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"Formulation and calculation of isoparametric finite element matrixes"

-Formulation of structural elements (plate and general shell elements)

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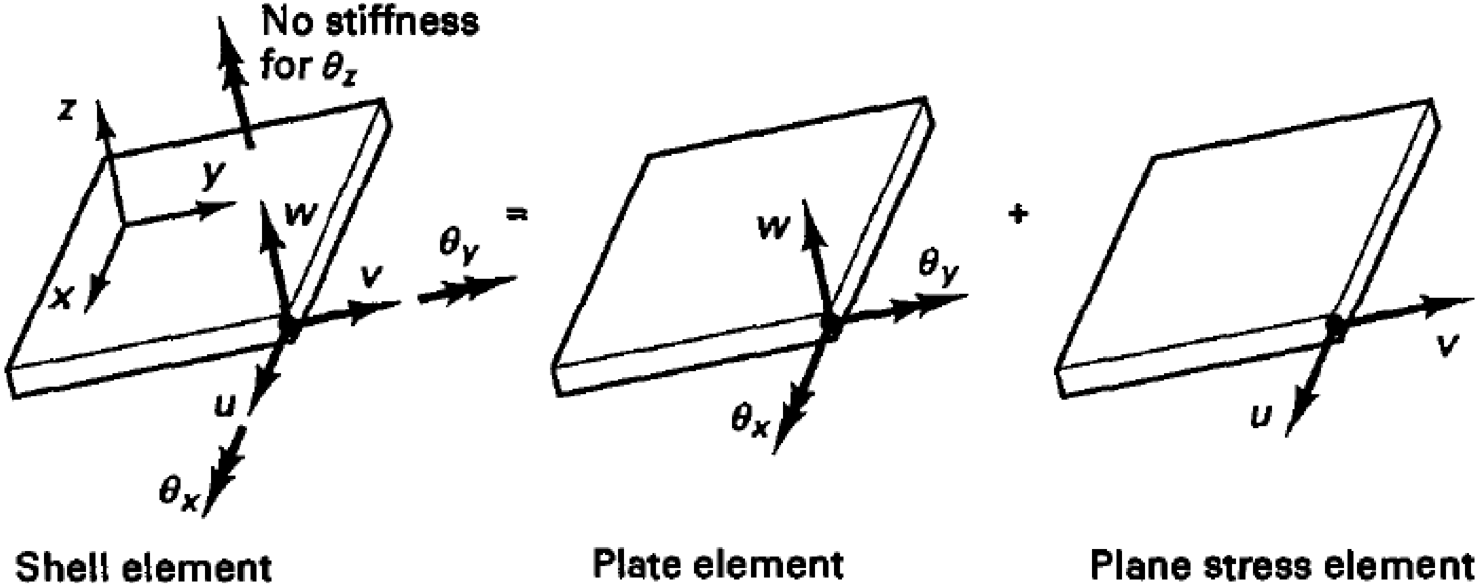
Shell element definition



$\tau_{zz} = 0$ (stress perpendicular to the midsurface)

Content

- Formulation of the plane stress element

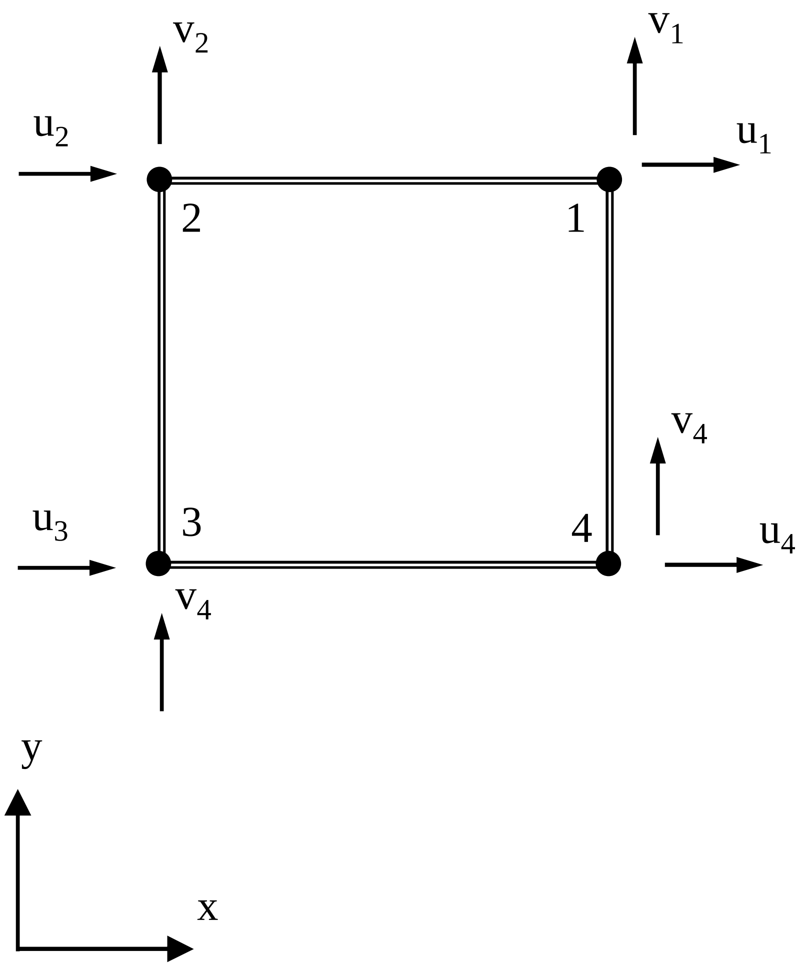
- Formulation of the plate element

- Rotational stiffness perpendicular to the element surface

- The patch test and the incompatible displacement modes

- The general shell element

Plane stress element (4 nodes)



Linear static analysis

$$K\,\delta = F$$

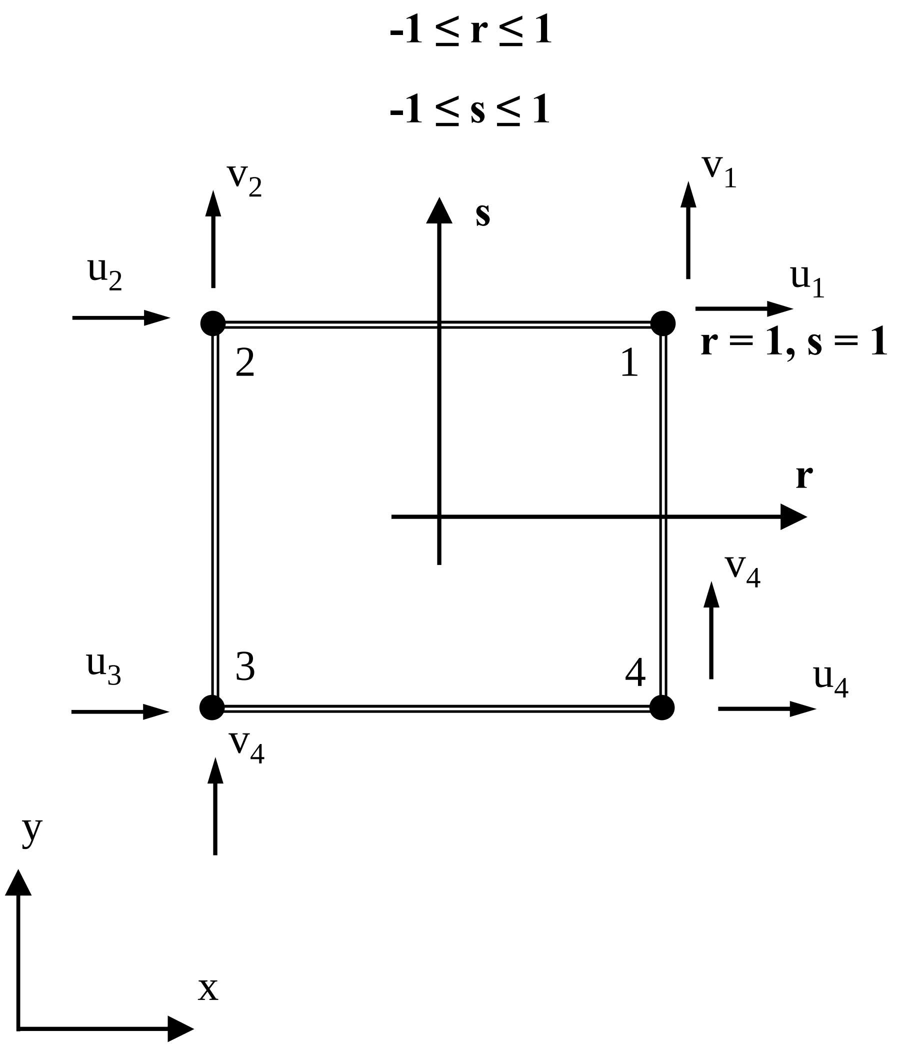
Stiffness matrix (K)

