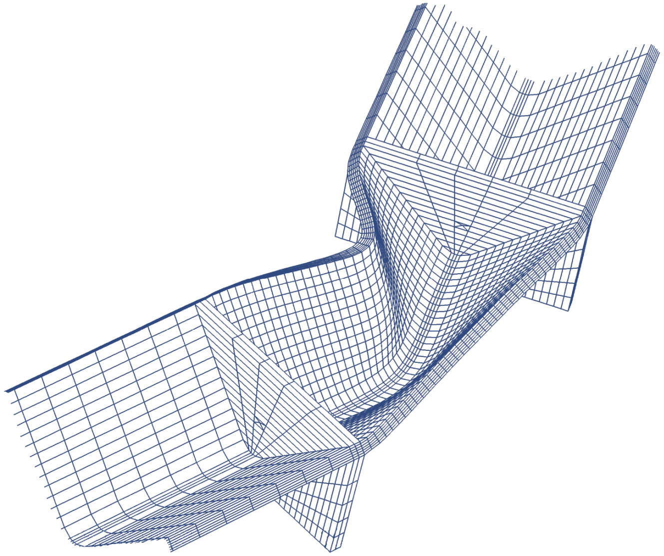
The Finite Element Method for the Analysis of Linear Systems



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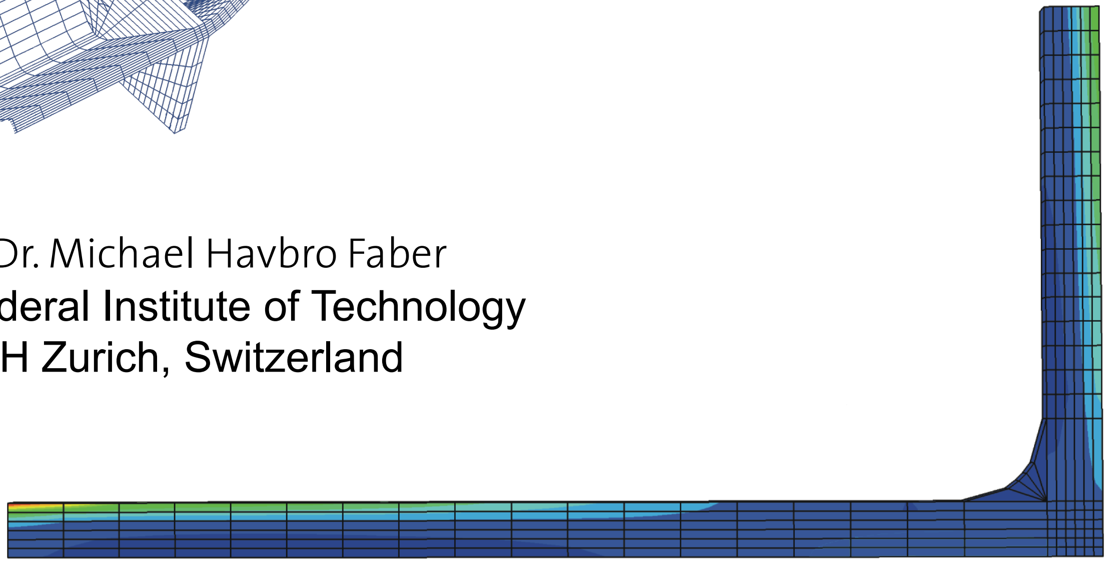
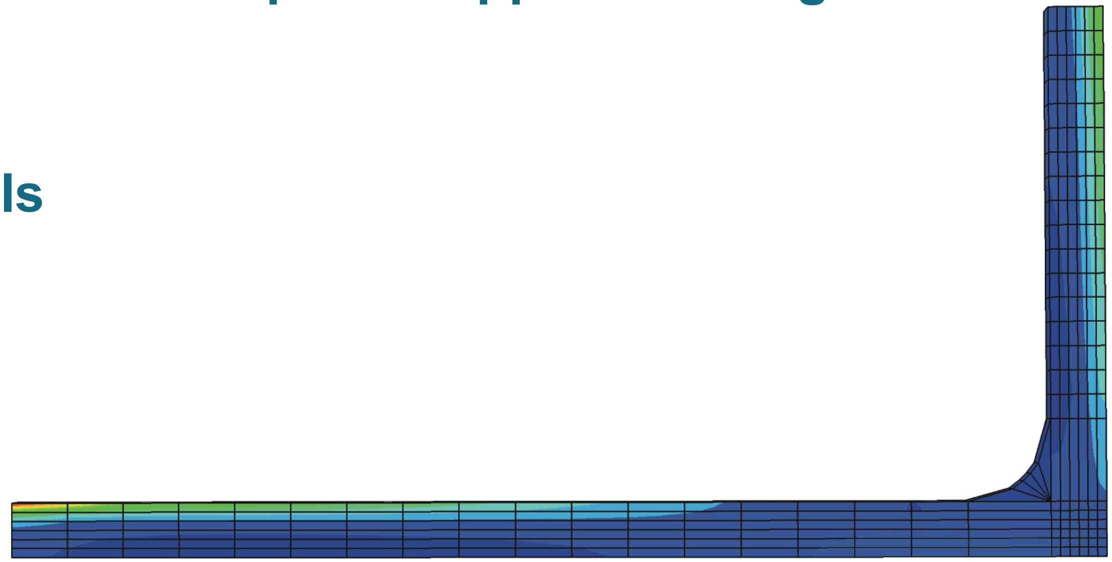


Contents of Today's Lecture

- **Motivation, overview and organization of the course**
- **Introduction to the use of finite element**

  - Physical problem, mathematical modeling and finite element solutions

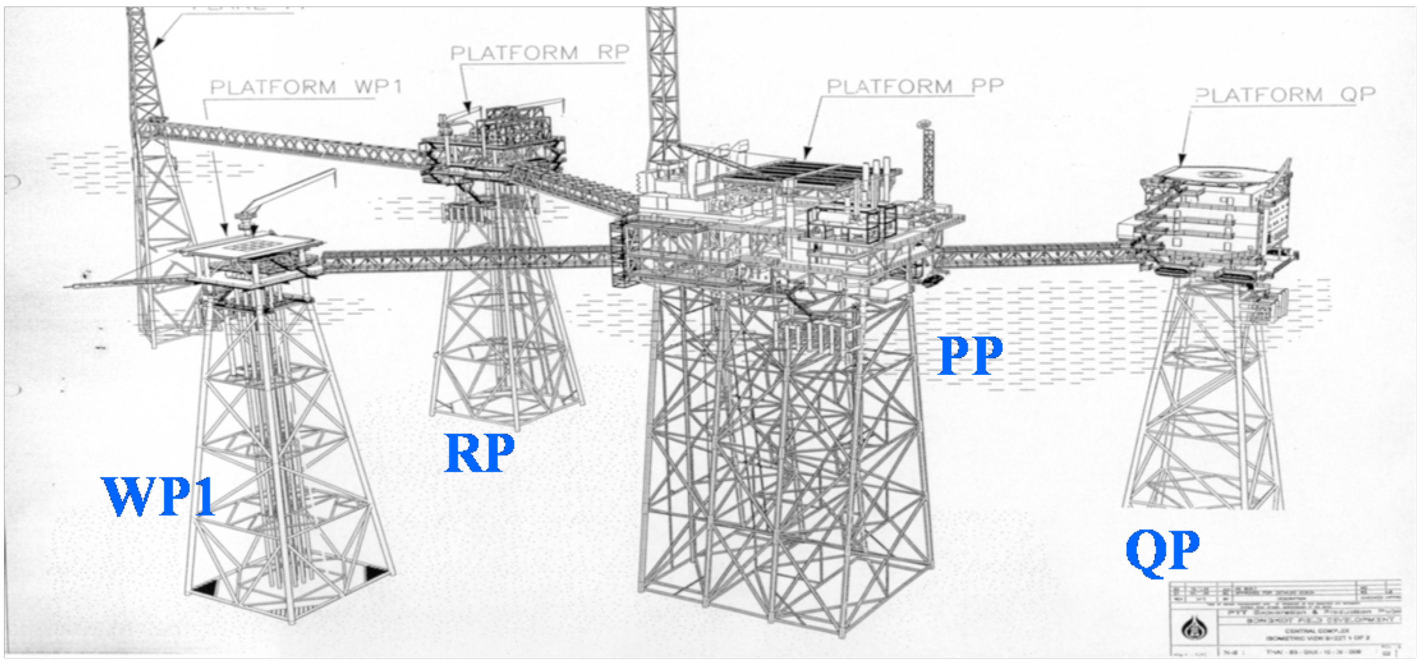
  - Finite elements as a tool for computer supported design and assessment
- **Basic mathematical tools**

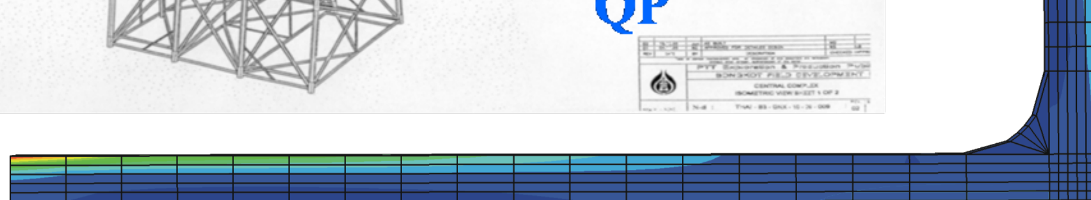


Motivation, overview and organization of the course

- **Motivation**

In this course we are focusing on the assessment of the response of engineering structures





Motivation, overview and organization of the course

- **Motivation**

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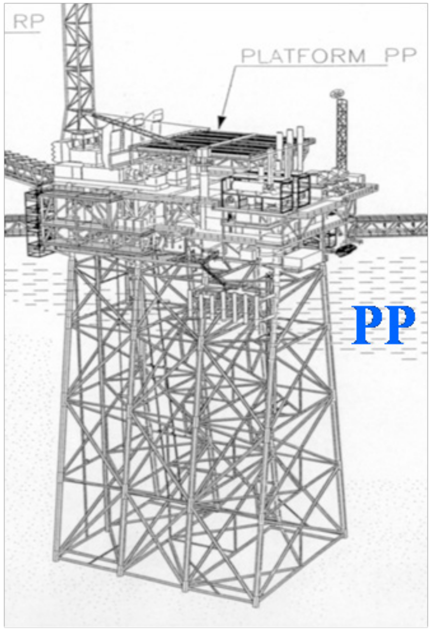
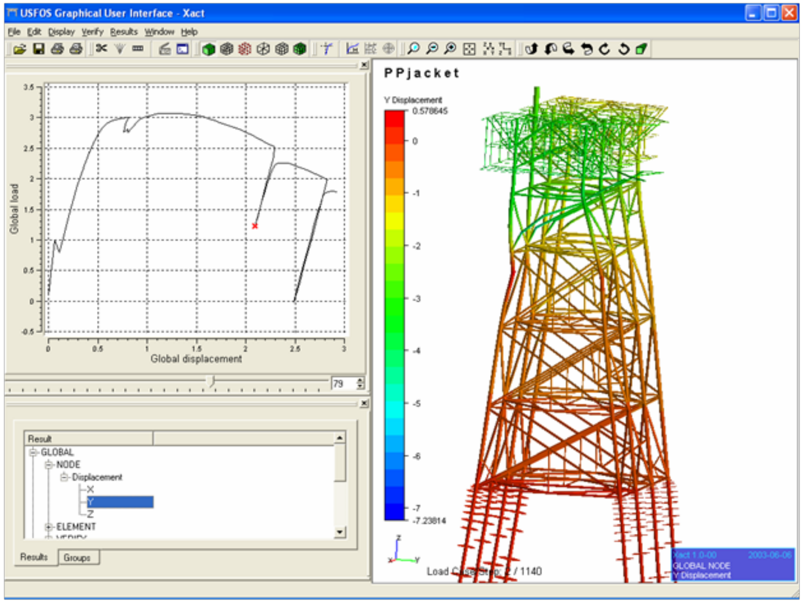
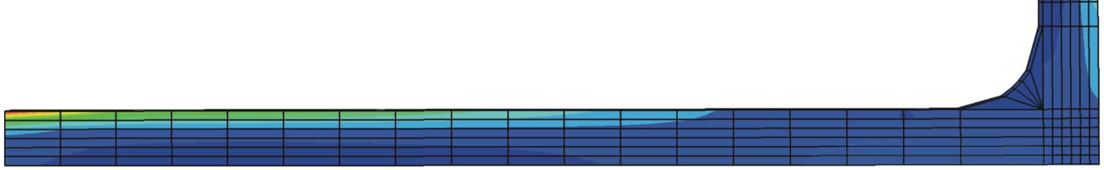






