Basic Statistics and Probability Theory in Civil, Surveying and Environmental Engineering

Prof. Dr. Michael Havbro Faber
Swiss Federal Institute of Technology
ETH Zürich, Switzerland

Interpretation of Probability

States of nature of which we have interest such as:

- a bridge failing due to excessive traffic loads
- a water reservoir being over-filled
- an electricity distribution system „falling out"
- a project being delayed

are in the following denoted „events"

we are generally interested in quantifying the probability that such events take place within a given „time frame"

Interpretation of Probability

- There are in principle three different interpretations of probability

  - **Frequentistic** $P(A) = \lim \frac{N_A}{n_{\exp}}$ **for** $n_{\exp} \to \infty$

  - **Classical** $P(A) = \frac{n_A}{n_{tot}}$

  - **Bayesian** $P(A) =$ degree of belief that A will occur

Interpretation of Probability

Consider the probability of getting a „head" when flipping a coin

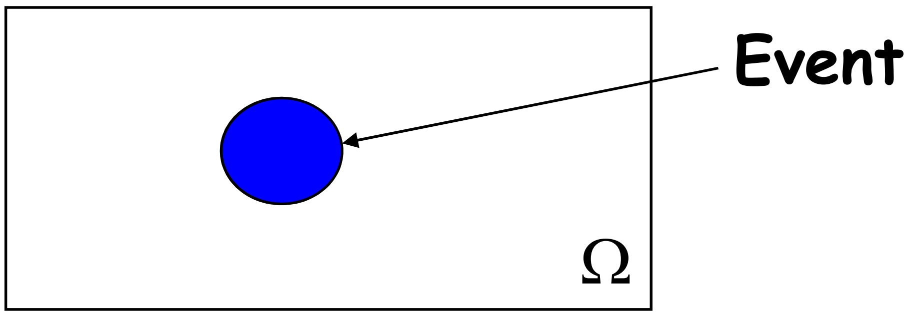
- **Frequentistic** $$P(A) = \frac{510}{1000} = 0.51$$
- **Classical** $$P(A) = \frac{1}{2}$$
- **Bayesian** $$P(A) = 0.5$$

Sample Space and Events

The set of all possible outcomes of the state of nature e.g. concrete compressive strength test results is called the sample space Ω. For concrete compressive strength test results the sample space can be written as $\Omega =]0;\infty[$

A sample space can be continuous or discrete.

Typically we illustrate the sample space and events using Venn diagrams

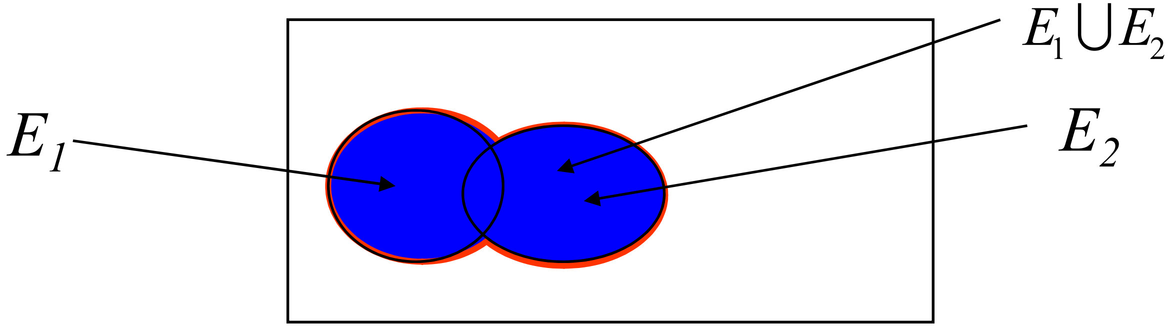


Sample Space and Events

An event is a sub-set of the sample space

- if the sub-set is empty the event is impossible
- if the sub-set contains all of the sample space the event is certain

Consider the two events E_1 and E_2:
The sub-set of sample points belonging to the event E_1 and/or the event E_2 is called the union of E_1 and E_2 and is written as : $E_1 \cup E_2$

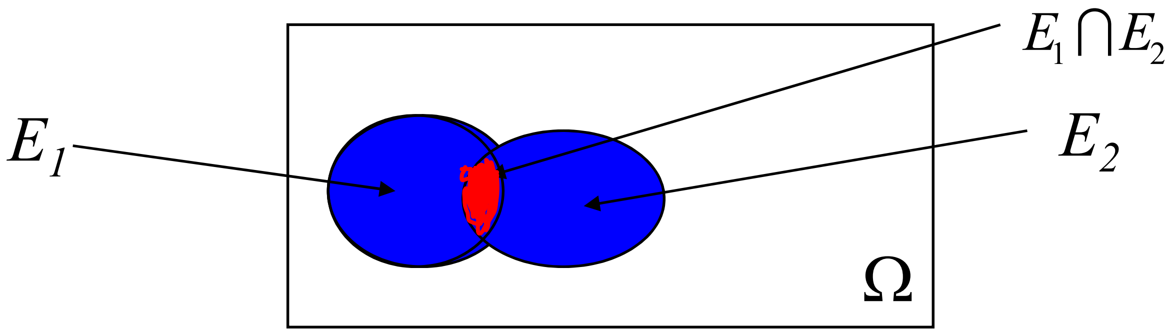


Sample Space and Events

An event is a sub-set of the sample space

- if the sub-set is empty the event is impossible
- if the sub-set contains all of the sample space the event is certain

Consider the two events E_1 and E_2 :
The sub-set of sample points belonging to the event E_1 and the event E_2 is called the intersection of E_1 and E_2 and is written as: $E_1 \cap E_2$

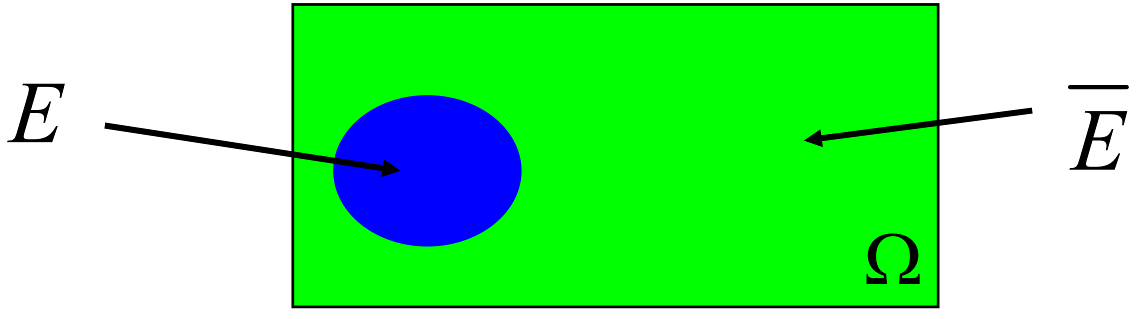


Sample Space and Events

The event containing all sample points in Ω not included in the event E is called the complementary event to E and written as : $\overline{E}$