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdot & \cdot & k_{17} & k_{18} \\ & k_{22} & k_{23} & \cdot & \cdot & k_{27} & k_{28} \\ & & k_{33} & \cdot & & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & \text{symm} & & & & k_{77} & k_{78} \\ & & & & & & k_{88} \end{bmatrix}$$

$$\delta^T = [u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4]$$

$$F^T = [F_{u1}, F_{v1}, F_{u2}, F_{v2}, F_{u3}, F_{v3}, F_{u4}, F_{v4}]$$

Plane stress element (4 nodes)



Displacement and Coordinate Interpolation

$$u(r,s) = \sum_{i=1}^{4} \mathbf{h_i}\,(r,s)\,u_i$$

$$v(r,s) = \sum_{i=1}^{4} \mathbf{h_i}\,(r,s)\,v_i$$

$$\mathbf{h_1 = (1+r)\,(1+s)\,/\,4}$$

$$\mathbf{h_2 = (1-r)\,(1+s)\,/\,4}$$

$$\mathbf{h_3 = (1-r)\,(1-s)\,/\,4}$$

$$\mathbf{h_4 = (1+r)\,(1-s)\,/\,4}$$

Plate and general shell element

Target:

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} \, dV$$

Volume evaluated in natural coordinates

$$dV = \det \mathbf{J} \, dr \, ds \, dt$$

$$\mathbf{K} = \int_V \mathbf{F} \, dr \, ds \, dt \quad \text{where} \quad \mathbf{F} = \mathbf{B}^T \mathbf{C} \mathbf{B} \det \mathbf{J}$$

by using numerical integration instead of explicit integration,

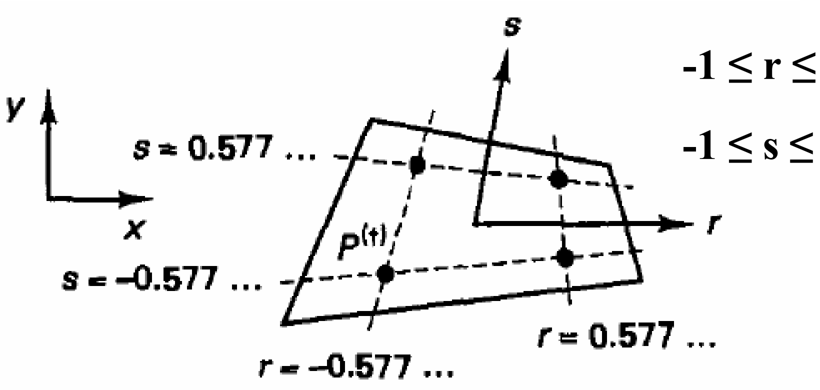
$$\mathbf{K} = \sum_{i,j} t_{ij} \alpha_{ij} \mathbf{F}_{ij}$$

where t_{ij} = thickness at the integration point

α_{ij} = weighting factor

$$\mathbf{F}_{ij} = \mathbf{B}_{ij}^T \mathbf{C} \mathbf{B}_{ij} \det \mathbf{J}_{ij}$$

Integration points (r_i, s_j) to evaluate F_{ij} of a 4-node plane stress element



$-1 \leq r \leq 1$

$-1 \leq s \leq 1$

B_{ij} , J_{ij} , C are unknowns ?????

Plate and general shell element

Matrixes $\mathbf{J}_{ij}$ and $\mathbf{B}_{ij}$

$$\boldsymbol{\epsilon}^T = [\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy}]$$

$$\boldsymbol{\epsilon} = \mathbf{B}\hat{\mathbf{u}}$$

$$\epsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}; \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x}$$

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \quad \text{or} \quad \frac{\partial}{\partial \mathbf{r}} = \mathbf{J}\frac{\partial}{\partial \mathbf{x}}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}_{\substack{\text{at } r=r_i \\ s=s_j}} = \mathbf{J}_{ij}^{-1} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix}_{\substack{\text{at } r=r_i \\ s=s_j}}$$

<u>Shape functions to calculate the Jacobian (J_{ij})</u>

$$x(r,s) = \sum_{i=1}^{4} \mathbf{h_i}(r,s)\, x_i$$

$$\frac{\partial x}{\partial r} = \frac{1}{4}(1+s)x_1 - \frac{1}{4}(1+s)x_2 - \frac{1}{4}(1-s)x_3 + \frac{1}{4}(1-s)x_4$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x}$$

$$\mathbf{J}_{ij}^{-1}$$

<u>Shape functions to interpolate displacements</u>

$$u(r,s) = \sum_{i=1}^{4} \mathbf{h_i}(r,s)\, u_i$$

$$\frac{\partial u}{\partial r} = \frac{1}{4}(1+s)u_1 - \frac{1}{4}(1+s)u_2 - \frac{1}{4}(1-s)u_3 + \frac{1}{4}(1-s)u_4$$

Plate and general shell element

$$\boldsymbol{\epsilon}^T = [\epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{xy}]$$

$$\boldsymbol{\epsilon} = \mathbf{B}\hat{\mathbf{u}}$$

$$\epsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}; \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}_{\substack{\text{at } r=r_i \\ s=s_j}} = \frac{1}{4}\mathbf{J}_{ij}^{-1}\begin{bmatrix} 1+s_j & 0 & -(1+s_j) & 0 & -(1-s_j) & 0 & 1-s_j & 0 \\ 1+r_i & 0 & 1-r_i & 0 & -(1-r_i) & 0 & -(1+r_i) & 0 \end{bmatrix}\hat{\mathbf{u}}$$

and

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}_{\substack{\text{at } r=r_i \\ s=s_j}} = \frac{1}{4}\mathbf{J}_{ij}^{-1}\begin{bmatrix} 0 & 1+s_j & 0 & -(1+s_j) & 0 & -(1-s_j) & 0 & 1-s_j \\ 0 & 1+r_i & 0 & 1-r_i & 0 & -(1-r_i) & 0 & -(1+r_i) \end{bmatrix}\hat{\mathbf{u}}$$

(

$$\begin{vmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{vmatrix}_{3\text{x}1} = \underset{3\text{x}8}{B_{ij}} \; * \; \underset{8\text{x}1}{\hat{\mathbf{u}}^T} = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4]$$