Motivation, overview and organization of the course

- **Motivation**

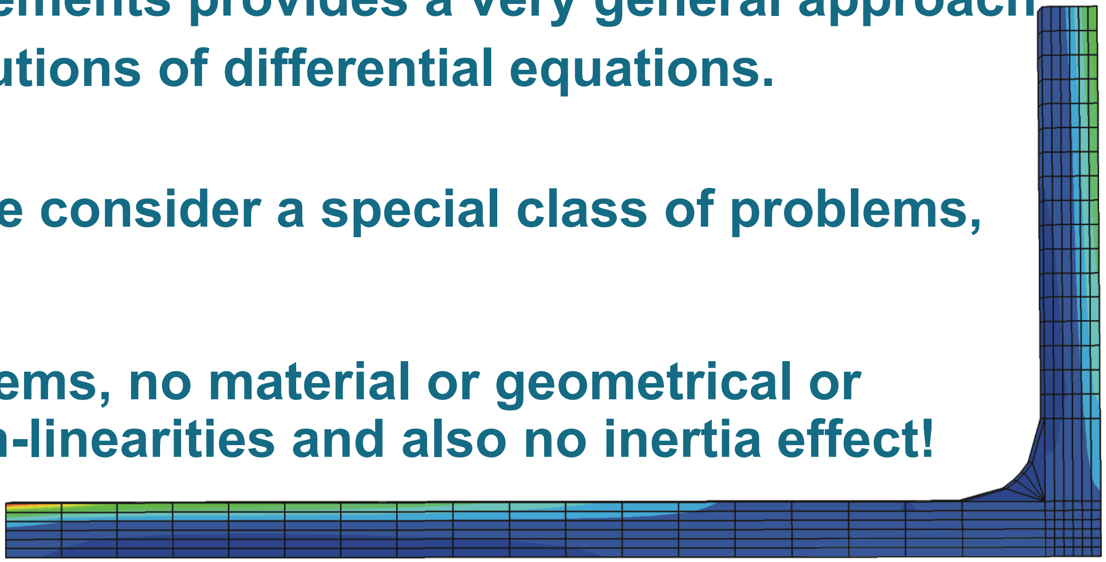
What we would like to establish is the response of a structure subject to "loading".

The Method of Finite Elements provides a framework for the analysis of such responses – however for very general problems.

The Method of Finite Elements provides a very general approach to the approximate solutions of differential equations.

In the present course we consider a special class of problems, namely:

Linear quasi-static systems, no material or geometrical or boundary condition non-linearities and also no inertia effect!



Motivation, overview and organization of the course

- **Organisation**

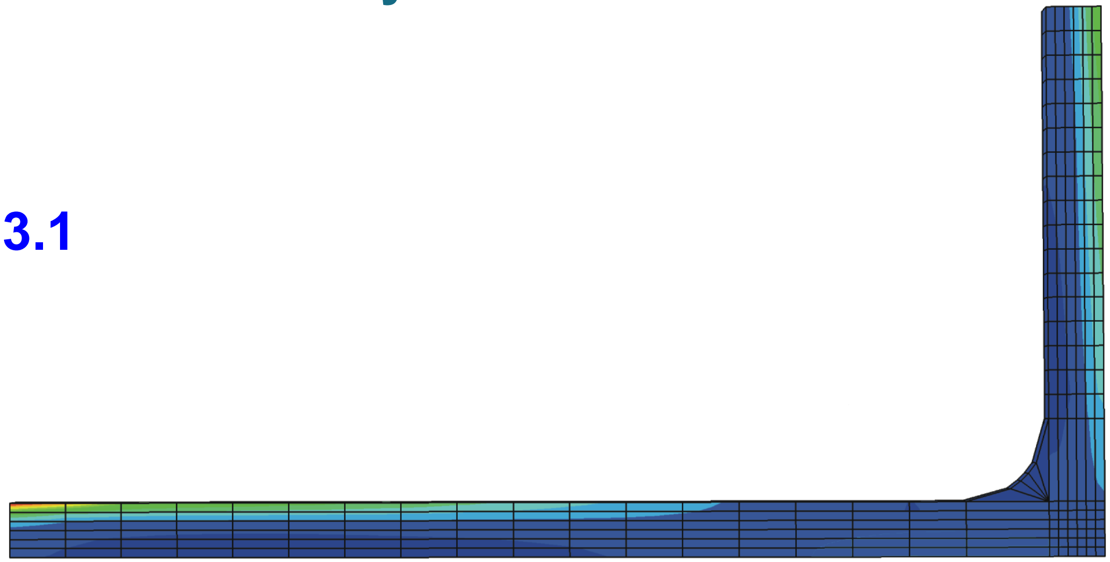
  The lectures will be given by:

  M. H. Faber

  Exercises will be organized/attended by:

  J. Qin

  By appointment, HIL E13.1



Motivation, overview and organization of the course

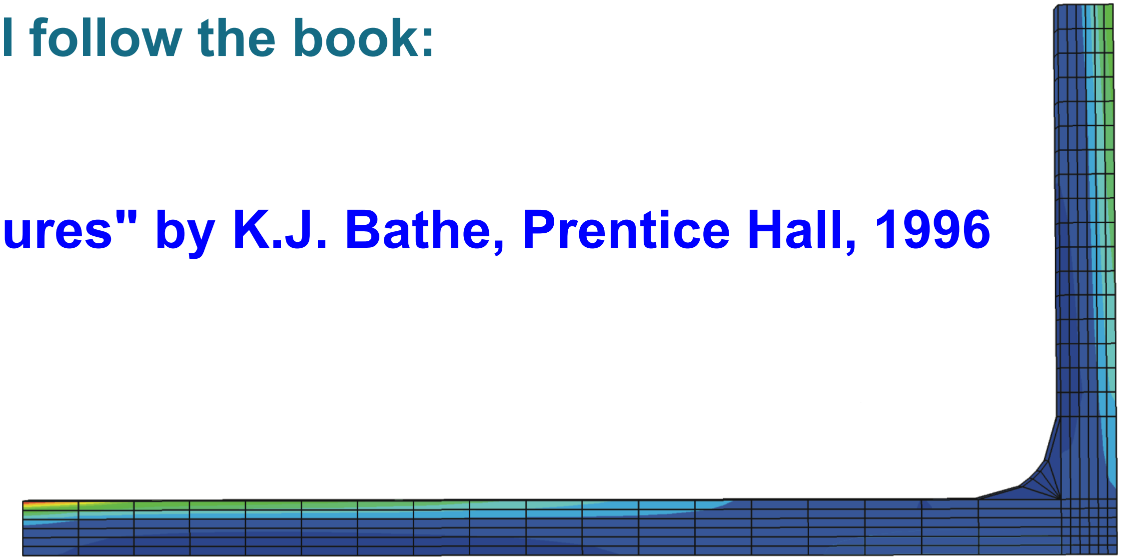
- **Organisation**

PowerPoint files with the presentations will be uploaded on our homepage one day in advance of the lectures

http://www.ibk.ethz.ch/fa/education/ss_FE

The lecture as such will follow the book:

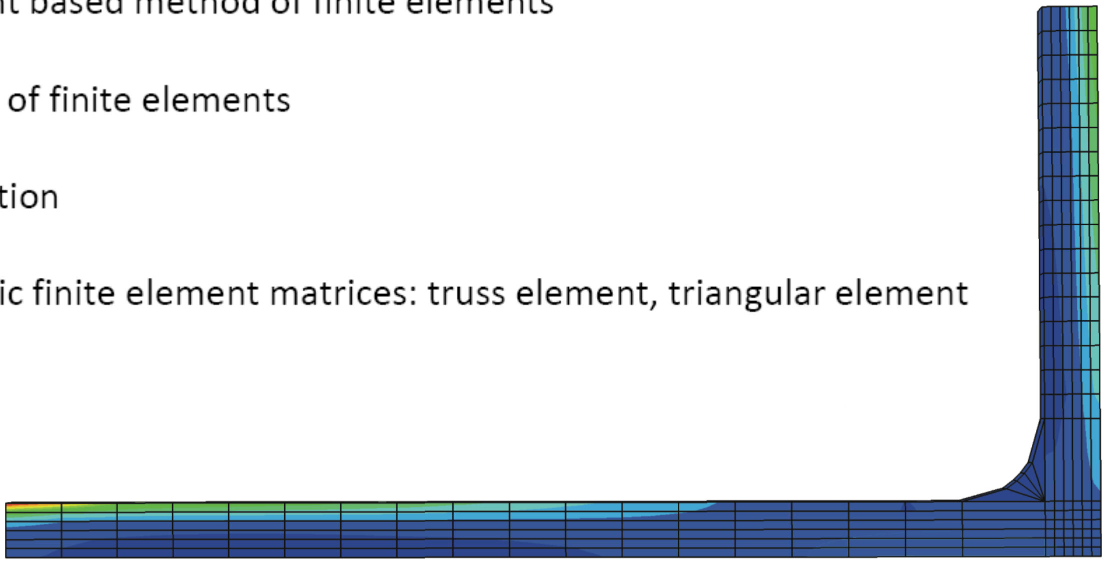
"Finite Element Procedures" by K.J. Bathe, Prentice Hall, 1996