It follows that $E \cup \overline{E} = \Omega$

and $E \cap \overline{E} = \varnothing$



Sample Space and Events

It can be show that the intersection and union operations obey the following commutative, associative and distributive laws:

$$E_1 \cap E_2 = E_2 \cap E_1$$

Commutative law

$$E_1 \cap (E_2 \cap E_3) = (E_1 \cap E_2) \cap E_3$$

$$E_1 \cup (E_2 \cup E_3) = (E_1 \cup E_2) \cup E_3$$

Associative law

$$E_1 \cap (E_2 \cup E_3) = (E_1 \cap E_2) \cup (E_1 \cap E_3)$$

$$E_1 \cup (E_2 \cap E_3) = (E_1 \cup E_2) \cap (E_1 \cup E_3)$$

Distributive law

Sample Space and Events

From the commutative, associative and distributive laws the so-called De Morgan's laws may be derived:

$$E_1 \cap E_2 = E_2 \cap E_1$$

$$E_1 \cap (E_2 \cap E_3) = (E_1 \cap E_2) \cap E_3$$

$$E_1 \cup (E_2 \cup E_3) = (E_1 \cup E_2) \cup E_3$$

$$E_1 \cap (E_2 \cup E_3) = (E_1 \cap E_2) \cup (E_1 \cap E_3)$$

$$E_1 \cup (E_2 \cap E_3) = (E_1 \cup E_2) \cap (E_1 \cup E_3)$$

$$E_1 \cap E_2 = \overline{\overline{E_1} \cup \overline{E_2}}$$

$$E_1 \cup E_2 = \overline{\overline{E_1} \cap \overline{E_2}}$$

The Three Axioms of Probability Theory

The probability theory is built up on – only – three axioms due to Kolmogorov:

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(\Omega) = 1$

Axiom 3: $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$

When E_1, E_2 ,... are mutually exclusive

Conditional Probability and Bayes's Rule

Conditional probabilities are of special interest as they provide the basis for utilizing new information in decision making.

The conditional probability of an event E_1 given that event E_2 has occured is written as:

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Not defined if $P(E_2) = 0$

The events E_1 and E_2 are said to be statistically independent if:

$$P(E_1|E_2) = P(E_1)$$

Conditional Probability and Bayes's Rule

From $P(E_1|E_2) = \dfrac{P(E_1 \cap E_2)}{P(E_2)}$

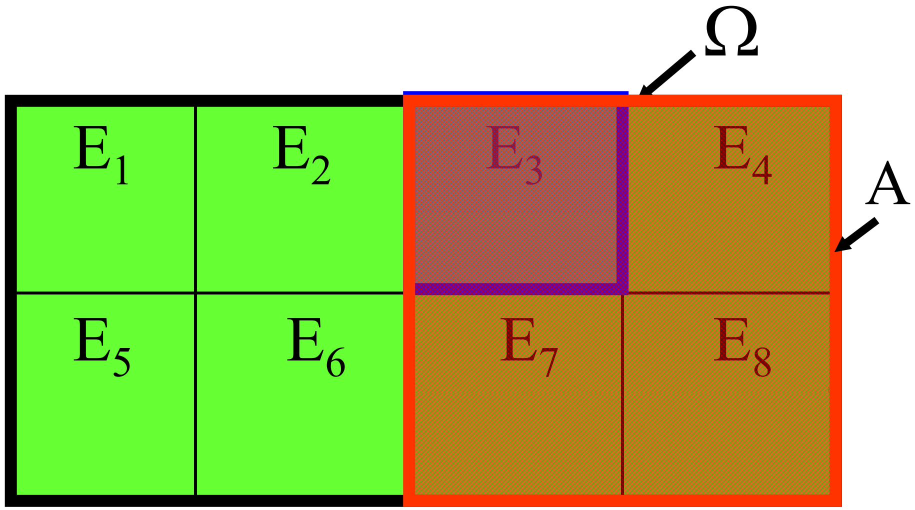
it follows that $P(E_1 \cap E_2) = P(E_2)P(E_1|E_2)$

and when E_1 and E_2 are statistically independent there is

$$P(E_1 \cap E_2) = P(E_2)P(E_1)$$

Conditional Probability and Bayes's Rule

Consider the sample space Ω divided up into n mutually exclusive events $E_1, E_2, \ldots, E_n$



$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n)$$

$$P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \ldots + P(A|E_n)P(E_n) =$$

$$\sum_{i=1}^{n} P(A|E_i)P(E_i)$$

Conditional Probability and Bayes's Rule

as there is $P(A \cap E_i) = P(A|E_i)P(E_i) = P(E_i|A)P(A)$

we have

Likelihood

Prior

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{i=1}^{n} P(A|E_i)P(E_i)}$$

Posterior

Bayes Rule

Reverend Thomas Bayes (1702-1764)

Uncertainties in Engineering Problems

Different types of uncertainties influence decision making

- Inherent natural variability – aleatory uncertainty
  - result of throwing dices
  - variations in material properties
  - variations of wind loads
  - variations in rain fall

- Model uncertainty – epistemic uncertainty
  - lack of knowledge (future developments)
  - inadequate/imprecise models (simplistic physical modelling)

- Statistical uncertainties – epistemic uncertainty
  - sparse information/small number of data

Uncertainties in Engineering Problems

- Consider as an example a dike structure

  - the design (height) of the dike will be determining the frequency of floods

  - if exact models are available for the prediction of future water levels and our knowledge about the input parameters is perfect then we can calculate the frequency of floods (per year) - a deterministic world !

  - even if the world would be deterministic – we would not have perfect information about it – so we might as well consider the world as random

Uncertainties in Engineering Problems

In principle the so-called

inherent physical uncertainty (aleatory – Type I)

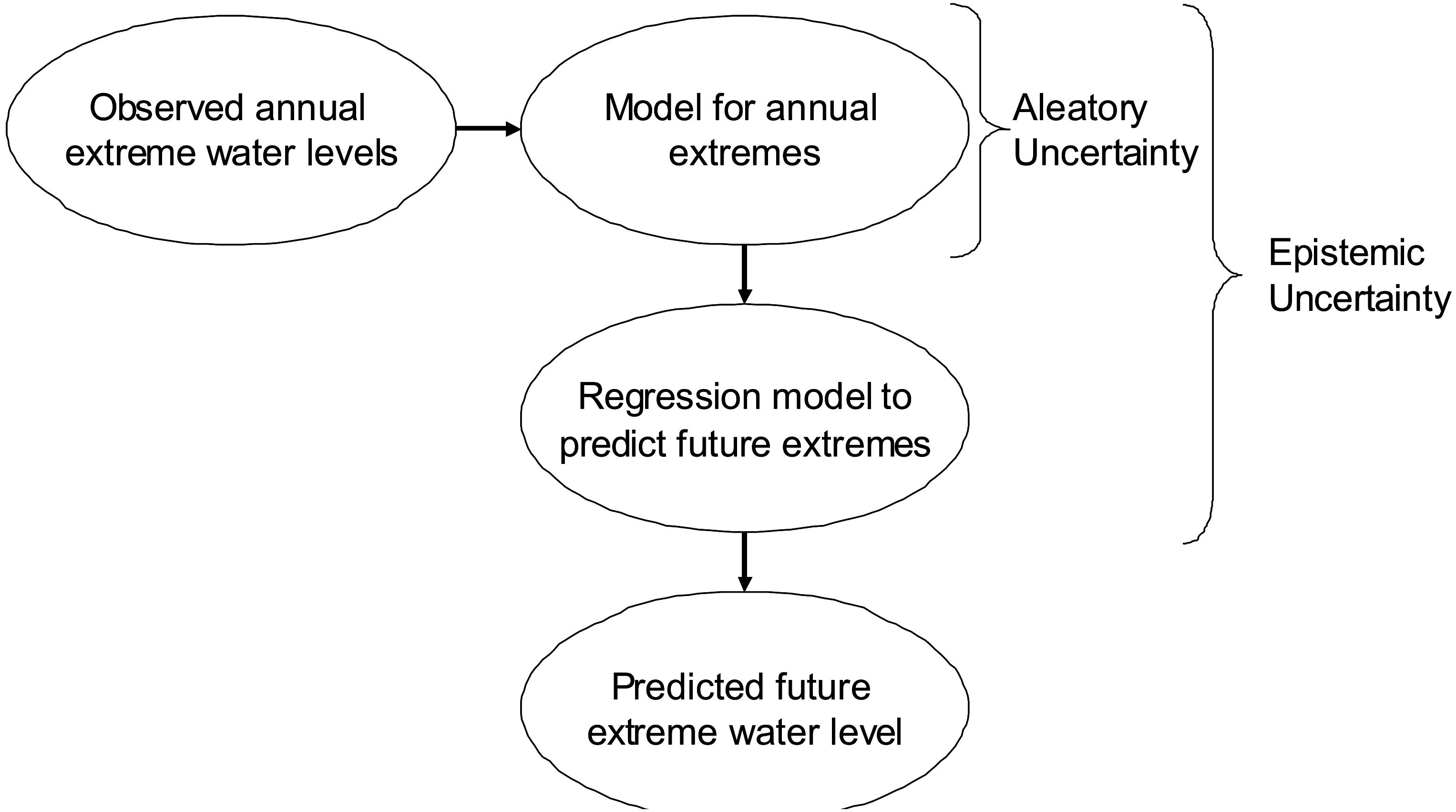
is the uncertainty caused by the fact that the world is random, however, another pragmatic viewpoint is to define this type of uncertainty as