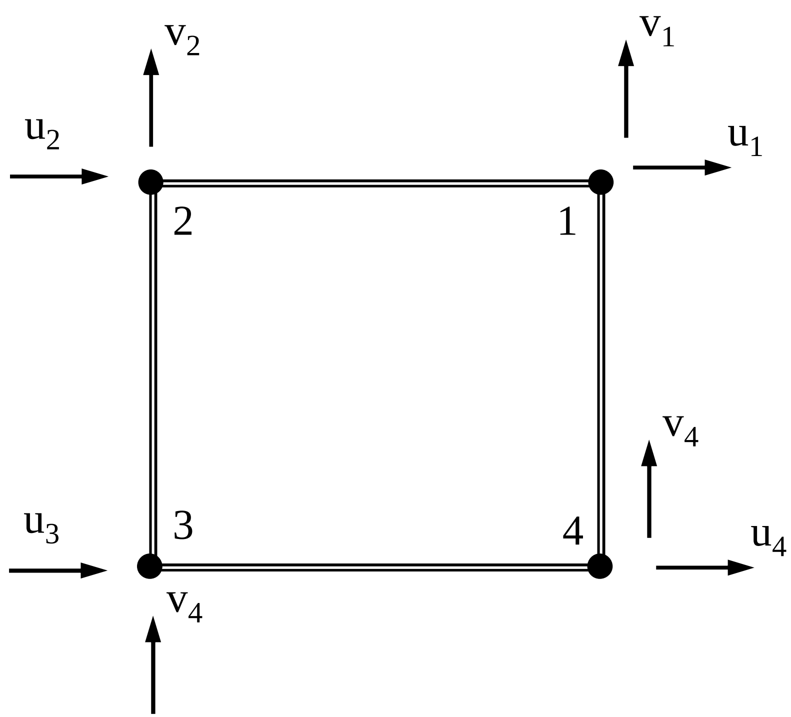
B_{ij} evaluated at the integration points (r_i,s_j)

Plate and general shell element

Plane stress and isotropic material

$$\underset{3x3}{\text{Material matrix } \mathbf{C}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\underset{8x8}{\mathbf{K}} = \int_V \underset{8x3}{\mathbf{B}^T}\,\underset{3x3}{\mathbf{C}}\,\underset{3x8}{\mathbf{B}}\,dV$$



Stiffness matrix (K)

$$\underset{8x8}{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdot & \cdot & k_{17} & k_{18} \\ & k_{22} & k_{23} & \cdot & \cdot & k_{27} & k_{28} \\ & & k_{33} & \cdot & & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & \text{symm} & & & & k_{77} & k_{78} \\ & & & & & & k_{88} \end{bmatrix}$$

Plate and general shell element

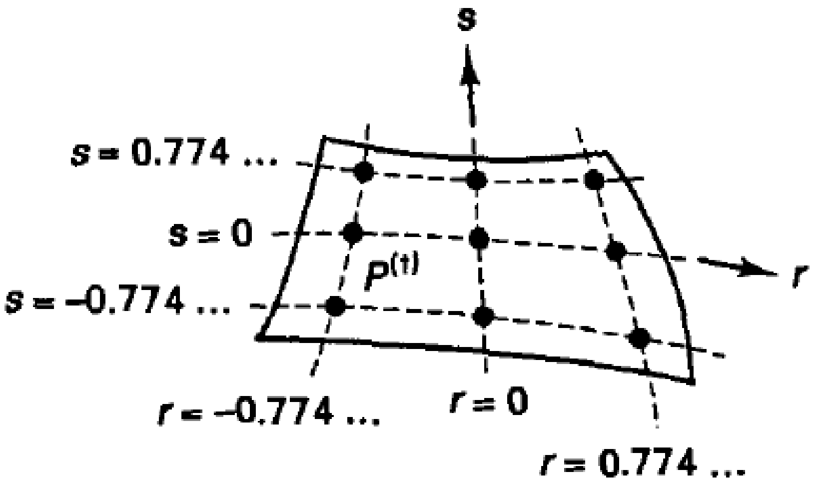
Calculation of stress

$$\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\epsilon} \qquad [\tau_{xx}\ \tau_{yy}\ \tau_{xy}]^T = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} [\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}]^T$$

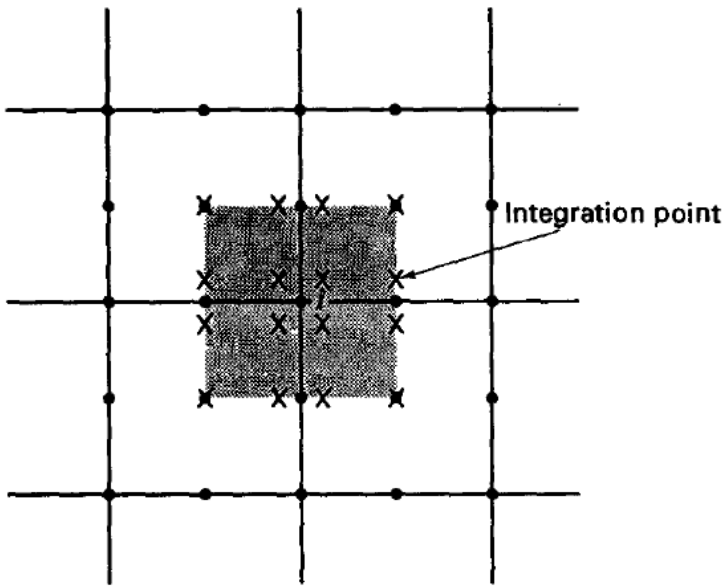
$$\boldsymbol{\epsilon} = \mathbf{B}\hat{\mathbf{u}}$$

Material matrix **C**

Strains and stresses are calculated at the integration points

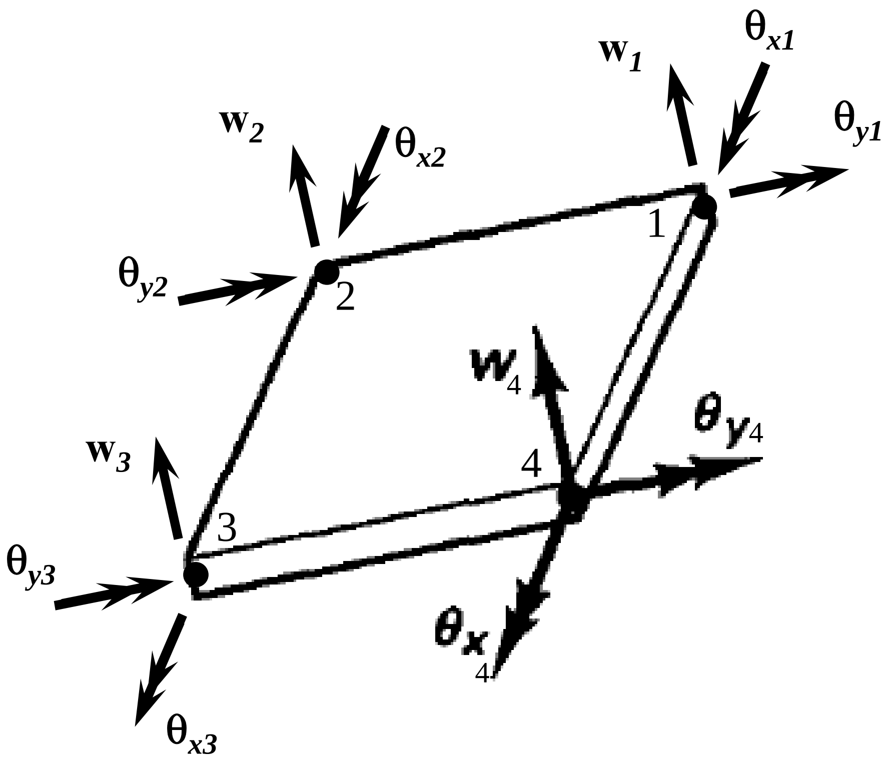


$$(\tau^h)_{\text{impr.}} = \sum_{i=1}^{9} h_i (\tau^h)^i_{\text{mean}}$$



This example correspond to a 9-node element with 3x3 Gauss points

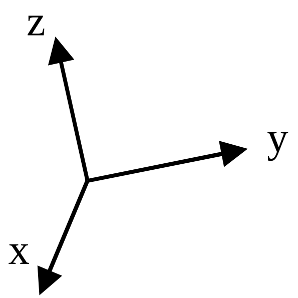
Plate bending element (4 nodes)



$$K\,\delta = F$$

Stiffness matrix (K)