Motivation, overview and organization of the course

- **Overview**

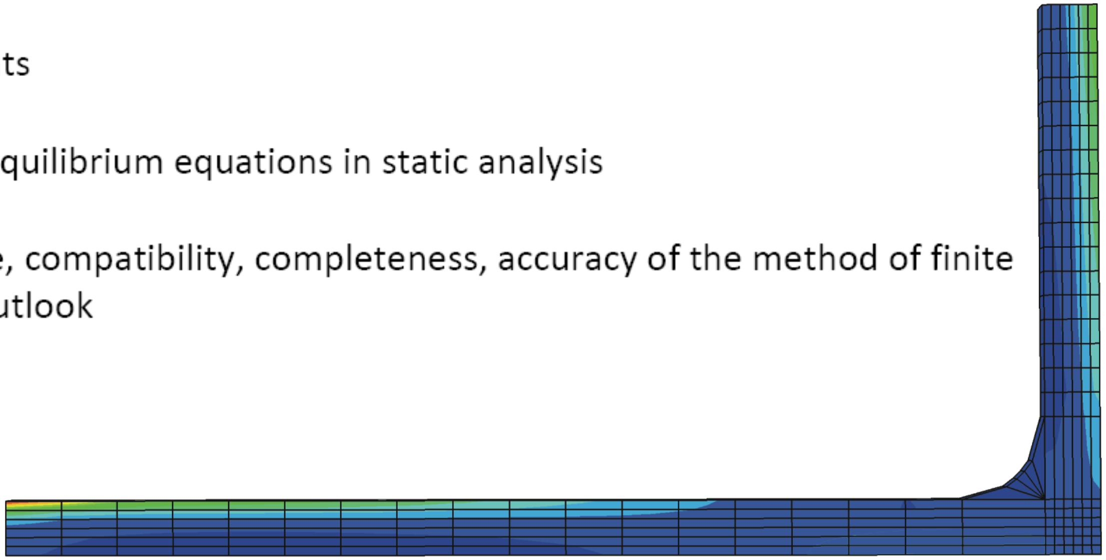
Date	Pages	Subject
20.02.2009	1-51	Introduction to the use of finite elements, basic mathematical tools
27.02.2009	105-147	Basic concepts of engineering analysis
06.03.2009	149-200	Displacement based method of finite elements
13.03.2009		Formulation of finite elements
20.03.2009	455-484	Implementation
27.03.2009	338-340, 363-376	Isoparametric finite element matrices: truss element, triangular element



Motivation, overview and organization of the course

- **Overview**

03.04.2009	342-363, 386-388	Quadrilateral elements; Element matrices in global coordinates
24.04.2009	397-420	Beam elements and axisymmetric shell elements
08.05.2009	420-436	Plate elements
15.05.2009	437-450	Shell elements
22.05.2009	695-741	Solution of equilibrium equations in static analysis
29.05.2009	225-259	Convergence, compatibility, completeness, accuracy of the method of finite elements; Outlook



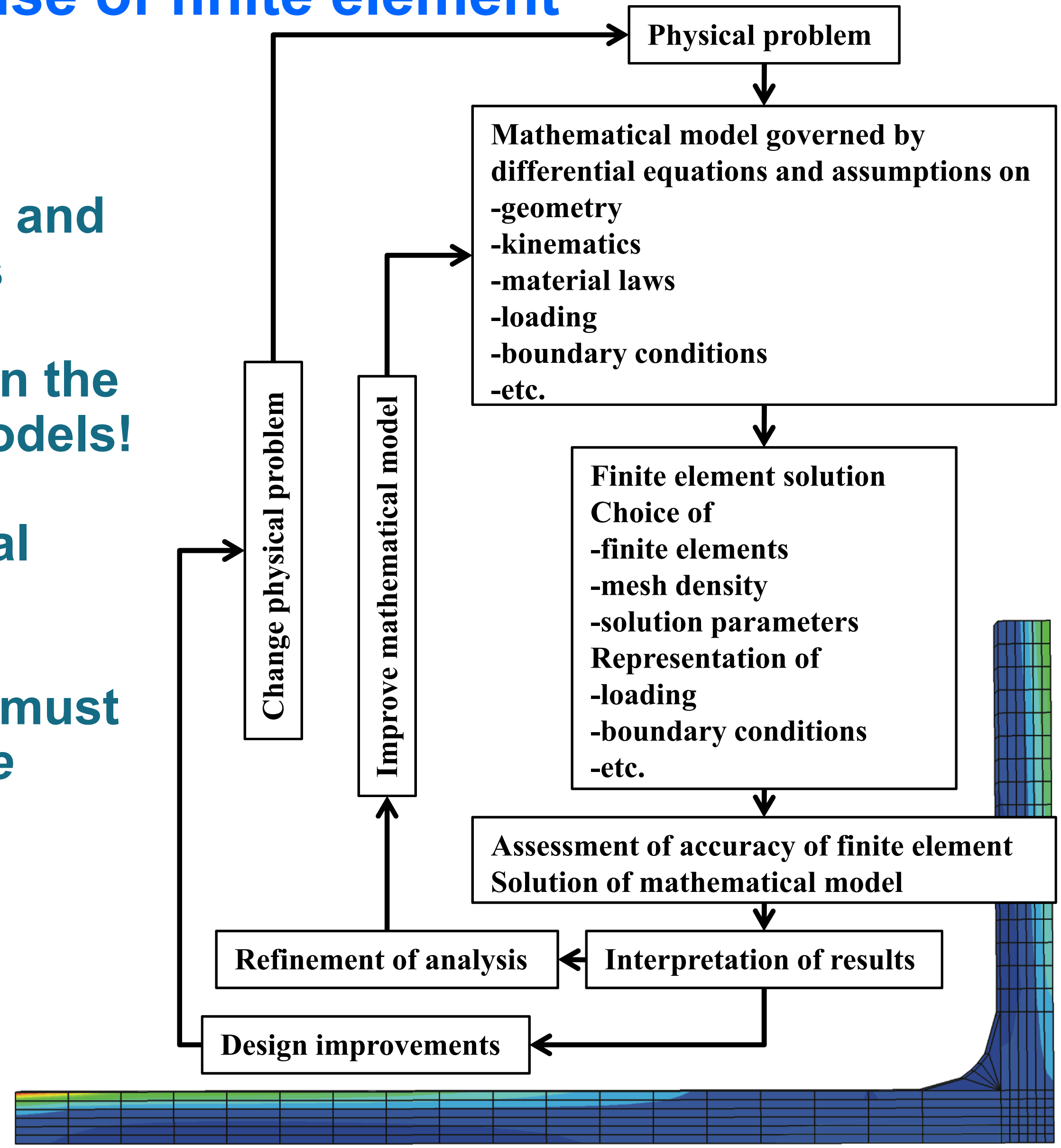
Introduction to the use of finite element

- Physical problem, mathematical modeling and finite element solutions

  - we are only working on the basis of mathematic models!

  - choice of mathematical model is crucial!

  - mathematical models must be *reliable* and *effective*



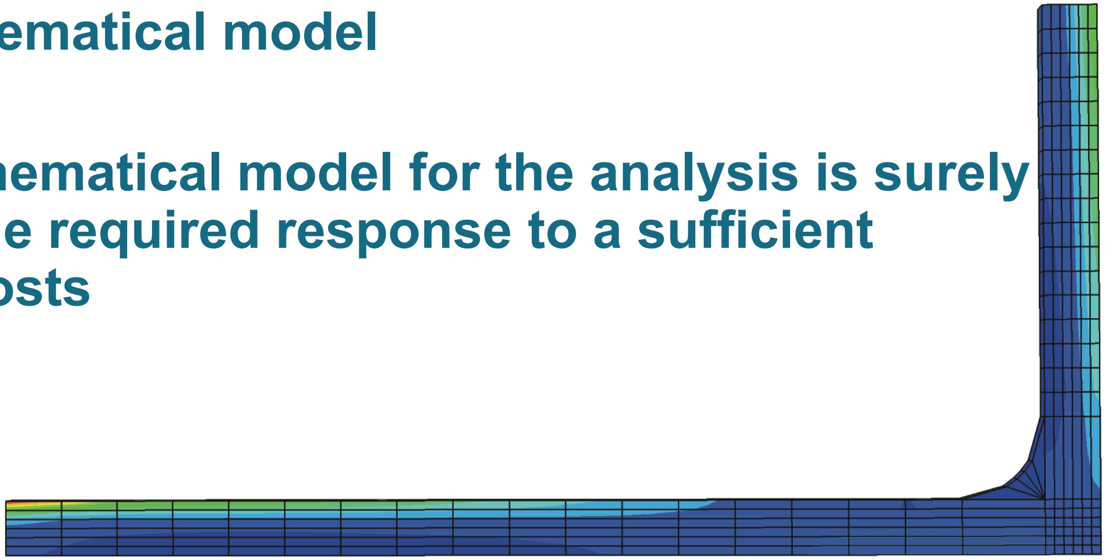
Introduction to the use of finite element

- ***Reliability*** **of a mathematical model**

  The chosen mathematical model is reliable if the required response is known to be predicted within a selected level of accuracy measured on the response of a very comprehensive mathematical model

- ***Effectiveness*** **of a mathematical model**

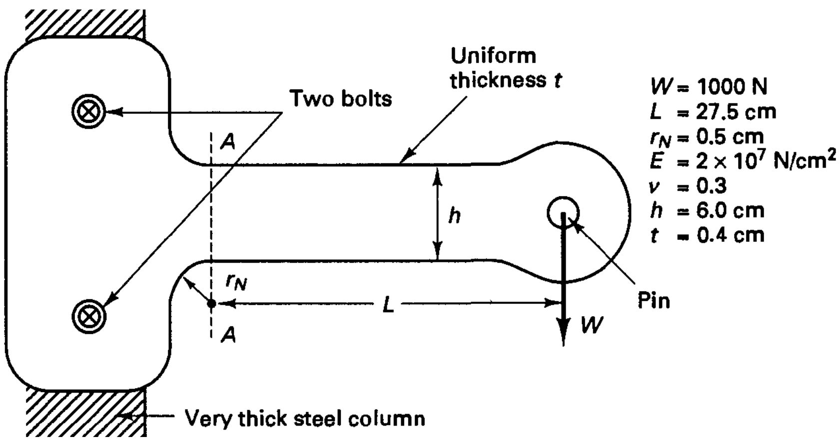
  The most effective mathematical model for the analysis is surely that one which yields the required response to a sufficient accuracy and at least costs

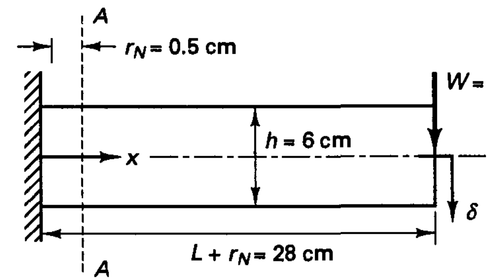


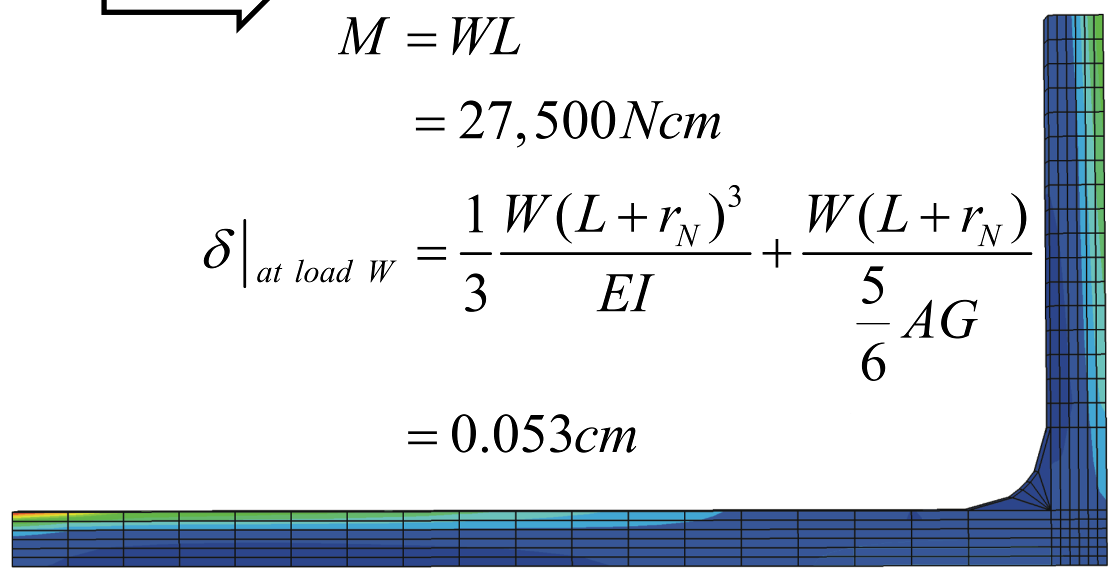
Introduction to the use of finite element