any uncertainty which cannot be reduced by means of collection of additional information

the uncertainty which can be reduced is then the

model and statistical uncertainties (epistemic – Type II)

Uncertainties in Engineering Problems



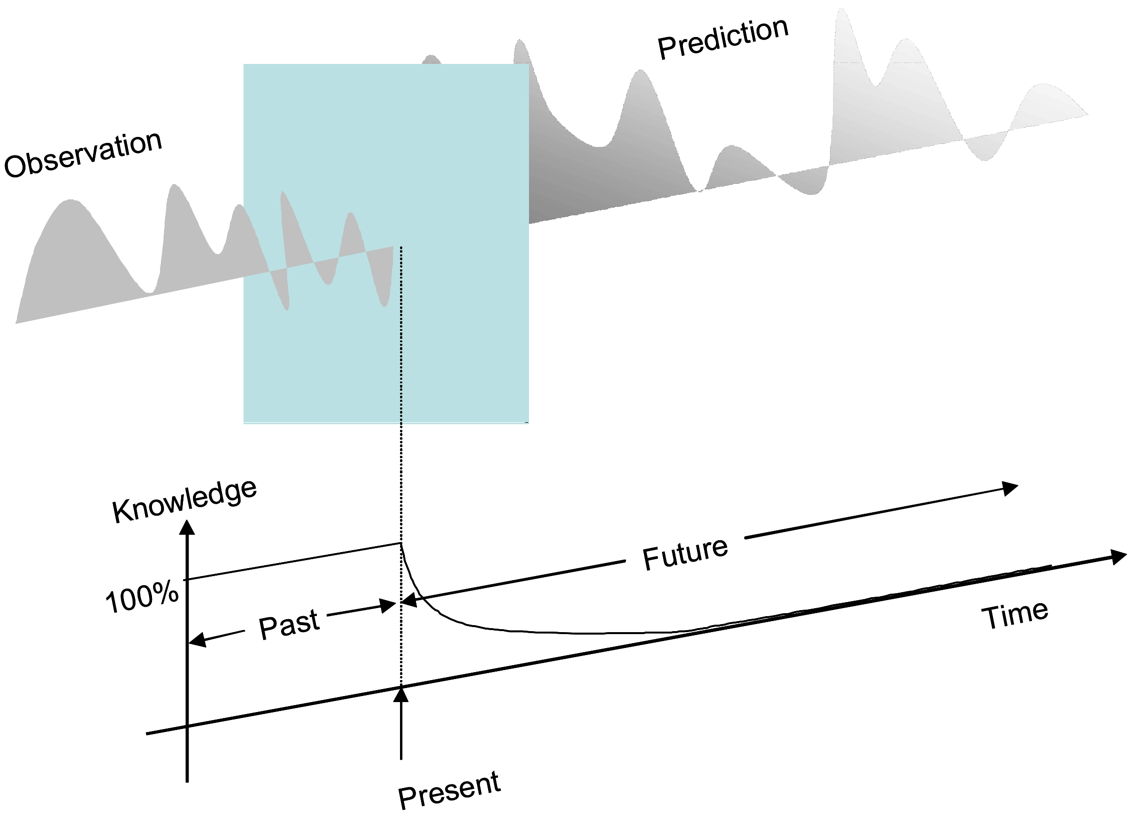
Uncertainties in Engineering Problems

The relative contribution of aleatory and epistemic uncertainty to the prediction of future water levels is thus influenced directly by the applied models

refining a model might reduce the epistemic uncertainty – but in general also changes the contribution of aleatory uncertainty

the uncertainty structure of a problem can thus be said to be scale dependent !

Uncertainties in Engineering Problems



The uncertainty structure changes also as function of time – is thus time dependent !

Random Variables

- Probability distribution and density functions

  A random variable is denoted with capital letters : X

  A realization of a random variable is denoted with small letters : x

  We distinguish between

  - *continuous random variables* : can take any value in a given range

  - *discrete random variables* : can take only discrete values

Random Variables

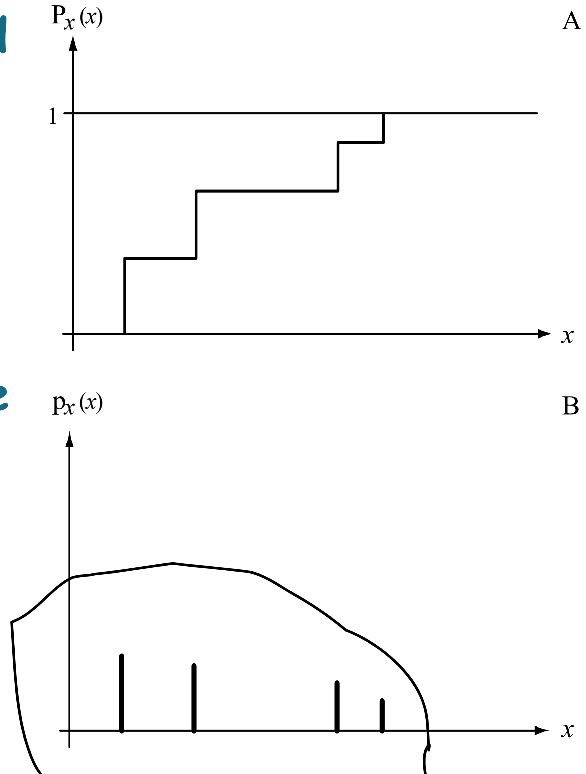
- Probability distribution and density functions