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdot & \cdot & k_{111} & k_{112} \\ & k_{22} & k_{23} & \cdot & \cdot & k_{211} & k_{212} \\ & & k_{33} & \cdot & & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & \text{symm} & & & & k_{1111} & k_{1112} \\ & & & & & & k_{1212} \end{bmatrix}$$



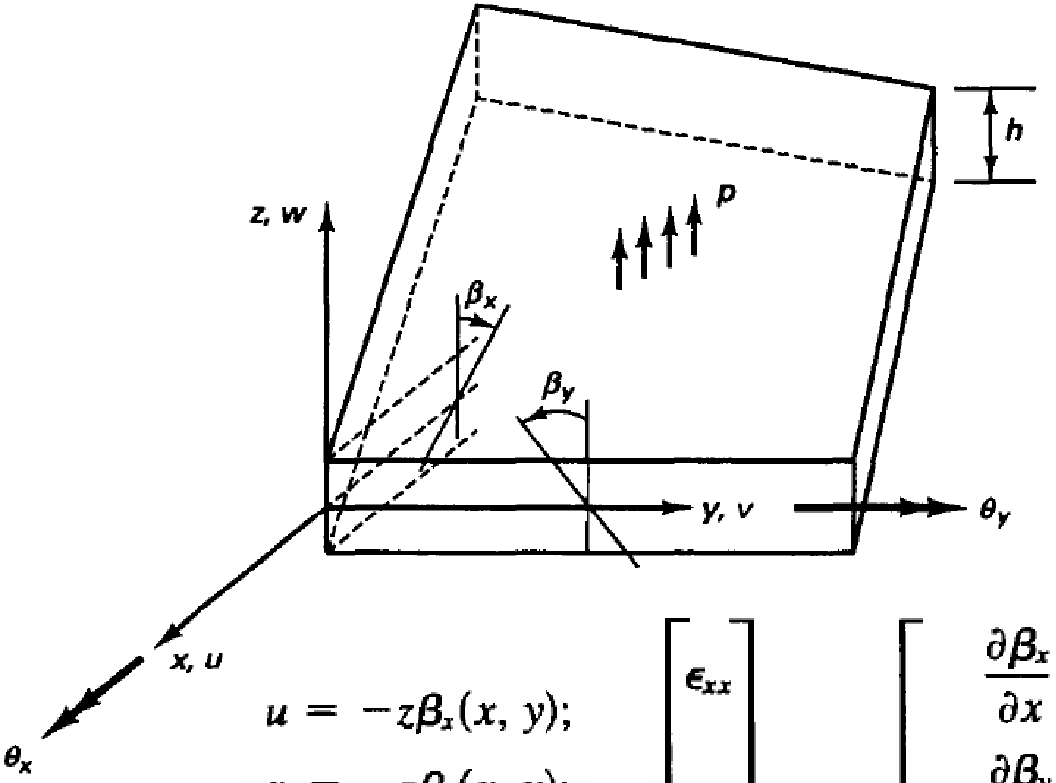
$$\delta^T = [w_1, \theta_{x1}, \theta_{y1}, w_2, \theta_{x2}, \theta_{y2}, w_3, \theta_{x3}, \theta_{y3}, w_4, \theta_{x4}, \theta_{y4}]$$

$$F^T = [F_{z1}, M_{x1}, M_{y1}, F_{z2}, M_{x2}, M_{y2}, F_{z3}, M_{x3}, M_{y3}, F_{z4}, M_{x4}, M_{y4}]$$

Plate bending element (4 nodes)

$-1 \le r \le 1$

$-1 \le s \le 1$



<u>Displacement, Rotation and Coordinate Interpolation</u>

$$w(r,s) = \sum_{i=1}^{4} \mathbf{h_i}\,(r,s)\; w_i$$

$$\beta_x(r,s) = -\sum_{i=1}^{4} \mathbf{h_i}\,(r,s)\; \theta_{yi}$$

$$\beta_y(r,s) = \sum_{i=1}^{4} \mathbf{h_i}\,(r,s)\; \theta_{xi}$$

$h_1 = (1+r)\,(1+s)\,/\,4$
h_2 …

$$u = -z\beta_x(x, y); \quad v = -z\beta_y(x, y); \quad w = w(x, y)$$

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = -z \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{bmatrix}$$

$$\begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{bmatrix}$$

Plate and general shell element

$$\int_A \bar{\boldsymbol{\kappa}}^T \mathbf{C}_b \boldsymbol{\kappa} \, dA + \int_A \bar{\boldsymbol{\gamma}}^T \mathbf{C}_s \boldsymbol{\gamma} \, dA = \int_A \bar{w} p \, dA$$

Area evaluated in natural coordinates

$$dA = \det \mathbf{J} \, dr \, ds$$

Bending moment

$$\boldsymbol{\kappa} = \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{bmatrix} \qquad \mathbf{C}_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Shear force

$$\boldsymbol{\gamma} = \begin{bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{bmatrix}$$

$$\mathbf{C}_s = \frac{Ehk}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

k (shear correction factor for rectangular cross sections) = 5/6

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x}$$

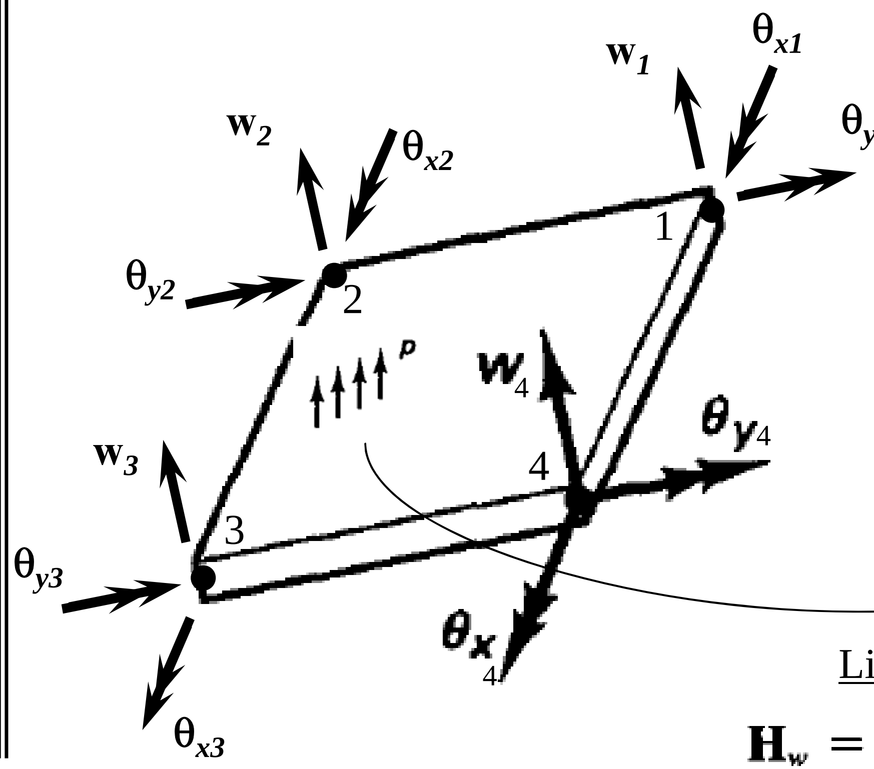
$$\begin{bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{4} \mathbf{J}_{ij}^{-1} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