- **Example**

Complex physical problem modeled by a simple mathematical model







$$M = WL$$

$$= 27{,}500 \, Ncm$$

$$\delta\Big|_{at\ load\ W} = \frac{1}{3}\frac{W(L+r_N)^3}{EI} + \frac{W(L+r_N)}{\frac{5}{6}AG}$$

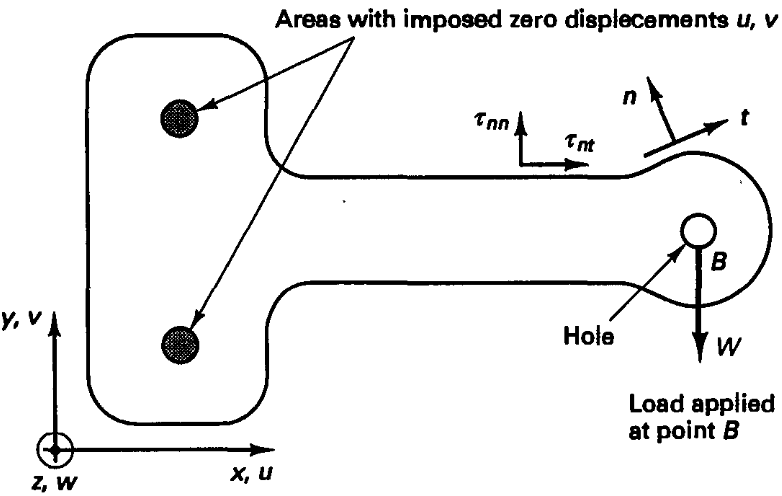
$$= 0.053 cm$$

Introduction to the use of finite element

- **Example**

Detailed reference model – 2D plane stress model – for FEM analysis

$$\left.\begin{aligned}\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0\\ \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}=0\end{aligned}\right\}\text{ in domain of bracket}$$

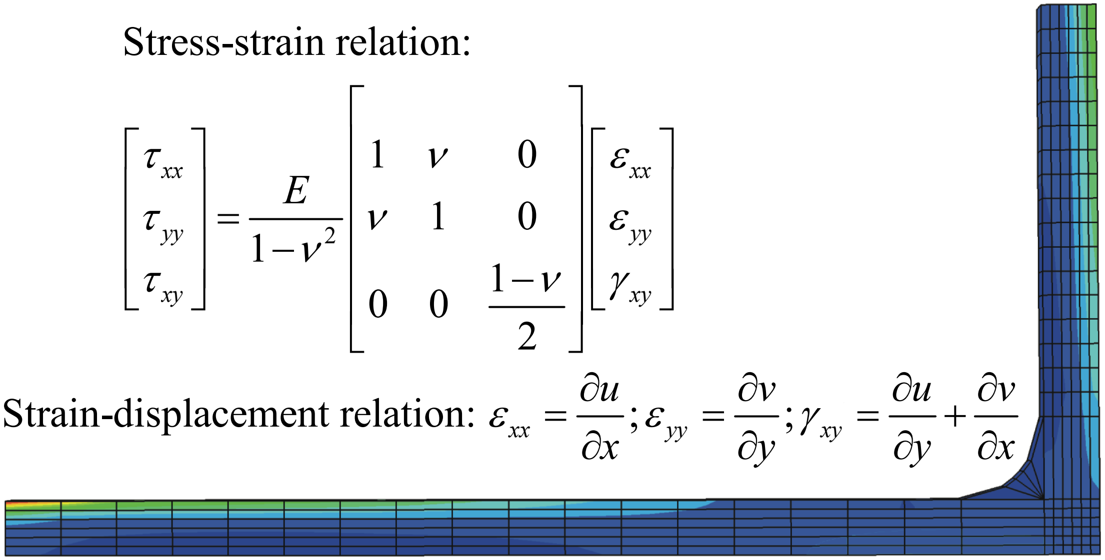


$\tau_{nn}=0,\ \tau_{nt}=0$ on surfaces except at point B and at imposed zero displacements

Stress-strain relation:

$$\begin{bmatrix}\tau_{xx}\\ \tau_{yy}\\ \tau_{xy}\end{bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix}$$

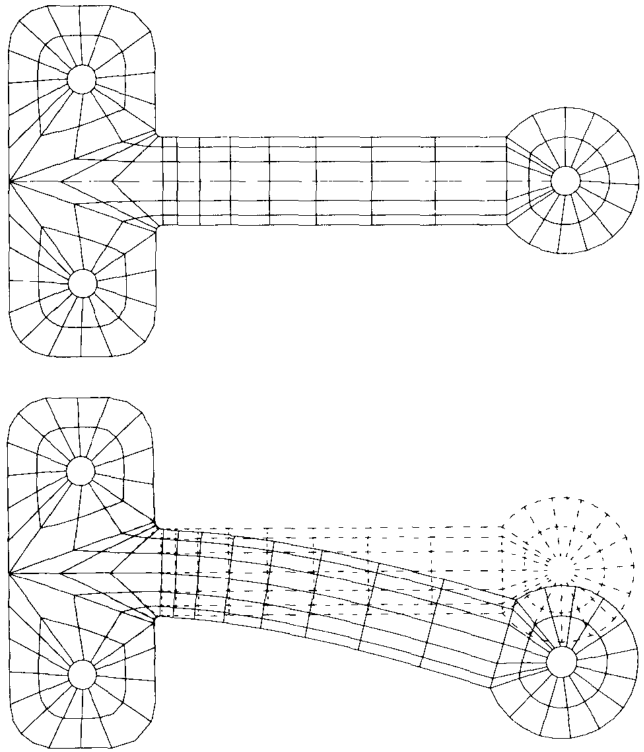
Strain-displacement relation: $\varepsilon_{xx}=\frac{\partial u}{\partial x};\varepsilon_{yy}=\frac{\partial v}{\partial y};\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$



Introduction to the use of finite element

- **Example**

Comparison between simple and more refined model results



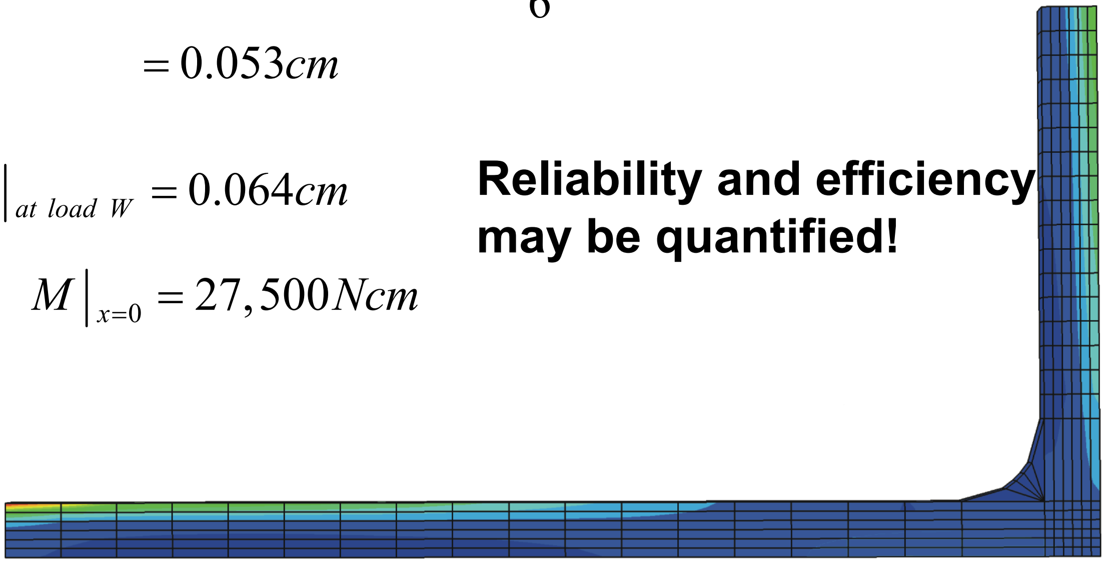
$$M = WL$$
$$= 27,500 Ncm$$

$$\delta\Big|_{at\ load\ W} = \frac{1}{3}\frac{W(L+r_N)^3}{EI} + \frac{W(L+r_N)}{\frac{5}{6}AG}$$
$$= 0.053cm$$

$$\delta\Big|_{at\ load\ W} = 0.064cm$$

$$M\Big|_{x=0} = 27,500 Ncm$$

Reliability and efficiency may be quantified!



Introduction to the use of finite element

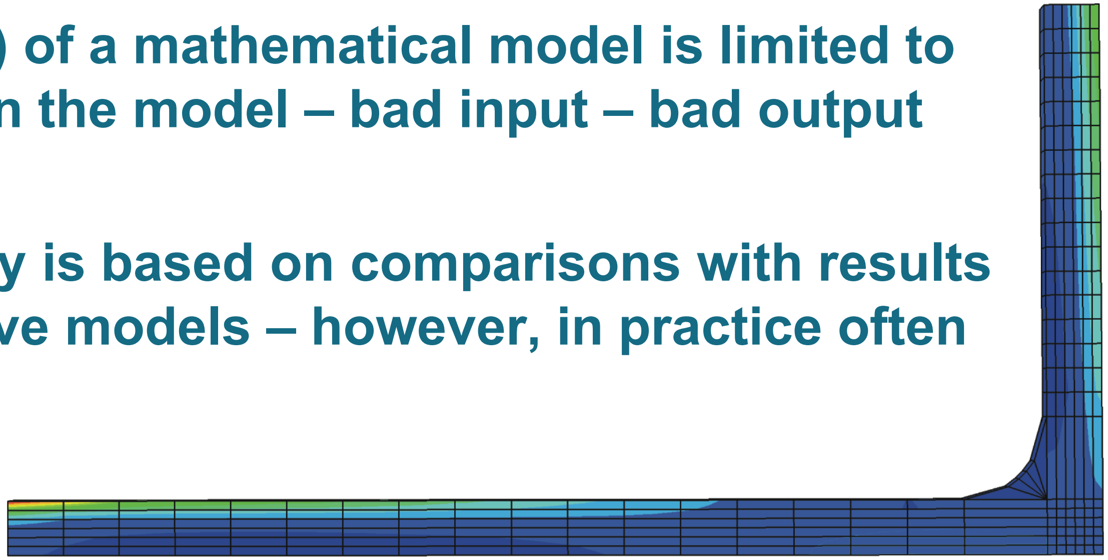
- **Observations**

  Choice of mathematical model must correspond to desired response measures

  The most effective mathematical model delivers reliable answers with the least amount of efforts