  The probability that the outcome of a discrete random variable X is smaller than x is denoted the *cumulative distribution function*

$$P_X(x) = \sum_{x_i < x} p_X(x_i)$$

  The *probability density function* for a discrete random variable is defined by

$$p_X(x_i) = P(X = x_i)$$



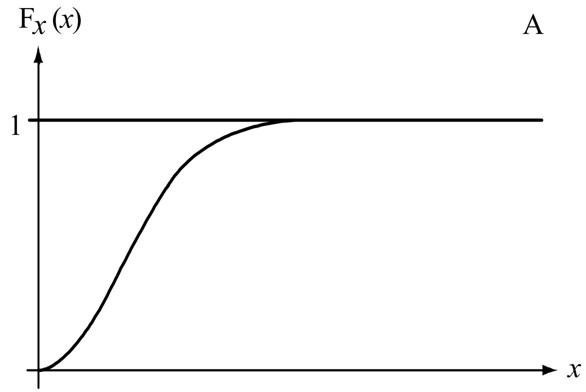
Sum of these must equal 1

Random Variables

- Probability distribution and density functions

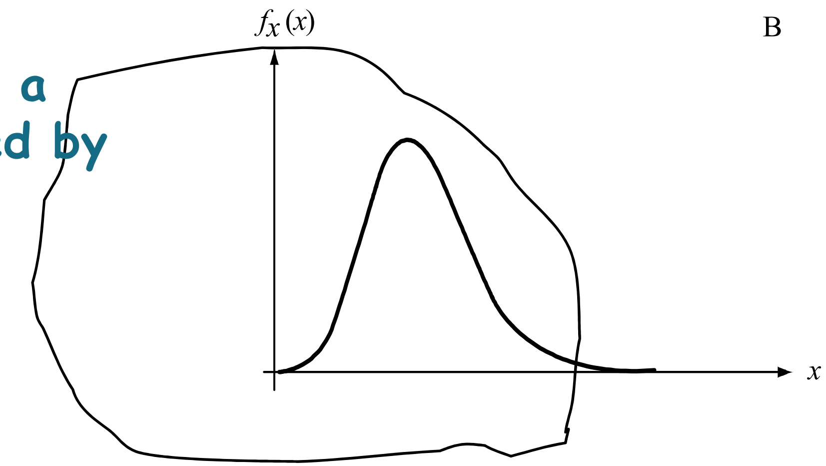
The probability that the outcome of a continuous random variable X is smaller than x is denoted the *cumulative distribution function*

$$F_X(x) = P(X < x)$$



The *probability density function* for a continuous random variable is defined by

$$f_X(x) = \frac{\partial F_X(x)}{\partial x}$$



Integral of this must equal 1

Random Variables

- **Moments of random variables and the expectation operator**

  Cumulative distributions and density functions can be described in terms of their paramaters $\mathbf{p}$ or their moments

  Often we write

$$F_X(x,\mathbf{p}) \qquad f_X(x,\mathbf{p})$$

Parameters

The parameters can be related to the moments and visa versa

Random Variables

- **Moments of random variables and the expectation operator**

The i'th moment m_i for a continuous random variable X is defined through

$$m_i = \int_{-\infty}^{\infty} x^i \cdot f_X(x)dx$$

The *expected value E[X]* of a continuous random variable X is defined accordingly as the first moment

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$$

Random Variables

- Moments of random variables and the expectation operator

  The i'th moment m_i for a discrete random variable X is defined through

$$m_i = \sum_{j=1}^{n} x_j^i \cdot p_X(x_j)$$

  The *expected value E[X]* of a discrete random variable X is defined accordingly as the first moment

$$\mu_X = E\left[X\right] = \sum_{j=1}^{n} x_j \cdot p_X(x_j)$$

Random Variables

- **Moments of random variables and the expectation operator**

  The *standard deviation* σ_X of a continuous random variable is defined as the second central moment i.e. for a continuous random variable *X* we have

$$\sigma_X{}^2 = \mathrm{Var}[\mathrm{X}] = E\left[(X-\mu_X)^2\right] = \int_{-\infty}^{\infty}(x-\mu_X)^2 \cdot f_X(x)dx$$

Variance **Mean value**

for a discrete random variable we have correspondingly

$$\sigma_X^2 = Var[X] = \sum_{j=1}^{n}(x_j-\mu_X)^2 \cdot p_X(x_j)$$

Random Variables

- Moments of random variables and the expectation operator

The ratio between the standard deviation and the expected value of a random variable is called the *Coefficient of Variation CoV* and is defined as

$$CoV\left[X\right]=\frac{\sigma_X}{\mu_X}$$

Dimensionless

a useful characteristic to indicate the variability of the random variable around its expected value

Random Variables

- Random vectors and joint moments

Now we consider not just one continuous random variable but a vector of continuous random variables

$$\mathbf{X} = \left(X_1, X_2, \ldots, X_n\right)^T$$

The *joint cumulative distribution function* is given by

$$F_{\mathbf{X}}\left(\mathbf{x}\right) = P\left(X_1 \le x_1 \cap X_2 \le x_2 \cap \ldots \cap X_n \le x_n\right)$$

and the *joint probability density function* is given by

$$f_{\mathbf{X}}\left(\mathbf{x}\right) = \frac{\partial^n}{\partial z_1 \partial z_2 \ldots \partial z_n} F_{\mathbf{X}}\left(\mathbf{x}\right)$$

Random Variables

- **Random vectors and joint moments**

  The *marginal probability density function* of a random variable X_i is defined by

$$f_{X_i}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (n-1 \text{ fold}) f_{\mathbf{X}}(\mathbf{x})\, dx_1..dx_{i-1} dx_{i+1}..dx_n$$

Random Variables

- Random vectors and joint moments

The *covariance* between the i'th and the j'th component of the random vector of continuous random variables is defined as the *joint central moment* i.e. by

$$C_{X_iX_j} = E\left[(X_i - \mu_{X_i})(X_j - \mu_{X_j})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(x_i - \mu_{X_i}\right)\left(x_j - \mu_{X_j}\right) f_{X_iX_j}\left(x_i, x_j\right) dx_i dx_j$$

$$C_{X_iX_i} = Var\left[X_i\right]$$

From where we see that for i = j we get the variance for X_i

Correlation coefficient $\rho_{X_iX_j} = \dfrac{C_{X_iX_j}}{\sigma_{X_i}\sigma_{X_j}}$ $\qquad \rho_{X_iX_i} = 1$

Random Variables

- Random vectors and joint moments

The expected value and the variance of a linear function

$$Y = a_0 + \sum_{i=1}^{n} a_i X_i$$

are given by

$$E[Y] = a_0 + \sum_{i=1}^{n} a_i E[X_i]$$

$$Var[Y] = \sum_{i=1}^{n} a_i^2 Var[X_i] + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i a_j C_{X_i X_j}$$

Random Variables

- **Conditional distributions and conditional moments**

  The *conditional probability density function* for the random variable X_1 given the outcome of the random variable X_2 is given by

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)}$$

  where if X_1 and X_2 are independent

$$f_{X_1|X_2}(x_1|x_2) = f_{X_1}(x_1)$$

  The *conditional cumulative distribution function* is obtained by integration as

$$F_{X_1|X_2}(x_1|x_2) = \frac{\int_{-\infty}^{x_1} f_{X_1,X_2}(z,x_2)dz}{f_{X_2}(x_2)}$$

