# Formulation and calculation of isoparametric finite element matrixes

1. Numerical integration

2. Implementation of a finite element computer code

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Finite Element Method and the Analysis of Systems with Uncertain Properties

# Numerical integration

**Goal: Integration of the matrix F(r)**

Numerical integration

$$\int_a^b F(r)dr = \sum_i \alpha_i F(r_i)$$

With a=-1 and b=1 (isoparametric elements)

**Approximation of F(r) with ψ(r)**

Polynomial:

$$\psi(r) = a_0 + a_1 r + a_2 r^2 + \ldots + a_n r^n$$

Lagragian interp. :

$$\psi(r) = F_0 l_0(r) + F_1 l_1(r) + F_2 l_2(r) + \ldots + F_n l_n(r)$$

With

$$l_j(r) = \frac{(r - r_0)(r - r_1)\cdots(r - r_{j-1})(r - r_{j+1})\cdots(r - r_n)}{(r_j - r_0)(r_j - r_1)\cdots(r_j - r_{j-1})(r_j - r_{j+1})\cdots(r_j - r_n)}$$

# Numerical integration II

**Netwon:** equally spaced sampling points

$$\int_a^b F(r)dr = (b-a)\sum_{i=0}^{n} C_i^n F(r_i)$$

With $C_i^n$=Newton-Cotes constant

**Gauss:** variation of sampling interval

$$\int_a^b F(r)dr = \sum_i \alpha_i F(r_i)$$

With $\alpha_i = \int_{-1}^{1} l_j(r)dr$

**Multidimensional integration**

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} F(r,s,t)\; dr\; ds\; dt = \sum_{i,j,k} \alpha_i \alpha_j \alpha_k F(r_i, s_j, t_k)$$

# Numerical integration III

**Kind and order of integration**

=> cost of analysis, errors, degree of local variables, linearity, …

=> For displacement matrix => full integration

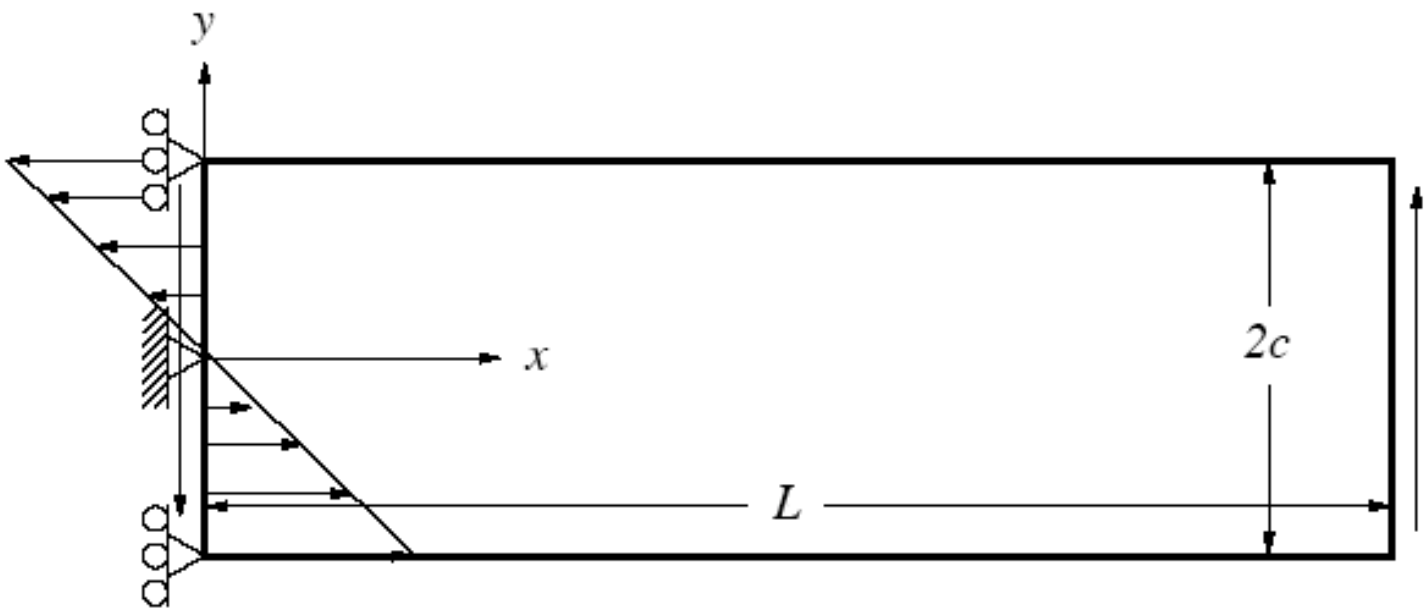
**Reduced and Selective**

=> Improving overall results by reducing the order of certain matrix evaluation

=> Different strains terms integrated with different orders

## => Stability and convergence

# Example for FEM code




**Displacement boundary conditions:**

$u_x$ = 0 @ (0; c), (0,0) and (0; -c) , $u_y$ = 0 @ (0; 0).