  Any solution (also FEM) of a mathematical model is limited to information contained in the model – bad input – bad output

  Assessment of accuracy is based on comparisons with results from very comprehensive models – however, in practice often based on experience



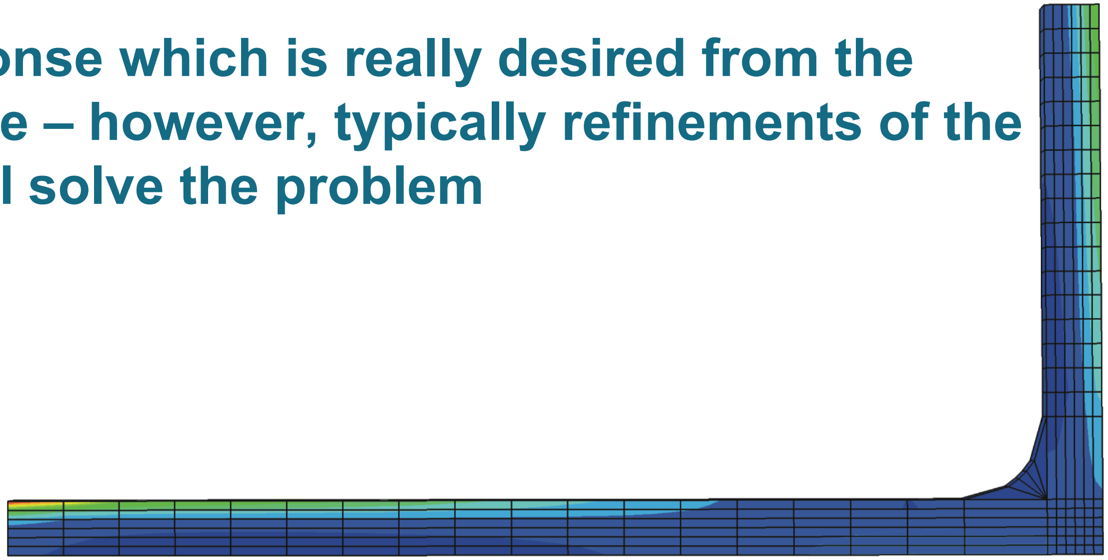
Introduction to the use of finite element

- **Observations**

Sometimes the chosen mathematical model results in problems such as singularities in stress distributions

The reason for this is that simplifications have been made in the mathematical modeling of the physical problem

Depending on the response which is really desired from the analysis this may be fine – however, typically refinements of the mathematical model will solve the problem

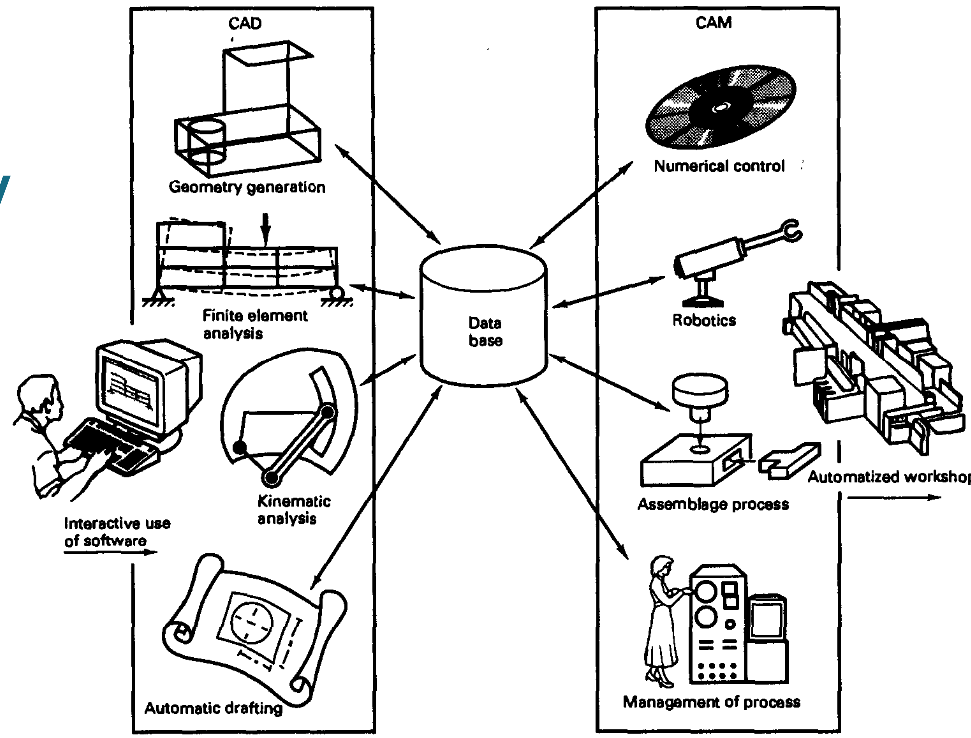


Introduction to the use of finite element

- **Finite elements as a tool for computer supported design and assessment**

FEM forms a basic tool framework in research and applications covering many different areas

- Fluid dynamics
- Structural engineering
- Aeronautics
- Electrical engineering
- etc.



Introduction to the use of finite element

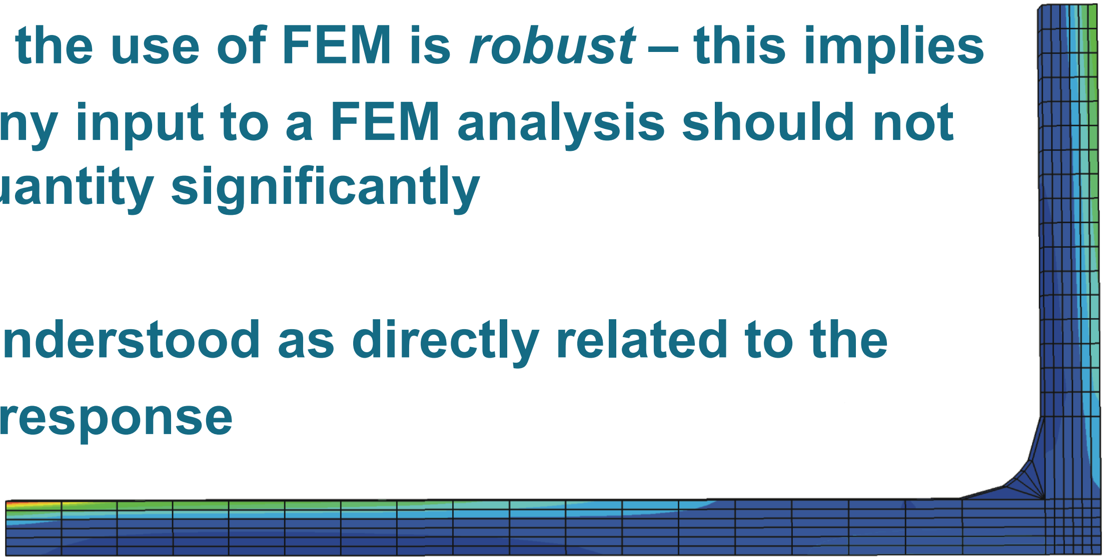
- **Finite elements as a tool for computer supported design and assessment**

  The practical application necessitates that solutions obtained by FEM are reliable and efficient

  however

  also it is necessary that the use of FEM is *robust* – this implies that minor changes in any input to a FEM analysis should not change the response quantity significantly

  Robustness has to be understood as directly related to the desired type of result – response



Basic mathematical tools

- ## Vectors and matrices

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1i} & & a_{1n} \\ \vdots & \ddots & & & \\ a_{i1} & & a_{ii} & & \vdots \\ \vdots & & & \ddots & \\ a_{m1} & & \cdots & & a_{mn} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

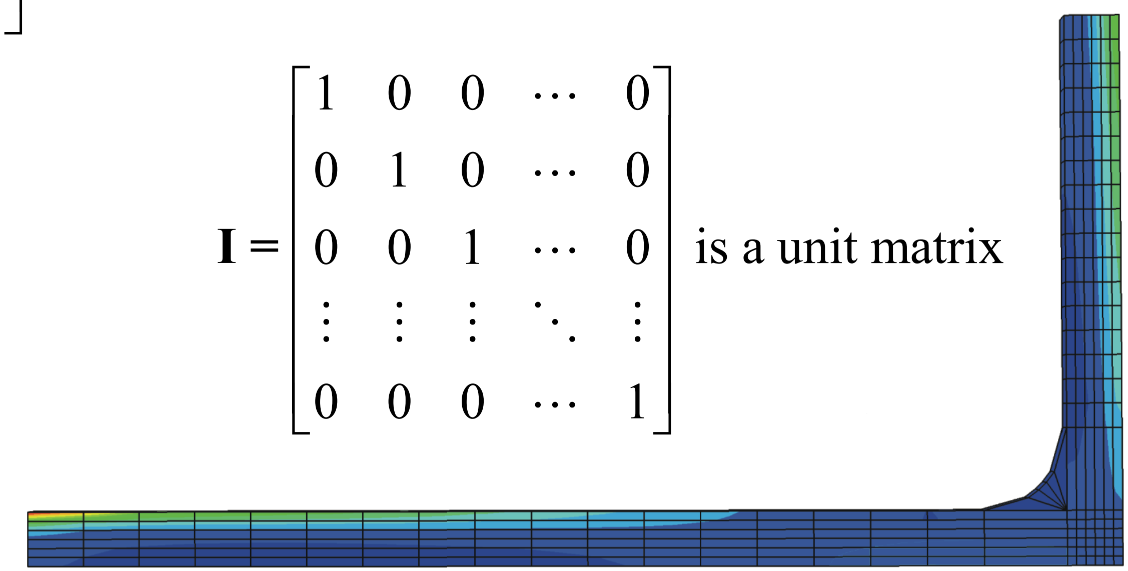
$\mathbf{A}^T$ is the transpose of $\mathbf{A}$

if $\mathbf{A} = \mathbf{A}^T$ there is

$m = n$ (square matrix)

and $a_{ij} = a_{ji}$ (symmetrical matrix)