Random Variables

- Conditional distributions and conditional moments

The *un-conditional cumulative distribution function* for the random variable X_1 can be derived from the conditional comulative distribution function by use of the *total probability theorem*

$$F_{X_1}(x_1) = \int_{-\infty}^{\infty} F_{X_1|X_2}(x_1|x_2) f_{X_2}(x_2) dx_2$$

The *conditional expected value* is defined by

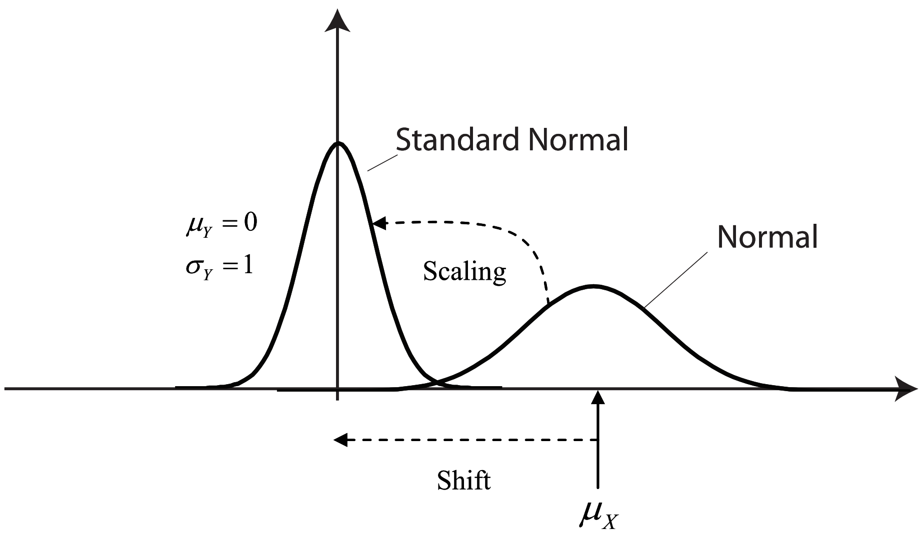
$$\mu_{X_1|X_2} = E\left[X_1 \middle| X_2 = x_2\right] = \int_{-\infty}^{\infty} x_1 f_{X_1|X_2}\left(x \middle| x_2\right) dx_1$$

Random Variables

- The Normal distribution:

  In the case where the mean value is equal to zero and the standard deviation is equal to 1 the random variable is said to be *standardized*.

$$Y = \frac{X - \mu_X}{\sigma_X}$$ **Standardized random variable**



Random Variables

- The Normal distribution:

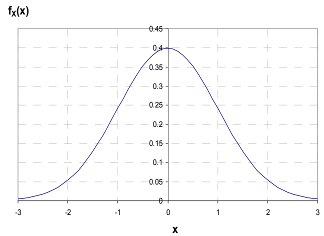
  In the case where the mean value is equal to zero and the standard deviation is equal to 1 the random variable is said to be *standardized.*

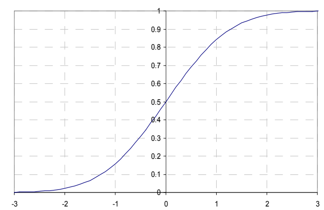
$$Y = \frac{X - \mu_X}{\sigma_X}$$ **Standardized random variable**

$$f_Y(y) = \varphi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right)$$

$$F_Y(y) = \Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{1}{2}x^2\right) dx$$

Standard normal





Random Variables

Where the normal distribution follows from the sum of random variables – Central Limit Theorem

the log-normal distribution follows from the product of random variables

$$Y = X_1 \cdot X_2 \cdots X_n$$

$$\Downarrow$$

$$\ln(Y) = \ln(\prod_{i=1}^{n} X_i) = \sum_{i=1}^{n} \ln(X_i)$$

$\uparrow$

Normal distributed $\Rightarrow$ Y is Lognormal distributed

Random Variables

When the logarithm of a random variable X i.e.

Y = ln(X), Y : N(μ_Y,σ_Y)

is normal distributed the random variable X is said to be log-normal distributed

X : LN(λ,ζ)

$$f_X(x) = \frac{1}{x\zeta\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln(x)-\lambda}{\zeta}\right)^2\right) \qquad \mu_X = \exp\left(\lambda + \frac{\zeta^2}{2}\right)$$

$$F_X(x) = \Phi\left(\frac{\ln(x)-\lambda}{\zeta}\right) \qquad \sigma_X = \exp\left(\lambda + \frac{\zeta^2}{2}\right)\sqrt{\exp(\zeta^2)-1}$$

Random Variables

There exist a large number of different cumulative probability functions:

Uniform
Normal
Log-normal
Exponential
Beta
Gamma
...
...

Distribution type	Parameters	Moments
Uniform, $a \le x \le b$ $f_X(x) = \frac{1}{b-a}$ $F_X(x) = \frac{x-a}{b-a}$	a b	$\mu = \frac{a+b}{2}$ $\sigma = \frac{b-a}{\sqrt{12}}$
Normal $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$ $F_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right)dt$	μ $\sigma > 0$	μ σ
Shifted Lognormal, $x > \varepsilon$ $f_X(x) = \frac{1}{(x-\varepsilon)\zeta\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln(x-\varepsilon)-\lambda}{\zeta}\right)^2\right)$ $F_X(x) = \Phi\left(\frac{\ln(x-\varepsilon)-\lambda}{\zeta}\right)$	λ $\zeta > 0$ ε	$\mu = \varepsilon + \exp\left(\lambda + \frac{\zeta^2}{2}\right)$ $\sigma = \exp\left(\lambda + \frac{\zeta^2}{2}\right)\sqrt{\exp(\zeta^2)-1}$
Shifted Exponential, $x \ge \varepsilon$ $f_X(x) = \lambda\exp(-\lambda(x-\varepsilon))$ $F_X(x) = 1-\exp(-\lambda(x-\varepsilon))$	ε $\lambda > 0$	$\mu = \varepsilon + \frac{1}{\lambda}$ $\sigma = \frac{1}{\lambda}$
Gamma, $x \ge 0$ $f_X(x) = \frac{b^p}{\Gamma(p)}\exp(-bx)x^{p-1}$ $F_X(x) = \frac{\Gamma(bx,p)}{\Gamma(p)}$	$p > 0$ $b > 0$	$\mu = \frac{p}{b}$ $\sigma = \frac{\sqrt{p}}{b}$

Stochastic Processes and Extremes

- Random quantities may be "time variant" in the sense that they take new values at different times or at new trials.