Plate and general shell element

$$\underset{3x1}{\boldsymbol{\kappa}(r,s)} = \underset{3x12}{\mathbf{B}_\kappa}\,\underset{12x1}{\hat{\mathbf{u}}} \qquad \underset{2x1}{\boldsymbol{\gamma}(r,s)} = \underset{2x12}{\mathbf{B}_\gamma}\,\underset{12x1}{\hat{\mathbf{u}}} \qquad \underset{12x1}{\hat{\mathbf{u}}^T} = [w_1 \quad \theta_x^1 \quad \theta_y^1;\, w_2 \quad \ldots \quad \theta_y^4]$$

$$\underset{12x12}{\mathbf{K}} = \int_{-1}^{+1}\int_{-1}^{+1} (\underset{12x3}{\mathbf{B}_\kappa^T}\,\underset{3x3}{\mathbf{C}_b}\,\underset{3x12}{\mathbf{B}_\kappa} + \underset{12x2}{\mathbf{B}_\gamma^T}\,\underset{2x2}{\mathbf{C}_s}\,\underset{2x12}{\mathbf{B}_\gamma}) \det \mathbf{J}\, dr\, ds$$

B_k and B_γ are valuated at the integration points (r_i,s_j)



$$\underset{12x1}{F_{ext}} = \underset{12x12}{K} * \underset{12x1}{u}$$

$$F_{ext} = \int_{-1}^{+1}\int_{-1}^{+1} \mathbf{H}_w^T p \quad \det \mathbf{J}\, dr\, ds$$

Linear shape functions to interpolate area forces

$$\mathbf{H}_w = \tfrac{1}{4}[(1 + r)(1 + s) \quad 0 \quad 0 \quad | \quad \ldots \quad 0]$$

Shear locking elimination

$$w(r,s) = \sum_{i=1}^{4} \mathbf{h_i}\,(r,s)\, w_i$$

$$\beta_x(r,s) = -\sum_{i=1}^{4} \mathbf{h_i}\,(r,s)\, \theta_{yi}$$

$$\beta_y(r,s) = \sum_{i=1}^{4} \mathbf{h_i}\,(r,s)\, \theta_{xi}$$

Shear strains

$$\begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{bmatrix}$$

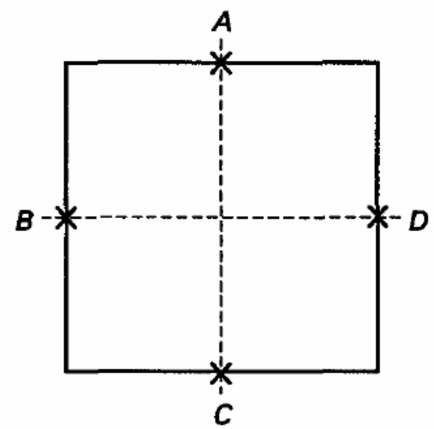
Incorrect interpolation at the corner nodes →

$\mathbf{h_1 = (1+r)\,(1+s)\,/\,4}$
$\mathbf{h_2}$ **…**

Correct evaluation that eliminates the shear locking in the 4-node plate bending element

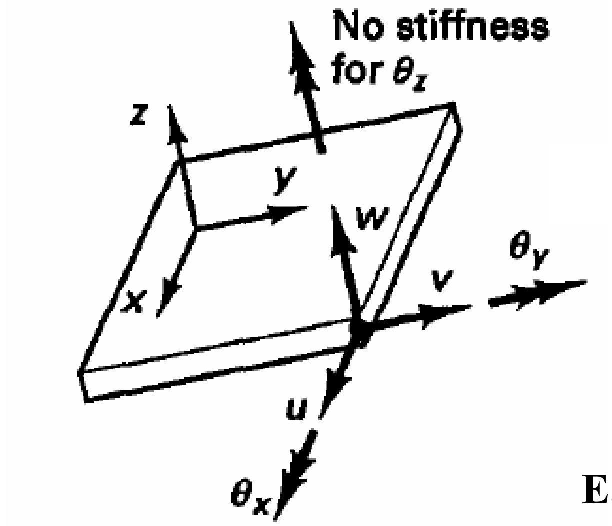
$$\gamma_{xz} = \gamma_{rz} \sin \beta - \gamma_{sz} \sin \alpha$$

$$\gamma_{yz} = -\gamma_{rz} \cos \beta + \gamma_{sz} \cos \alpha$$



where α and β are the angles between the r and x axes and s and x axes.

Shell element



$$\underset{20\times20}{\tilde{\mathbf{K}}_S} = \begin{bmatrix} \underset{12\times12}{\tilde{\mathbf{K}}_B} & \mathbf{0} \\ \mathbf{0} & \underset{8\times8}{\tilde{\mathbf{K}}_M} \end{bmatrix}$$

Rotational stiffness perpendicular to the element surface is not defined

$$\underset{24\times24}{\tilde{\mathbf{K}}_S^*} = \begin{bmatrix} \underset{20\times20}{\tilde{\mathbf{K}}_S} & \mathbf{0} \\ \mathbf{0} & \underset{4\times4}{\mathbf{0}} \end{bmatrix}$$

Easy solution

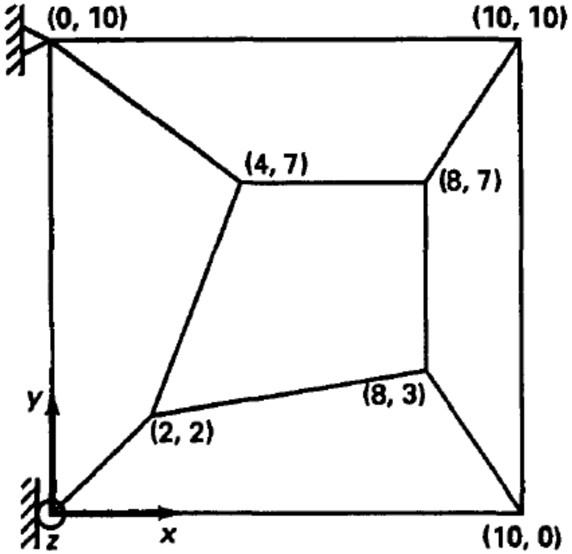
$$\tilde{\mathbf{K}}_S^{*\prime} = \begin{bmatrix} \underset{20\times20}{\tilde{\mathbf{K}}_S} & \mathbf{0} \\ \mathbf{0} & \underset{4\times4}{k\mathbf{I}} \end{bmatrix}$$

where k is about one-thousandth of the smallest diagonal element of $\tilde{\mathbf{K}}_S$.

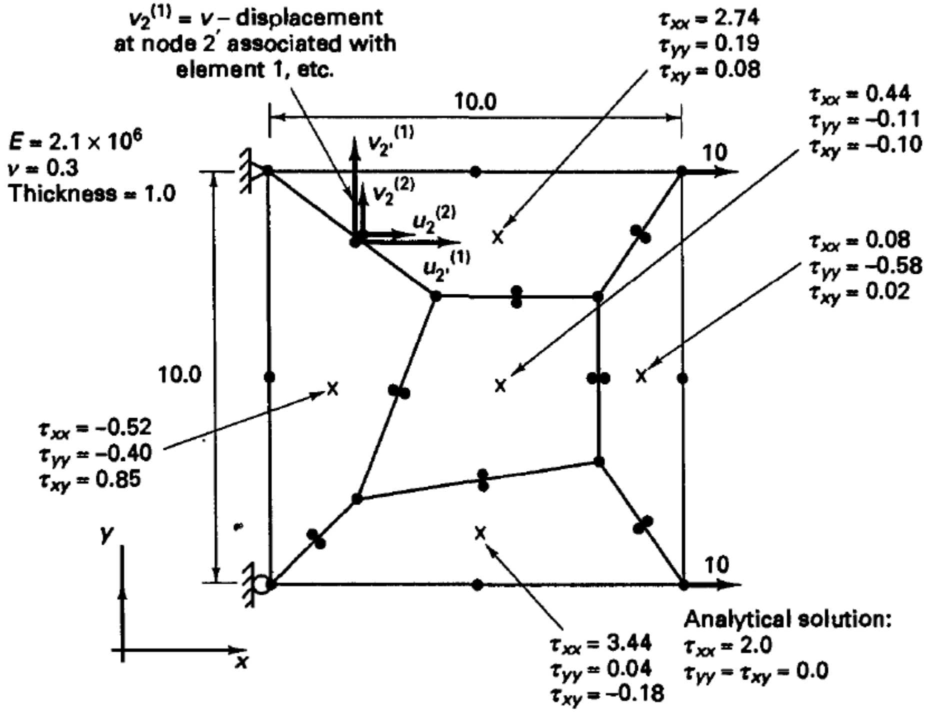
One elegant solutions is to add a "real" rotational stiffness for θ_z,

A. Ibrahimbegovic, R. Taylor, L. Wilson. "A robust quadrilateral membrane finite element with drilling degrees of freedom". *Int. J. Num. Met. Engr* 1990: 30: 445-457.