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$ is a unit matrix



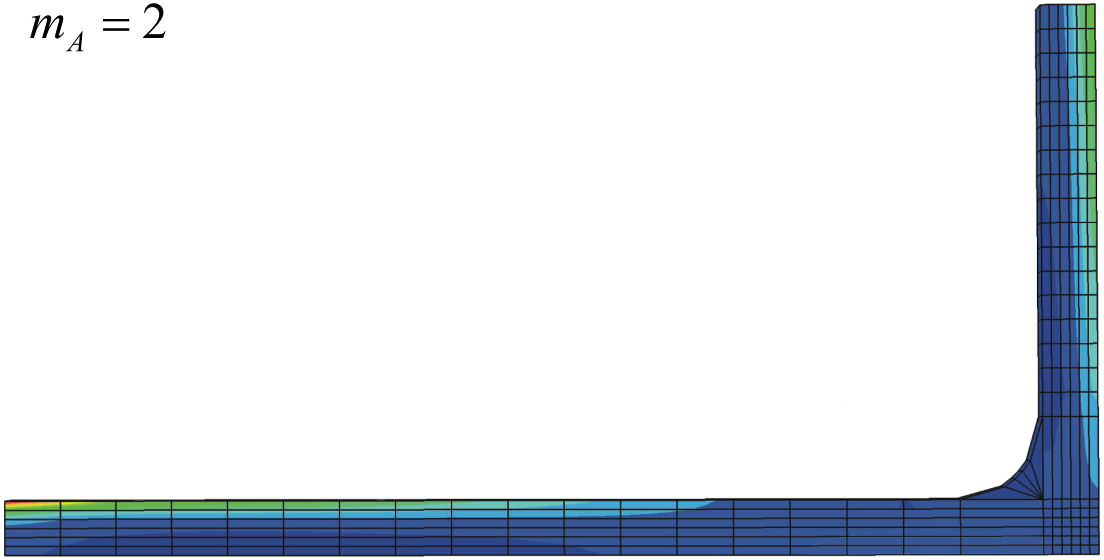
Basic mathematical tools

- **Banded matrices**

symmetric banded matrices

$$a_{ij} = 0 \text{ for } j > i + m_A, \; 2m_A + 1 \text{ is the bandwidth}$$

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 2 & 3 & 4 & 1 & 0 \\ 1 & 4 & 5 & 6 & 1 \\ 0 & 1 & 6 & 7 & 4 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix} \qquad m_A = 2$$

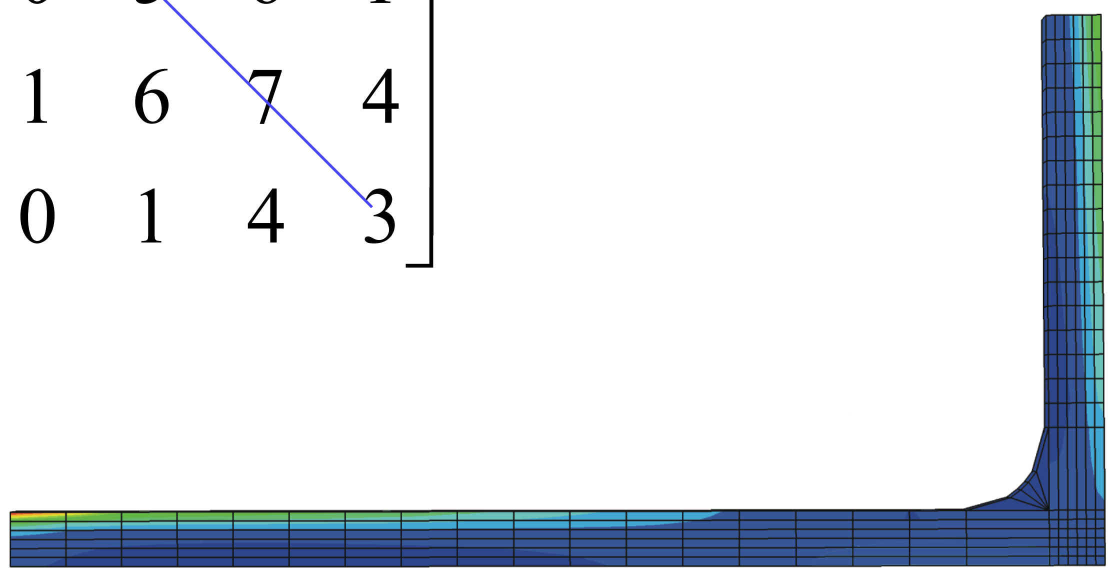


Basic mathematical tools

- **Banded matrices and skylines**

$m_A + 1$

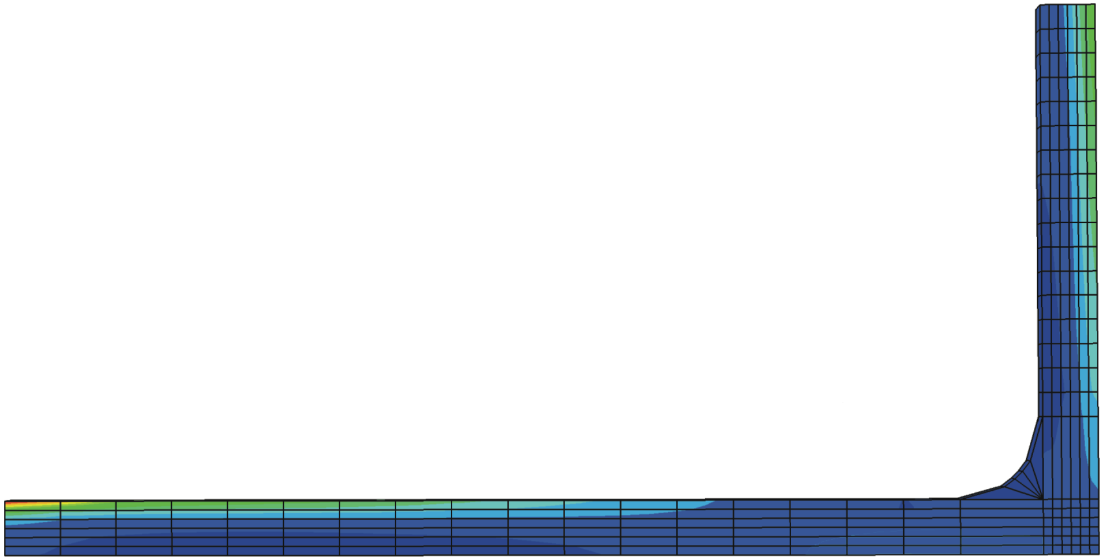
$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 6 & 1 \\ 0 & 1 & 6 & 7 & 4 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix}$$



Basic mathematical tools

- **Matrix equality**

$$\mathbf{A}(m \times p) = \mathbf{B}(n \times q) \text{ if and only if}$$
$$m = n,$$
$$p = q,$$
$$\text{and } a_{ij} = b_{ij}$$



Basic mathematical tools

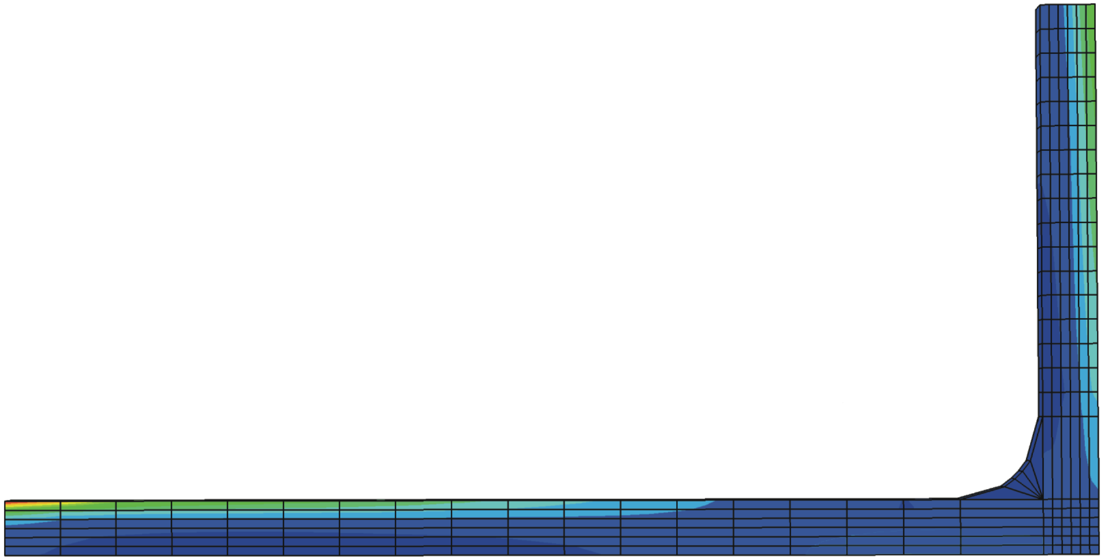
- **Matrix addition**

$\mathbf{A}(m \times p), \mathbf{B}(n \times q)$ can be added if and only if

$m = n, p = q$, and

if $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then

$$c_{ij} = a_{ij} + b_{ij}$$



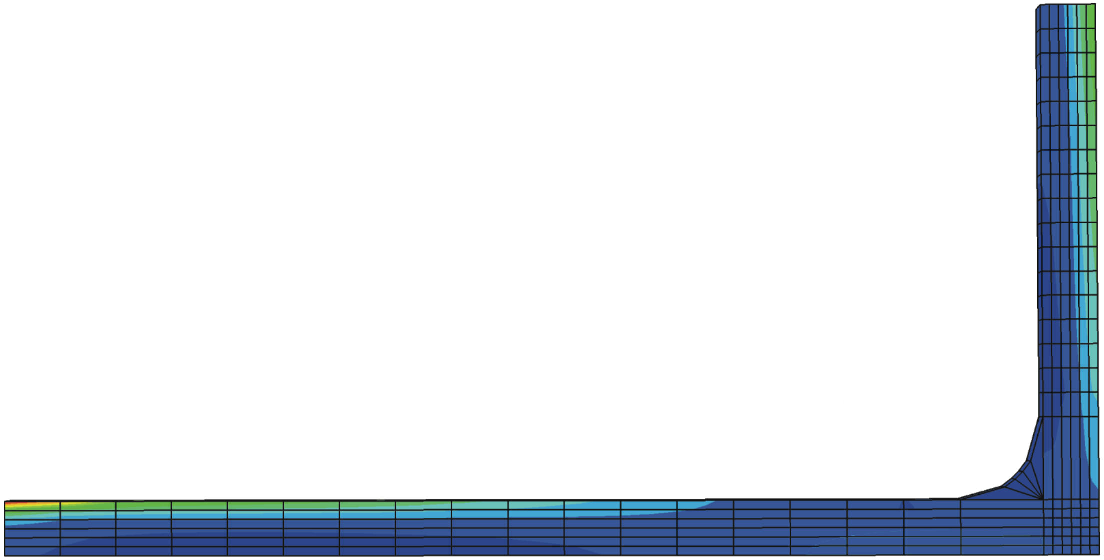
Basic mathematical tools

- **Matrix multiplication with a scalar**

A matrix $\mathbf{A}$ multiplied by a scalar c by multiplying all elements of $\mathbf{A}$ with c

$$\mathbf{B} = c\mathbf{A}$$

$$b_{ij} = c a_{ij}$$



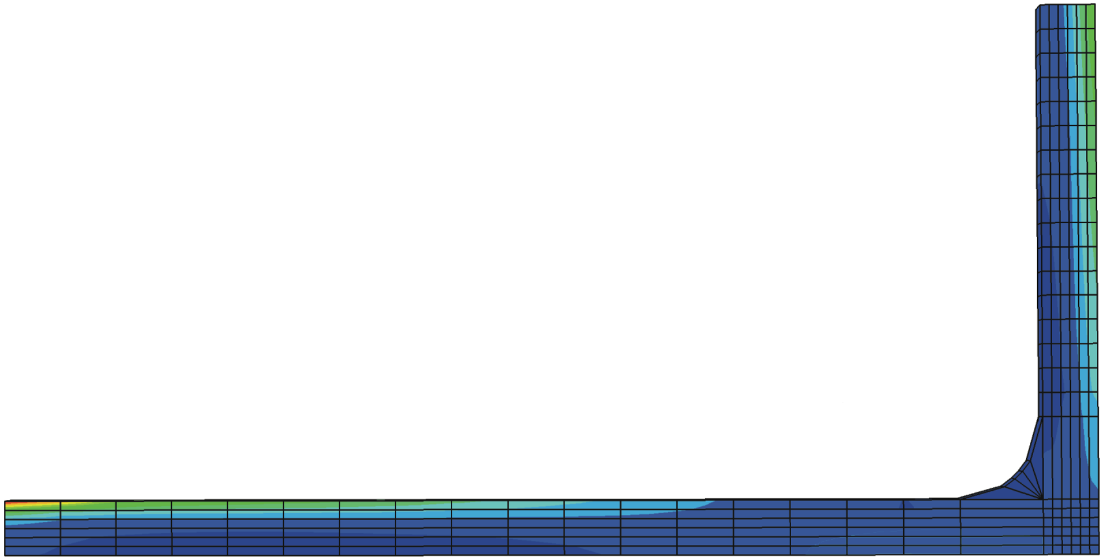
Basic mathematical tools

• Multiplication of matrices

Two matrices $\mathbf{A}(p \times m)$ and $\mathbf{B}(n \times q)$ can be multiplied only if $m = n$

$$\mathbf{C} = \mathbf{BA}$$

$$c_{ij} = \sum_{r=1}^{m} a_{ir} b_{rj}, \ \mathbf{C}(p \times q)$$



Basic mathematical tools

- **Multiplication of matrices**

The commutative law does not hold, i.e.