  - If the new realizations occur at discrete times and have discrete values the random quantity is called a **random sequence**

    failure events, traffic congestions,…

  - If the new realizations occur continuously in time and take continues values the random quantity is called a **random process** or **stochastic process**

    wind velocity, wave heights,…

Stochastic Processes and Extremes

- Random sequences

  - A sequence of experiments with only two possible and mutually exclusive outcomes is called a **Bernoulli trial**

  - Typically the outcomes of Bernoulli trials are denoted **successes** or **failures**

  If the probability of success in one trial is constant and equal to p the probability density of Y successes in n trials, i.e. $p_Y(y)$ is given by:

$$p_Y(y) = \binom{n}{y} p^y (1-p)^{n-y}, \quad y = 0,1,2\ldots n \qquad \binom{n}{y} = \frac{n!}{y!(n-y)!}$$

Binomial probability density function

Binomial operator

Stochastic Processes and Extremes

- **Random sequences**

  - A sequence of experiments with only two possible and mutually exclusive outcomes is called a **Bernoulli trial**

The **Binomial probability distribution function** then follows as:

$$P_Y(y) = \sum_{i=0}^{y} \binom{y}{i} p^i (1-p)^{n-i}, \qquad y = 0,1,2,\ldots n$$

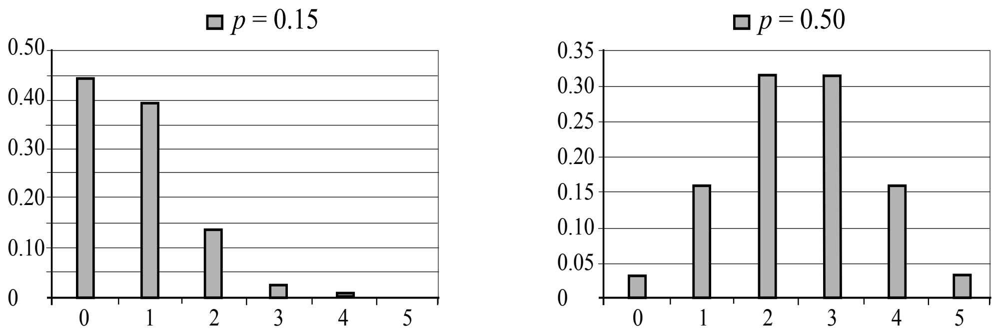
Stochastic Processes and Extremes

- **Random sequences**

  - A sequence of experiments with only two possible and mutually exclusive outcomes is called a Bernoulli trial

Illustration:

Binomial probability density function for n=5 and p=0.15 and p=0.5



Stochastic Processes and Extremes

- **Random sequences**

  The expected value and the variance of a binomially distributed random variable Y is given by:

$$E[Y] = np$$

$$Var[Y] = np(1-p)$$

Random Sequences

- The Poisson counting process was originally invented by Poisson

Poisson, Siméon-Denis
(1781-1840)
Student of Laplace
Former law clerk

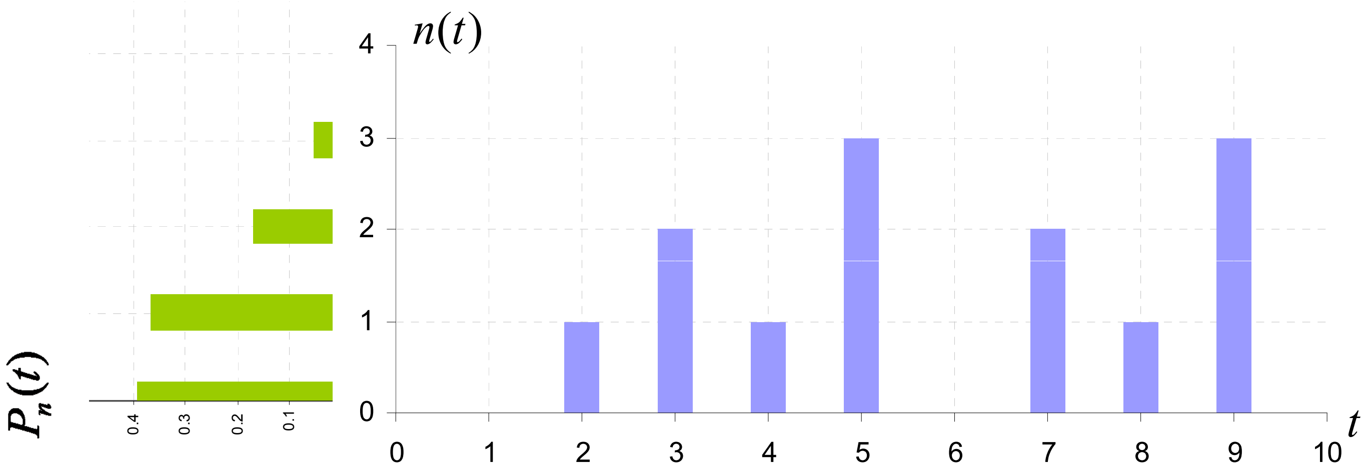
"life is only good for two things: to do mathematics and to teach it" (Boyer 1968, p. 569)

Poisson was originally interested in applying probability theory for the improvement of procedures of law

Random Sequences

- The **Poisson counting process** is one of the most commonly applied families of probability distributions applied in reliability theory

  The Poisson process provides a model for representing **rare events** – counting the number of events over time



Random Sequences

- The **Poisson counting process** is one of the most commonly applied families of probability distributions applied in reliability theory

  The process $N(t)$ denoting the number of events in a (time) interval $(t,t+\Delta t[$ is called a **Poisson process** if the following conditions are fulfilled:

  1) the probability of one event in the interval $(t,t+\Delta t[$ is asymptotically proportional to Δt.

  2) the probability of more than one event in the interval $(t,t+\Delta t[$ is a function of higher order of Δt *for* $\Delta t \rightarrow 0$.

  3) events in disjoint intervals are mutually independent.

Random Sequences

- The Poisson process can be described completely by its intensity $\nu(t)$

$$\nu(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(\text{one event in } [t, t+\Delta t[)$$

if $\nu(t)$ = constant, the Poisson process is said to be **homogeneous**, otherwise it is **inhomogeneous**.

The probability of n events in the time interval $(0,t[$ is:

$$P_n(t) = \frac{\left(\int_0^t \nu(\tau)d\tau\right)^n}{n!} \exp\left(-\int_0^t \nu(\tau)d\tau\right)$$

$$P_n(t) = \frac{(\nu t)^n}{n!} \exp(-\nu t)$$

Homogeneous case !

Random Sequences

- Early applications include the studies by:

  Ladislaus Bortkiewicz (1868-1931)
  - horse kick death in the Prussian cavalry
  - child suicide

  William Sealy Gosset ("Student") (1876-1937)
  - small sample testing of beer productions (Guinness)

  RD Clarke
  - study of distribution of V1/V2 hits under the London Raid

Random Sequences

- The **mean value** and **variance** of the random variable describing the number of events *N* in a given time interval (0,*t*[are given as:

$$E\left[N(t)\right]=Var\left[N(t)\right]=\int_0^t \nu(\tau)d\tau$$ **Inhomogeneous case !**

$$E\left[N(t)\right]=Var\left[N(t)\right]=\nu t$$ **Homogeneous case !**

Random Sequences

- **The Exponential distribution**

The probability of no events (N=0) in a given time interval (0,t[is often of special interest in engineering problems

- no severe storms in 10 years
- no failure of a structure in 100 years
- no earthquake next year
- ....