Patch elements to test the performance of the shell element



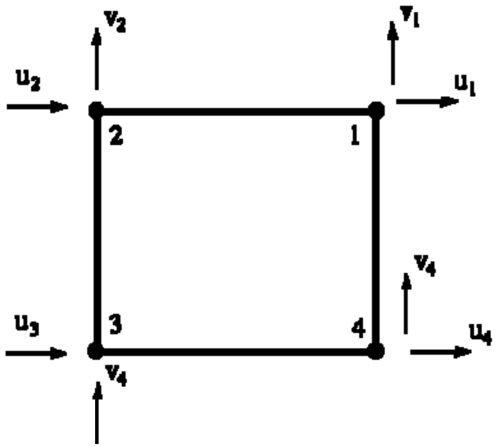
Test of the plane stress element



It shows incompatibility of displacements

To overcome this deficiency, incompatible displacement modes are added

Incompatible displacement modes



$$\hat{\mathbf{u}}^* = \begin{bmatrix} \hat{\mathbf{u}} \\ \boldsymbol{\alpha} \end{bmatrix}$$

$$\hat{\mathbf{u}}^T = [u_1 \quad \dots \quad u_4 \,\vdots\, v_1 \quad \dots \quad v_4]$$

$$\boldsymbol{\alpha}^T = [\alpha_1 \quad \dots \quad \alpha_4]$$

$$\boldsymbol{\epsilon} = [\mathbf{B} \,:\, \mathbf{B}_{IC}] \begin{bmatrix} \hat{\mathbf{u}} \\ \cdots \\ \boldsymbol{\alpha} \end{bmatrix}$$

Displacement interpolation functions

$$u = \sum_{i=1}^{4} h_i u_i + \alpha_1 \phi_1 + \alpha_2 \phi_2$$

$$v = \sum_{i=1}^{4} h_i v_i + \alpha_3 \phi_1 + \alpha_4 \phi_2$$

$$\phi_1 = (1 - x^2); \phi_2 = (1 - y^2)$$

Stiffness matrix (K)

$$\begin{bmatrix} \int_V \mathbf{B}^T \mathbf{C} \mathbf{B}\, dV & \int_V \mathbf{B}^T \mathbf{C} \mathbf{B}_{IC}\, dV \\ \int_V \mathbf{B}_{IC}^T \mathbf{C} \mathbf{B}\, dV & \int_V \mathbf{B}_{IC}^T \mathbf{C} \mathbf{B}_{IC}\, dV \end{bmatrix}_{12x12} \begin{bmatrix} \hat{\mathbf{u}} \\ \boldsymbol{\alpha} \end{bmatrix}_{12x1} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}_{12x1}$$

Correction of the incompatible strain-displacement matrix

$$\mathbf{B}_{IC}^{new} = \mathbf{B}_{IC} + \mathbf{B}_{IC}^{C}$$

$$\mathbf{B}_{IC}^{C} = -\frac{1}{V} \int_V \mathbf{B}_{IC}\, dV$$

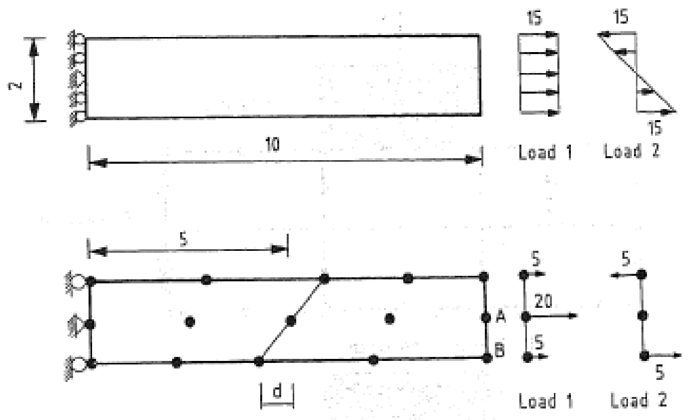
Stiffness matrix (8x8) is obtained by applying static condensation

$$\underbrace{(k_{CC} - k_{CI}\, k_{II}^{-1}\, k_{IC})}_{8x8} \; \underbrace{u}_{8x1} = \underbrace{R}_{8x1}$$

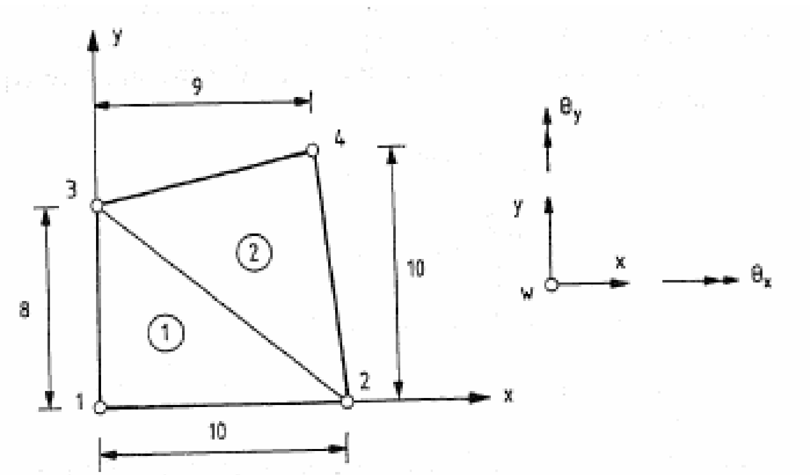
Higher order patch tests

R. L. Taylor, J. C. SIMO, O. C. Zienkiewicz, A. C. H. Chan. "The path test – A condition for assessing FEM convergence". *Int. J. Num. Met. Engr* 1986: 22: 39-62.