$\mathbf{AB} \neq \mathbf{BA}$, unless $\mathbf{A}$ and $\mathbf{B}$ commute

The distributive law hold, i.e.

$$\mathbf{E} = (\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

The associative law hold, i.e.

$$\mathbf{G} = (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \mathbf{ABC}$$

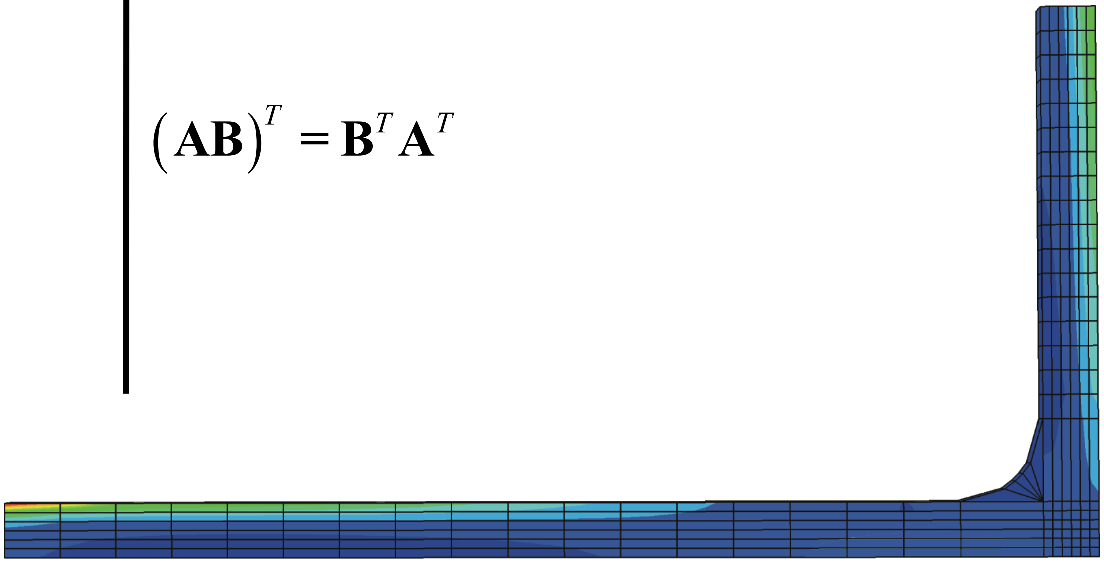
$\mathbf{AB} = \mathbf{CB}$ does not imply that $\mathbf{A} = \mathbf{C}$

however does hold for special cases

(e.g. for $\mathbf{B} = \mathbf{I}$)

Special rule for the transpose of matrix products

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$



Basic mathematical tools

- ## The inverse of a matrix

The inverse of a matrix $\mathbf{A}$ is denoted $\mathbf{A}^{-1}$

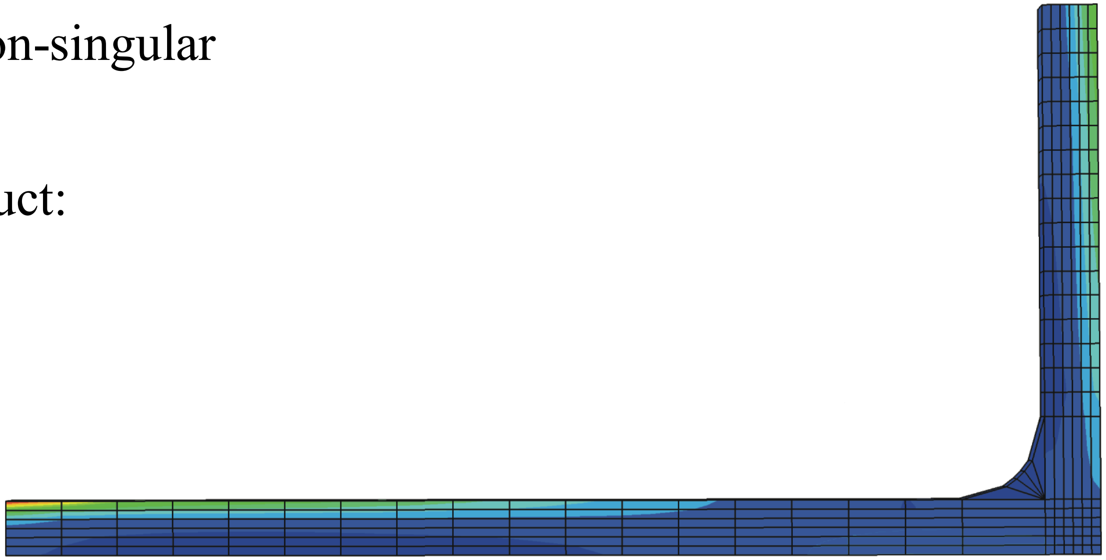
if the inverse matrix exist then there is:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The matrix $\mathbf{A}$ is said to be non-singular

The inverse of a matrix product:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$



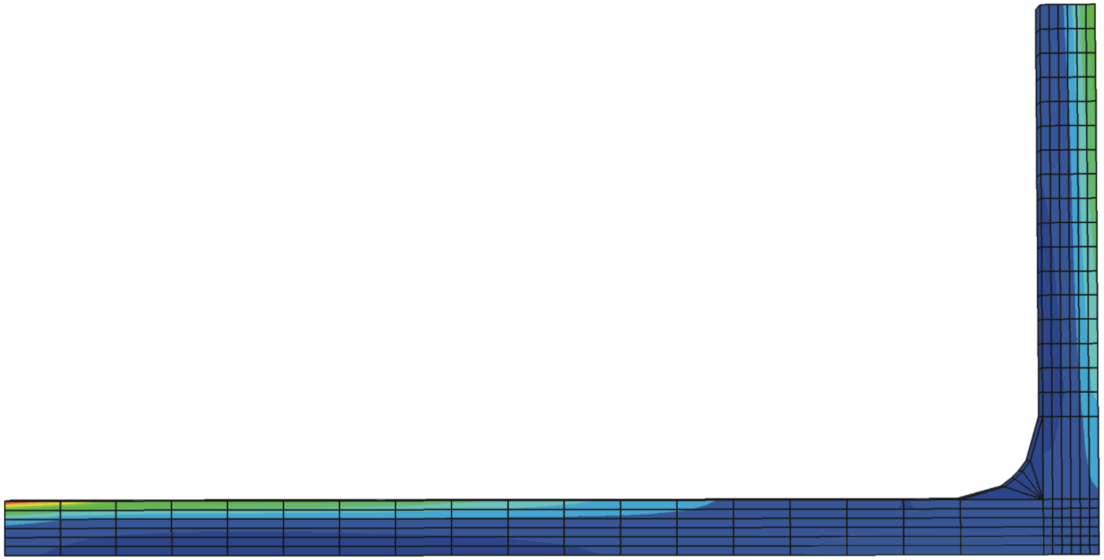
Basic mathematical tools

- **Sub matrices**

A matrix **A** may be sub divided as:

$$\mathbf{A} = \left[\begin{array}{c:cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hdashline a_{31} & a_{32} & a_{33} \end{array}\right]$$

$$\mathbf{A} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{bmatrix}$$

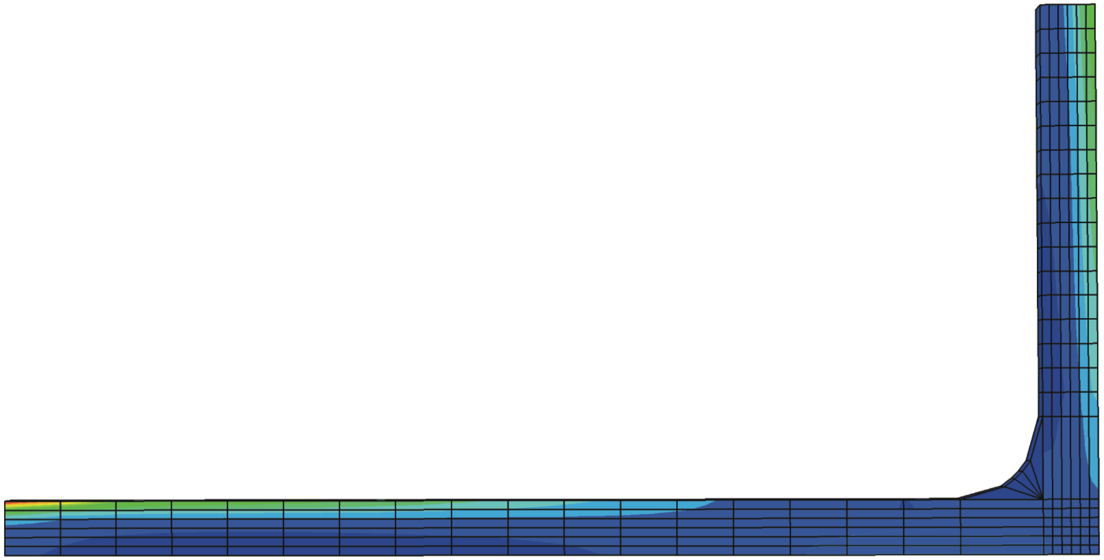


Basic mathematical tools

- **Trace of a matrix**

The trace of a matrix $\mathbf{A}(n \times n)$ is defined through:

$$tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$



Basic mathematical tools

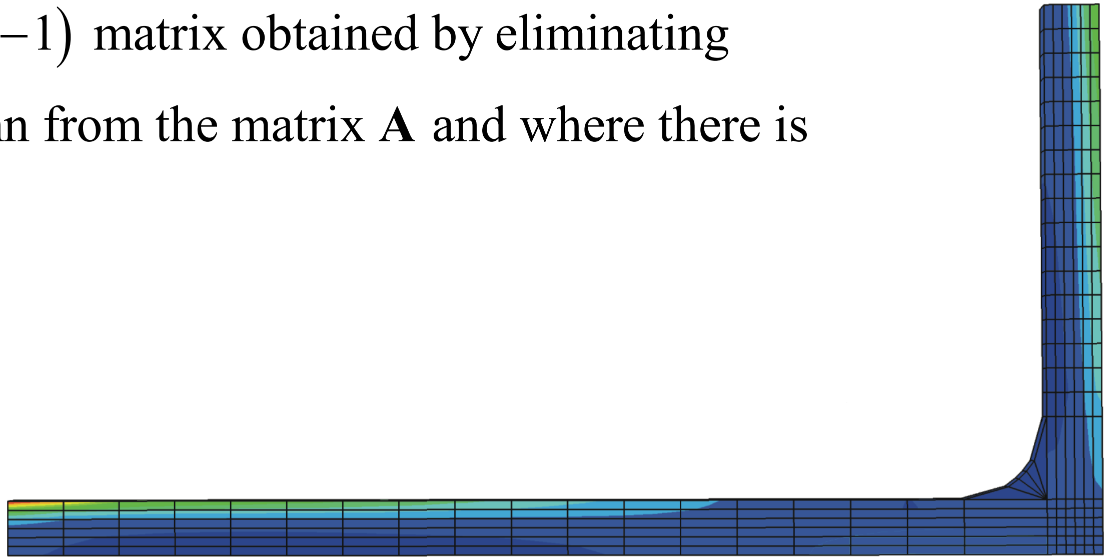
• The determinant of a matrix

The determinant of a matrix is defined through the recurrence formula

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(\mathbf{A}_{1j})$$

where $\mathbf{A}_{1j}$ is the $(n-1)\times(n-1)$ matrix obtained by eliminating the 1^{st} row and the j^{th} column from the matrix $\mathbf{A}$ and where there is

$$\text{if } \mathbf{A} = \left[a_{11}\right], \det \mathbf{A} = a_{11}$$



Basic mathematical tools

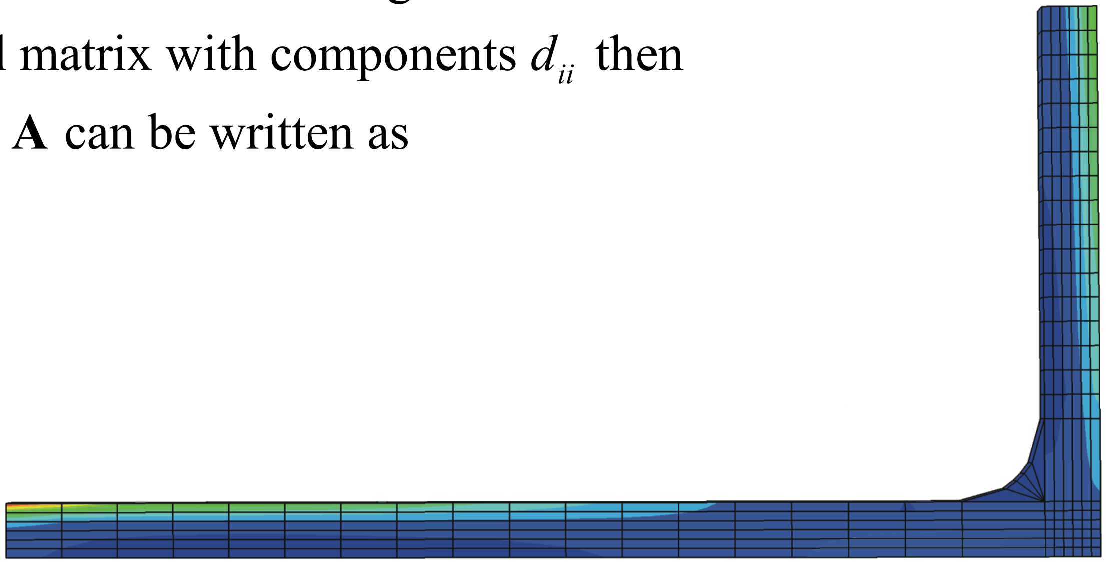
• The determinant of a matrix

It is convenient to decompose a matrix $\mathbf{A}$ by the so-called Cholesky decomposition

$$\mathbf{A} = \mathbf{LDL}^T \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

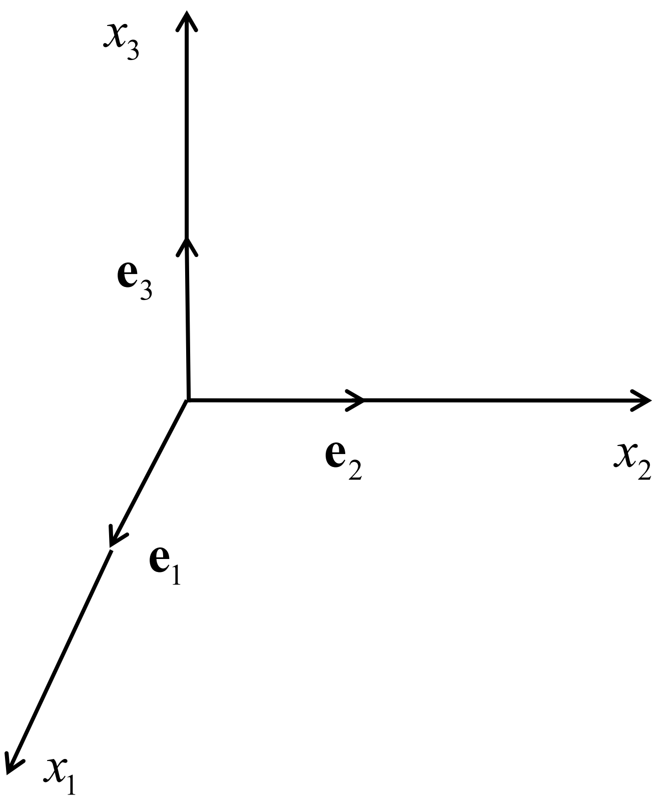
where $\mathbf{L}$ is a lower triangular matrix with all diagonal elements equal to 1 and $\mathbf{D}$ is a diagonal matrix with components d_{ii} then the determinant of the matrix $\mathbf{A}$ can be written as

$$\det \mathbf{A} = \prod_{i=1}^{n} d_{ii}$$



Basic mathematical tools

- **Tensors**



Let the Cartesian coordinate frame be defined by the unit base vectors $\mathbf{e}_i$

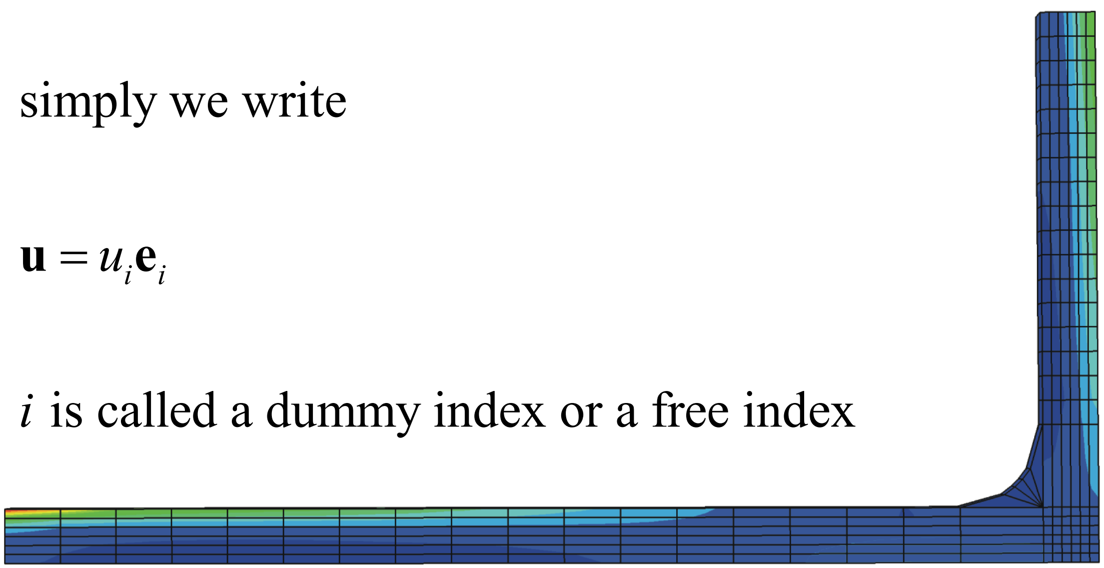
A vector $\mathbf{u}$ in this frame is given by

$$\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e}_i$$

simply we write

$$\mathbf{u} = u_i \mathbf{e}_i$$

i is called a dummy index or a free index



Basic mathematical tools

- **Tensors**

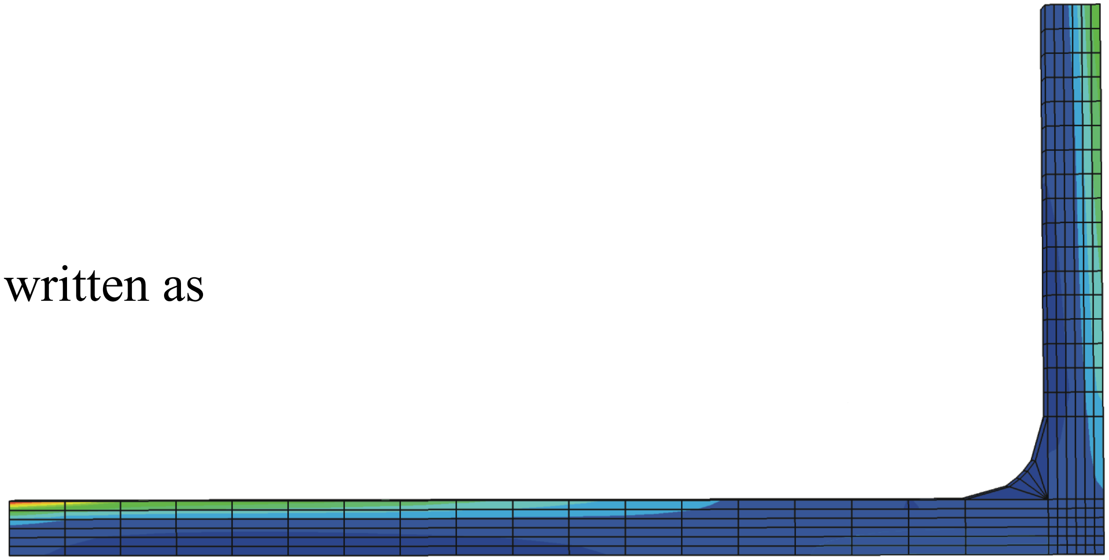
An entity is called a tensor of first order
if it has 3 components ξ_i in the unprimed frame
and 3 components ξ_i' in the primed frame,
and if these components are related by the characteristic law

$$\xi_i' = p_{ik}\xi_i$$

where $p_{ik} = \cos\left(\mathbf{e}_i', \mathbf{e}_k\right)$

In the matrix form, it can be written as

$$\boldsymbol{\xi}' = \mathbf{P}\boldsymbol{\xi}$$



Basic mathematical tools

- **Tensors**

An entity is called a second-order tensor
if it has 9 components t_{ij} in the unprimed frame
and 9 components t'_{ij} in the primed frame,
and if these components are related by the characteristic law

$$t'_{ij} = p_{ik} p_{jl} t_{kl}$$