This probability is directly achieved as:

$$P_0(t) = \frac{\left(\int_0^t \nu(\tau)d\tau\right)^0}{0!}\exp\left(-\int_0^t \nu(\tau)d\tau\right)$$

$$= \exp\left(-\int_0^t \nu(\tau)d\tau\right)$$

$$P_0(t) = \exp(-\nu t)$$

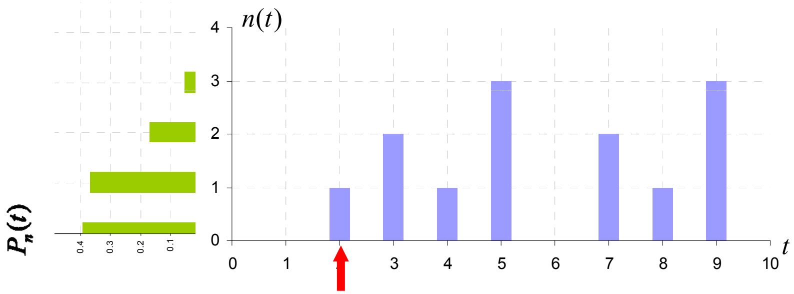
Homogeneous case !

Random Sequences

- **The probability distribution function of the (waiting) time till the first event T_1 is now easily derived recognizing that the probability of $T_1 > t$ is equal to $P_0(t)$ we get:**

$$F_{T_1}(t_1) = 1 - P_0(t_1)$$

$$= 1 - \exp(-\int_0^t \nu(\tau)d\tau)$$



Homogeneous case !

$$F_{T_1}(t_1) = 1 - \exp(-\nu t)$$

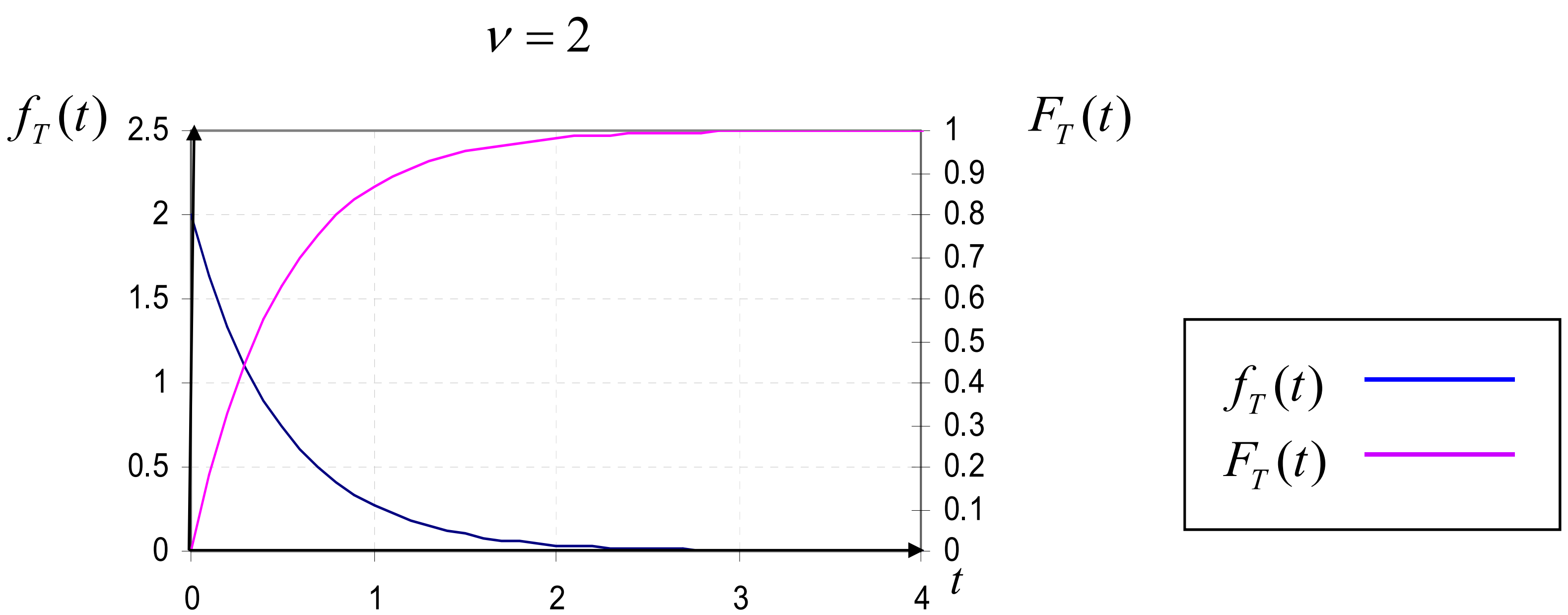
Exponential cumulative distribution

Exponential probability density

$$f_{T_1}(t_1) = \nu \exp(-\nu t)$$

Random Sequences

The **Exponential** probability density and cumulative distribution functions



Random Sequences

- The **exponential distribution** is frequently applied in the modeling of waiting times

  - time till failure
  - time till next earthquake
  - time till traffic accident
  - ....

$$\boxed{f_{T_1}(t_1) = \nu \exp(-\nu t)}$$

The expected value and variance of an exponentially distributed random variable T_1 are:

$$E[T_1] = \sqrt{Var[T_1]} = 1/\nu$$

Random Sequences

- **Sometimes also the time *T* till the *n*'th event is of interest in engineering modeling:**

  - repair events
  - flood events
  - arrival of cars at a roadway crossing

  If T_i, i=1,2,..n are independent exponentially distributed waiting times, then the sum *T* i.e.:

$$T = T_1 + T_2 + \ldots + T_{n-1} + T_n$$

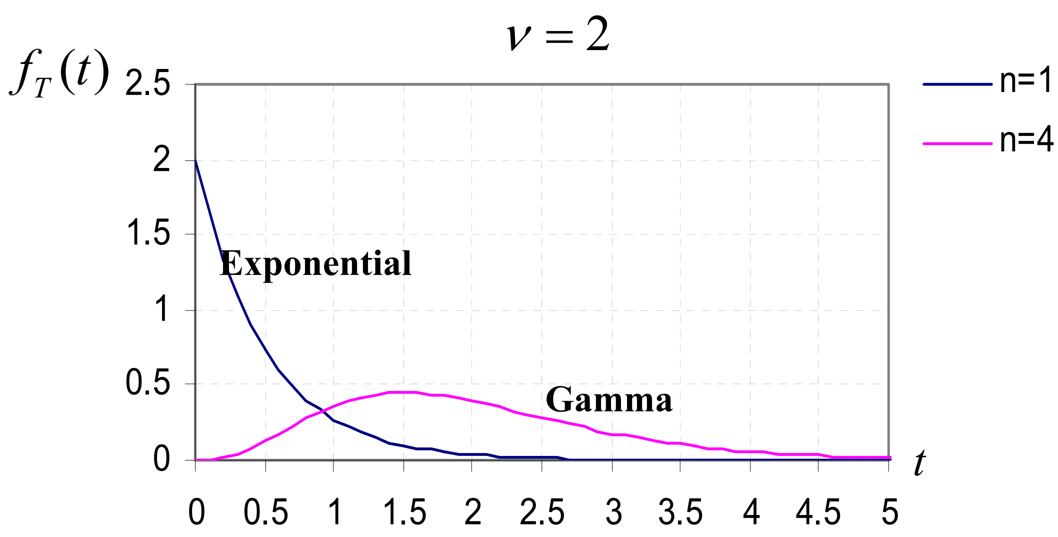
  follows a Gamma distribution:

$$f_T(t) = \frac{\nu(\nu t)^{(n\text{-}1)} \exp(-\nu t)}{(n-1)!}$$

This follows from repeated use of the result of the distribution of the sum of two random variables