For the plane stress element



For the plate bending element



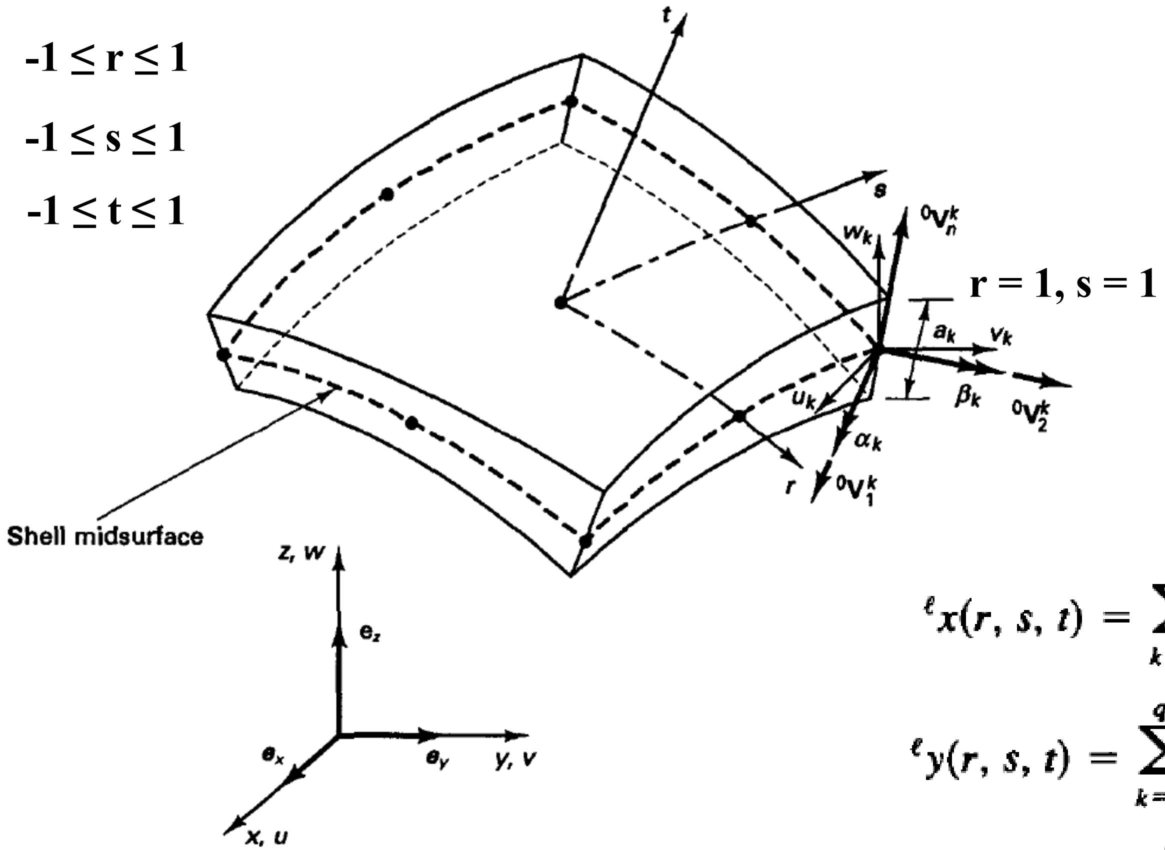
General Shell element

Analysis of complex shell geometries

-1 ≤ r ≤ 1

-1 ≤ s ≤ 1

-1 ≤ t ≤ 1



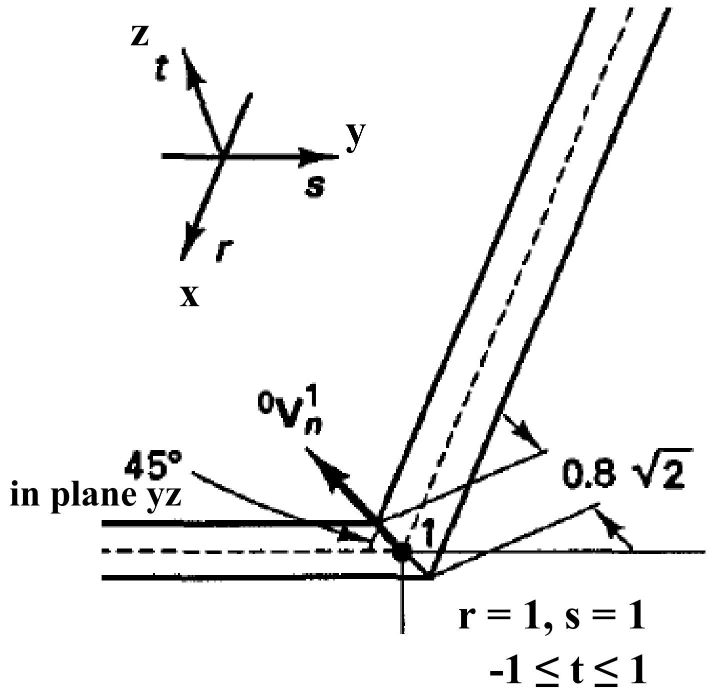
$$ {}^{\ell}x(r, s, t) = \sum_{k=1}^{q} h_k \, {}^{\ell}x_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{nx}^k $$

$$ {}^{\ell}y(r, s, t) = \sum_{k=1}^{q} h_k \, {}^{\ell}y_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{ny}^k $$

$$ {}^{\ell}z(r, s, t) = \sum_{k=1}^{q} h_k \, {}^{\ell}z_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{nz}^k $$

General Shell element

Coordinate interpolation



$$^{\ell}x(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{\ell}x_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{nx}^{k}$$

$$^{\ell}y(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{\ell}y_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{ny}^{k}$$

$$^{\ell}z(r,s,t) = \sum_{k=1}^{q} h_k \, {}^{\ell}z_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{nz}^{k}$$

$$^{0}\mathbf{V}_n^1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

<u>For t = 1</u>

$x = x_1$

$y = y_1 + 0.5 * 0.8\sqrt{0.2} * (-1/\sqrt{2})$

$z = z_1 + 0.5 * 0.8\sqrt{0.2} * (1/\sqrt{2})$

<u>For t = -1</u>

$x = x_1$

$y = y_1 + -0.5 * 0.8\sqrt{0.2} * (-1/\sqrt{2})$

$z = z_1 + -0.5 * 0.8\sqrt{0.2} * (1/\sqrt{2})$

General Shell element

Displacement interpolation

$$u(r,s,t) = \sum_{k=1}^{q} h_k u_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{nx}^k$$

$$v(r,s,t) = \sum_{k=1}^{q} h_k v_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{ny}^k$$

$$w(r,s,t) = \sum_{k=1}^{q} h_k w_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k V_{nz}^k$$

Strain – displacement matrix B(r,s,t)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x}$$

$\mathbf{J}_{ij}^{-1}$

$$\begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \\ \dfrac{\partial u}{\partial t} \end{bmatrix} = \sum_{k=1}^{q} \begin{bmatrix} \dfrac{\partial h_k}{\partial r}[1 & t g_{1x}^k & t g_{2x}^k] \\ \dfrac{\partial h_k}{\partial s}[1 & t g_{1x}^k & t g_{2x}^k] \\ h_k \; [0 & g_{1x}^k & g_{2x}^k] \end{bmatrix} \begin{bmatrix} u_k \\ \alpha_k \\ \beta_k \end{bmatrix}$$

B(r,s,t)

General Shell element

Stress-strain law

$$\boldsymbol{\tau} = \mathbf{C}_{sh}\boldsymbol{\epsilon}$$

$$\boldsymbol{\tau}^T = [\tau_{xx} \quad \tau_{yy} \quad \tau_{zz} \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}]$$

$$\boldsymbol{\epsilon}^T = [\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}]$$

$$\mathbf{C}_{sh} = \mathbf{Q}_{sh}^T \left(\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & \frac{1-\nu}{2} & 0 & 0 \\ & \text{Symmetric} & & & k\frac{1-\nu}{2} & 0 \\ & & & & & k\frac{1-\nu}{2} \end{bmatrix} \right) \mathbf{Q}_{sh}$$

$$\mathbf{Q}_{sh} = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & l_1 m_1 & m_1 n_1 & n_1 l_1 \\ l_2^2 & m_2^2 & n_2^2 & l_2 m_2 & m_2 n_2 & n_2 l_2 \\ l_3^2 & m_3^2 & n_3^2 & l_3 m_3 & m_3 n_3 & n_3 l_3 \\ 2l_1 l_2 & 2m_1 m_2 & 2n_1 n_2 & l_1 m_2 + l_2 m_1 & m_1 n_2 + m_2 n_1 & n_1 l_2 + n_2 l_1 \\ 2l_2 l_3 & 2m_2 m_3 & 2n_2 n_3 & l_2 m_3 + l_3 m_2 & m_2 n_3 + m_3 n_2 & n_2 l_3 + n_3 l_2 \\ 2l_3 l_1 & 2m_3 m_1 & 2n_3 n_1 & l_3 m_1 + l_1 m_3 & m_3 n_1 + m_1 n_3 & n_3 l_1 + n_1 l_3 \end{bmatrix}$$

where

$$l_1 = \cos(\mathbf{e}_x, \mathbf{e}_{\bar{r}}); \quad m_1 = \cos(\mathbf{e}_y, \mathbf{e}_{\bar{r}}); \quad n_1 = \cos(\mathbf{e}_z, \mathbf{e}_{\bar{r}})$$

$$l_2 = \cos(\mathbf{e}_x, \mathbf{e}_{\bar{s}}); \quad m_2 = \cos(\mathbf{e}_y, \mathbf{e}_{\bar{s}}); \quad n_2 = \cos(\mathbf{e}_z, \mathbf{e}_{\bar{s}})$$