**Traction boundary conditions:**

$t_x$ = y on x = 0 and $t_y$ = P($x^2$-$c^2$) on x = L.

# Implementation of computer code

PRE-PROCESSING

↓

GENERATING MESH

↓

INITIALIZING DATA STRUCTURES

↓

COMPUTING EXTERNAL LOADS

↓

COMPUTING STIFFNESS MATRIX

↓

APPLYING BOUNDARY CONDITIONS

↓

SOLVING SYSTEM

↓

POST-PROCESSING

# Implementation of computer code II

- PRE-PROCESSING
- GENERATING MESH
- INITIALIZING DATA STRUCTURES
- COMPUTING EXTERNAL LOADS
- COMPUTING STIFFNESS MATRIX
- APPLYING BOUNDARY CONDITIONS
- SOLVING SYSTEM
- POST-PROCESSING

- **Material properties**
- **Beam properties**
- **Mesh properties**
- **Model assumption**

# Implementation of computer code III

PRE-PROCESSING

**GENERATING MESH**

INITIALIZING
DATA STRUCTURES

COMPUTING
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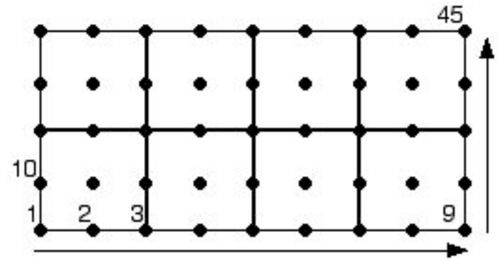
COMPUTING
STIFFNESS MATRIX

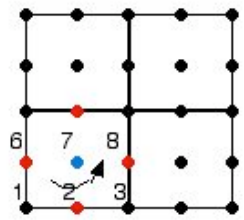
APPLYING BOUNDARY
CONDITIONS

SOLVING SYSTEM

POST-PROCESSING

- **Element connectivity matrix:** Node pattern




- **Boundary node pattern**

- **Displacement boundary**

# Implementation of computer code IV

PRE-PROCESSING

GENERATING MESH

**INITIALIZING DATA STRUCTURES**

COMPUTING EXTERNAL LOADS

COMPUTING STIFFNESS MATRIX

APPLYING BOUNDARY CONDITIONS

SOLVING SYSTEM

POST-PROCESSING

- **Nodal displacement vector**

```
U=zeros(2*numnode,1);
```

- **External load vector**

```
f=zeros(2*numnode,1);
```

- **Stiffness matrix**

```
K=sparse(2*numnode,2*numnode);
```

With numnode = total # nodes

# Implementation of computer code V

PRE-PROCESSING

GENERATING MESH

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DATA STRUCTURES

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EXTERNAL LOADS**

COMPUTING
STIFFNESS MATRIX

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POST-PROCESSING

- **Gaussian quadrature**

```
[W,Q]=quadrature( 4, 'GAUSS', 1 );
```

Where W and Q are the sampling points and weights

- **Shape function**

```
[N,dNdxi]=lagrange_basis(edgeElemType,pt);
```

- **Integration of the tractions on the left and right edges**

```
f(sctrx)=N*fxPt*detJ0*wt
```

With fxPt = x traction at quadrature point

# Implementation of computer code VI

PRE-PROCESSING

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STIFFNESS MATRIX**

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POST-PROCESSING

- **Shape function**

```
[W,Q]=quadrature( 4, 'GAUSS', 2 );

[N,dNdxi]=lagrange_basis(edgeElemType,pt);

J0=node(sctr,:)'*dNdxi;

dNdx=dNdxi*invJ0;
```

- **Compute Element Stiffness at quadrature point**

```
K(sctrB,sctrB)=B'*C*B*W(q)*det(J0)
```

# Implementation of computer code VII

PRE-PROCESSING

GENERATING MESH

INITIALIZING
DATA STRUCTURES

COMPUTING
EXTERNAL LOADS

COMPUTING
STIFFNESS MATRIX

**APPLYING BOUNDARY CONDITIONS**

SOLVING SYSTEM

POST-PROCESSING

**Enforcement of the essential boundary conditions**

Modifying the system but keeping it symmetric so that the boundary condition are satisfied according to the applied mapping.

# Implementation of computer code VIII



- **Solving system**

```
U=K\f
```

- **Compute Element strain and stress at the stress point**

```
strain=B*U(sctrB);
stress(e,q,:)=C*strain;
```

# Literature

Code:
http://www.tam.northwestern.edu/jfc795/Matlab/

The finite element method, O.C. Zienkiewicz, R.L. Taylor - Butterworth Heinemann (2000) Vol.1-3

Additional:
Matlab guide to finite elements: an interactive approach, (2005) P.I. Kattan

Introduction to finite and spectral element methoads using Matlab,(2003)
C. Pozrikidis

The finite element method using Matlab (2000), Y.W. Kwon, H. Bang