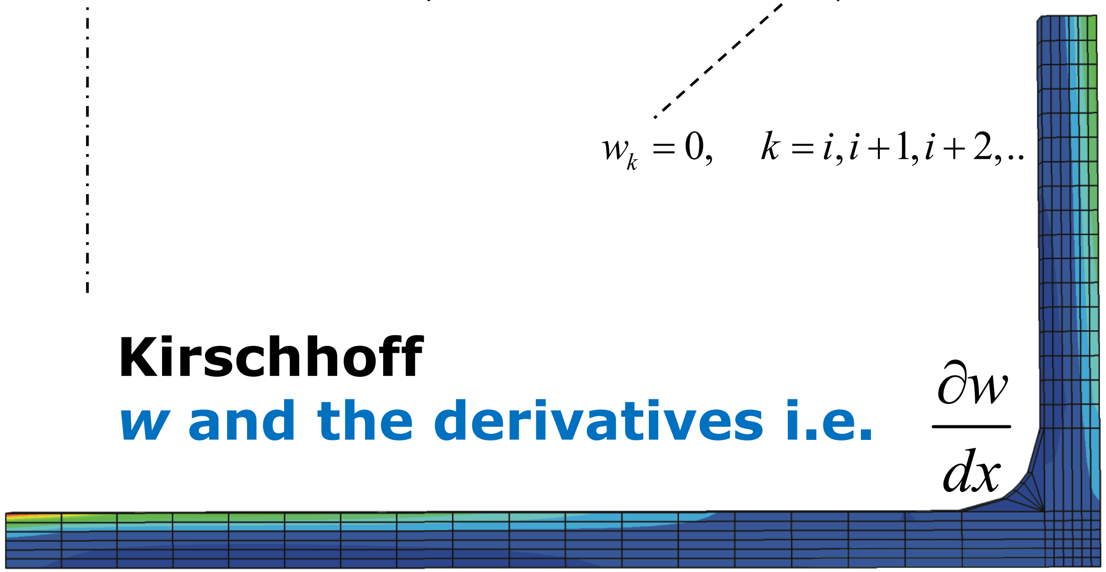
$$w_k = 0, \quad k = i, i+1, i+2, ..$$

3-D solid

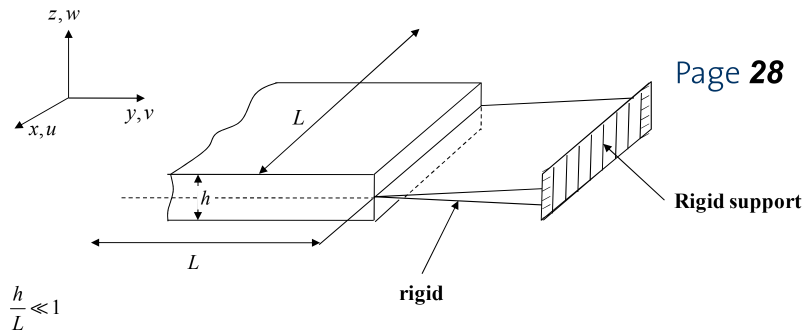
Displacement in all directions

Kirschhoff

***w* and the derivatives i.e.** $\dfrac{\partial w}{dx}$



Boundary Conditions



The main message here is that we have to specify boundary conditions in accordance with the characteristics of elements we are using !