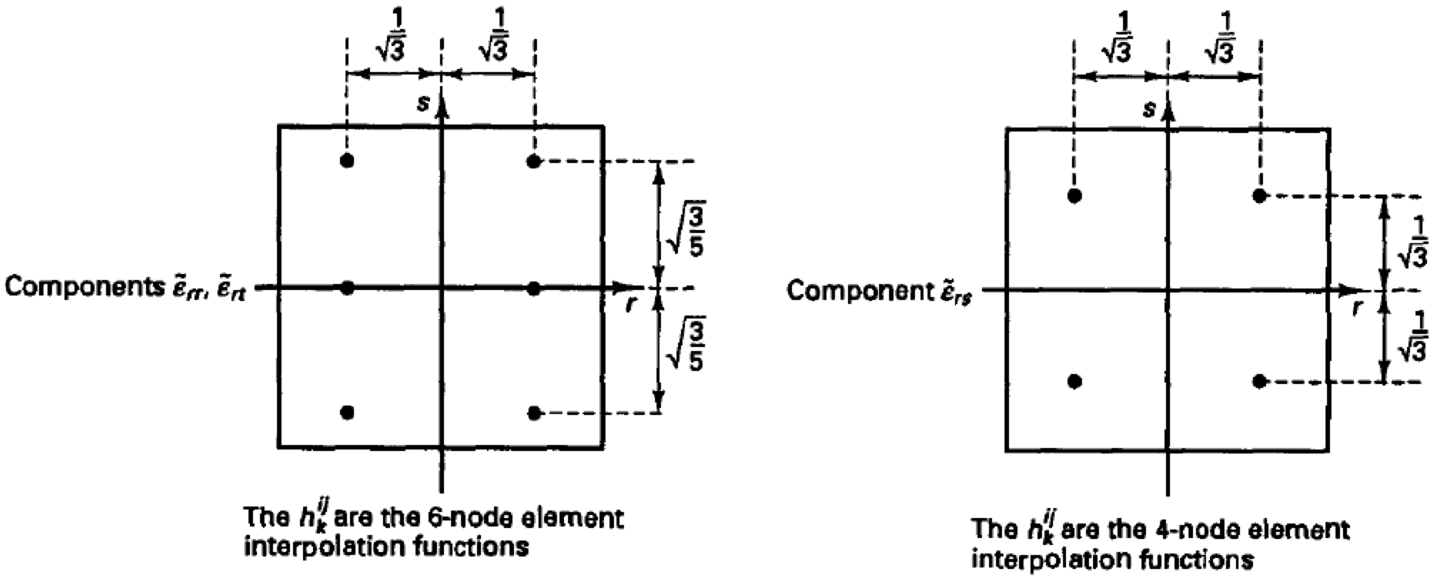
$$l_3 = \cos(\mathbf{e}_x, \mathbf{e}_t); \quad m_3 = \cos(\mathbf{e}_y, \mathbf{e}_t); \quad n_3 = \cos(\mathbf{e}_z, \mathbf{e}_t)$$

General Shell element

Shear locking

$$\boldsymbol{\epsilon} = \underbrace{\tilde{\epsilon}_{rr}\mathbf{g}^r\mathbf{g}^r + \tilde{\epsilon}_{ss}\mathbf{g}^s\mathbf{g}^s + \tilde{\epsilon}_{rs}(\mathbf{g}^r\mathbf{g}^s + \mathbf{g}^s\mathbf{g}^r)}_{\text{in-layer strains}} + \underbrace{\tilde{\epsilon}_{rt}(\mathbf{g}^r\mathbf{g}^t + \mathbf{g}^t\mathbf{g}^r) + \tilde{\epsilon}_{st}(\mathbf{g}^s\mathbf{g}^t + \mathbf{g}^t\mathbf{g}^s)}_{\text{transverse shear strains}}$$

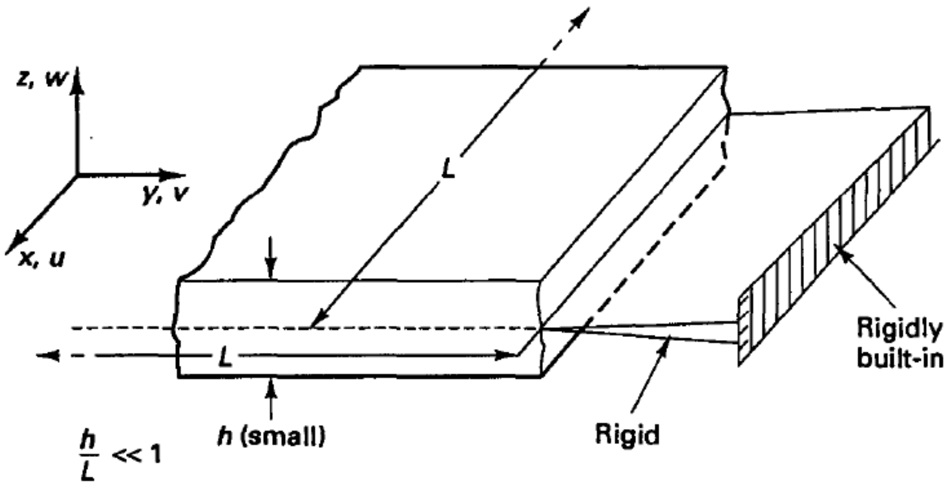
9-node shell element

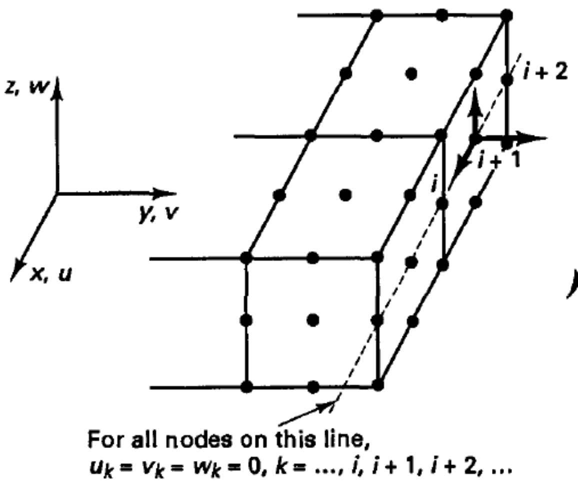


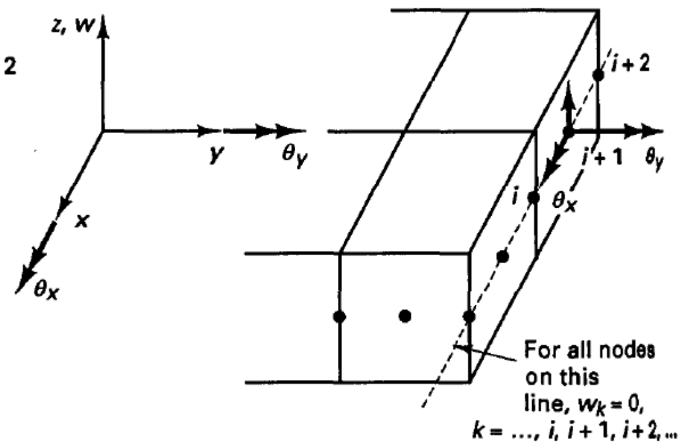
(a) MITC9 shell element

Plate and general shell element

Boundary conditions







Solid element

Shell element

CONCLUSIONS

- To select properly plane stress, bending plate or shell.

- Compatibility of element degrees of freedom.

- Methodology to overcome shear locking.

- Evaluate the element performance of FEM programs or codes using the patch tests.

- To use shell elements instead of 3D elements

- To use quadrilateral shell elements instead of triangular.

THANK YOU