Random Sequences

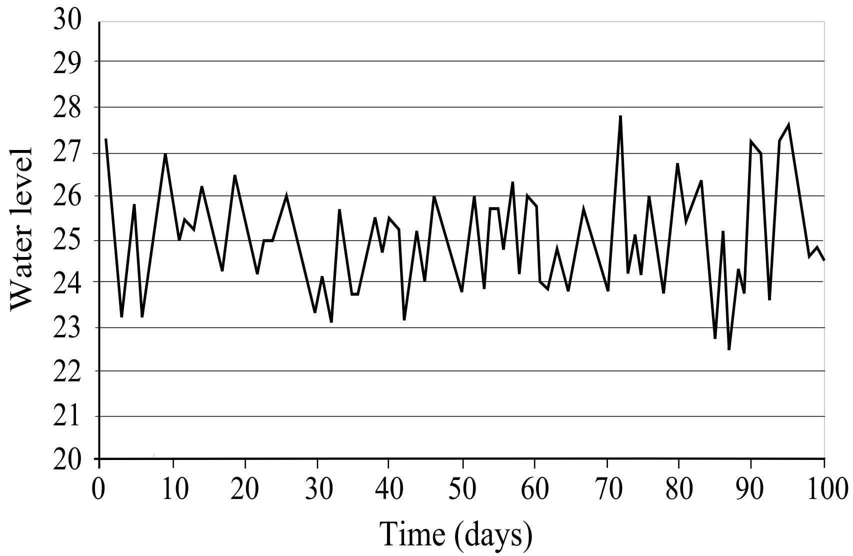
The **Gamma** probability density function



Random Processes

- **Continuous random processes**

  A continuous random process is a random process which has realizations continuously over time and for which the realizations belong to a continuous sample space.



Variations of;
water levels
wind speed
rain fall
.
.
.

Realization of continuous scalar valued random process

Random Processes

- **Continuous random processes**

  The mean value of the possible realizations of a random process is given as:

$$\mu_X(t) = E\left[X(t)\right] = \int_{-\infty}^{\infty} x\, f_X(x\,;t)dx$$

Function of time !

The correlation between realizations at any two points in time is given as:

$$R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\, x_2\, f_{XX}(x_1,x_2;t_1,t_2)dx_1dx_2$$

Auto-correlation function – refers to a scalar valued random process

Random Processes

- **Continuous random processes**

  The auto-covariance function is defined as:

$$C_{XX}(t_1,t_2)=E\left[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))\right]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-\mu_X(t_1))\;(x_2-\mu_X(t_2))\;f_{XX}(x_1,x_2;t_1,t_2)dx_1dx_2$$

for $t_1=t_2=t$ the auto-covariance function becomes the covariance function:

$$\sigma_X^2(t)=C_{XX}(t,t)=R_{XX}(t,t)-\mu_X^2(t)$$

$\sigma_X(t)$ **Standard deviation function**

Random Processes

- **Continuous random processes**

A vector valued random process is a random process with two or more components:

$$\mathbf{X}(t) = (X_1(t), X_2(t),.., X_n(t))^T$$

with covariance functions:

$$C_{X_iX_j}(t_1,t_2) = E\left[(X_i(t_1)-\mu_{X_i}(t_1))(X_j(t_2)-\mu_{X_j}(t_2))\right]$$

$i = j$ **auto-covariance functions**

$i \neq j$ **cross-covariance functions**

The correlation coefficient function is defined as:

$$\rho\left[X_i(t_1), X_j(t_2)\right] = \frac{C_{X_iX_j}(t_1,t_2)}{\sigma_{X_i}(t_1)\cdot\sigma_{X_j}(t_2)}$$

Random Processes

- **Normal or Gauss process**

A random process $X(t)$ is said to be Normal if:

for any set; $X(t_1), X(t_2),\ldots,X(t_j)$

the joint probability distribution of $X(t_1), X(t_2),\ldots,X(t_j)$

is the Normal distribution.

Random Processes

- **Stationarity and ergodicity**

A random process is said to be *strictly stationary* if all its moments are invariant to a shift in time.

A random process is said to be *weakly stationary* if the first two moments i.e. the mean value function and the auto-correlation function are invariant to a shift in time

$$\left.\begin{aligned} &\mu_X(t) = cst \\ &R_{XX}(t_1,t_2) = f(t_2 - t_1) \end{aligned}\right\} \quad \textbf{Weakly stationary}$$

Random Processes

- Stationarity and ergodicity

  - A random process is said to be ***strictly ergodic*** if it is strictly stationary and in addition all its moments may be determined on the basis of one realization of the process.
  - A random process is said to be ***weakly ergodic*** if it is weakly stationary *and in addition* its first two moments may be determined on the basis of one realization of the process.

- The assumptions in regard to stationarity and ergodicity are often very useful in engineering applications.

  - If a random process is ergodic we can extrapolate probabilistic models of extreme events within short reference periods to any longer reference period.

Random Processes

- **Markov Process**

Discrete case

A discrete Markov process (Markov chain) is sequence of random variables $X_1, X_2, \ldots X_n$ satisfying:

$$\Pr(X_{n+1} = x \mid X_n = x_n, \ldots, X_1 = x_1) = \Pr(X_{n+1} = x \mid X_n = x_n)$$

The possible values of X_1 form a countable set S – state space

Continuous Markov process have a continuous index

Random Processes

- **Markov Process**

  Time homogeneous Markov chains:

$$\Pr(X_{n+1} = x \mid X_n = y) = \Pr(X_{n+1} = x \mid X_{n-1} = y), \ \textbf{for all } n$$

  A Markov chain of *m*'th order:

$$\Pr(X_{n+1} = x \mid X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots) =$$
$$\Pr(X_n = x_n \,\|\, X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, .., X_{n-m} = x_{n-m})$$

Random Processes

- **Markov Process**

  Some properties of Markov chains

  The probability to go from state *i* to *j* in *n* steps is:

$$p_{ij}^{(n)} = \Pr(X_n = j \mid X_0 = i)$$

satisfying:

$$p_{ij}^{(n)} = \sum_{r \in S} p_{ir}^{(k)} p_{rj}^{(n-k)} \qquad \textbf{Chapman-Kolmogorov equation}$$

Random Processes

- **Markov Process**

Some properties of Markov chains

Reducibility:

$$\Pr(X_{n+1} = j \,|\, X_0 = i) > 0 \quad \textbf{accessible}$$

State *i*, *j* communicate if it is possible to come from state *i* to *j* and from *j* to *i* .

A set of states *C* is a communicating class if every pair of states in *C* communicate.

A communicating class is closed if the probability of leaving the class is zero

Random Processes

- **Markov Process**

  Some properties of Markov chains

  Recurrence:

$$T_i = \min\left\{n : X_n = i \middle| X_0 = i\right\}$$

  A state is transient if there exists a finite *T_i* such that:

$$\Pr(T_i < \infty) < 1$$

Random Processes

- **Markov Process**

  If the state space is finite the probability distribution can be represented by a transition matrix P with elements:

$$p_{ij} = \Pr(X_{n+1} = j \mid X_n = i)$$

  For a homogeneous Markov process P is independent of *n* and can be calculated from P^n.

$$\Pr(T_i < \infty) < 1$$