# Formulation of Structural Elements

Beam Elements

and Axial-symmetrical Shell Elements

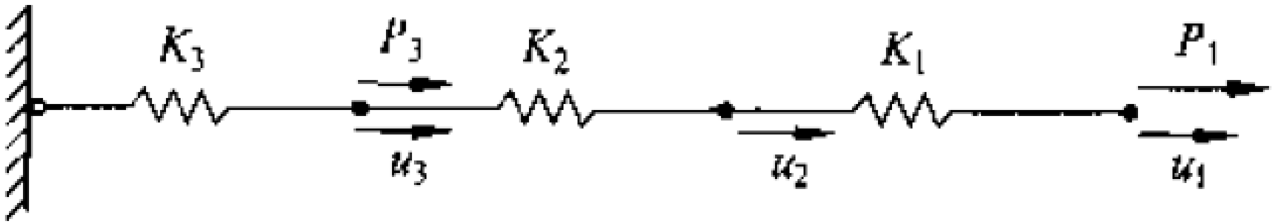
# Contents

- General
- Beam elements
- Axial-symmetrical Shell Elements

# General

Why not use the 2-D or 3-D continuum elements to analyze all the structures?

# Example




$K_1$=$K_2$=100000, $K_3$=1, $P_1$=$P_3$=1, $P_2$=0.

Calculate the $u_1$, $u_2$, $u_3$ with 5 significant digits.

$$\begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1+K_2 & -K_2 \\ 0 & -K_2 & K_2+K_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

$$\begin{bmatrix} 100\,000 & -100\,000 & 0 \\ -100\,000 & 200\,000 & -100\,000 \\ 0 & -100\,000 & 100\,000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.0 \\ 1.0 \end{Bmatrix}$$

$$\begin{bmatrix} 100\,000 & -100\,000 & 0 \\ 0 & 100\,000 & -100\,000 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 1.0 \\ 2.0 \end{Bmatrix}$$

$u_1 = 2.000\,02, \quad u_2 = 2.000\,01, \quad u_3 = 2.000\,00$

# Simplification

- Master-slave degrees of freedom

$$K_1 = K_2 \gg K_3 \qquad u_1 = u_2 = u_3$$

$$K_3 u_3 = P_1 + P_3$$

$$u_1 = u_2 = u_3 = 2.0$$

- Relative degrees of freedom

$$u_1 = u_2 + \Delta_1 \qquad u_2 = u_3 + \Delta_2$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ u_3 \end{pmatrix} = [\boldsymbol{T}] \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ u_3 \end{pmatrix}$$

$$\begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_1 \\ P_1 + P_3 \end{pmatrix}$$

$$u_1 = 2.000\,0 \quad u_2 = 2.000\,0 \quad u_3 = 2.0$$

# Basic Equations

$$\{u\} = [N]\{U\}$$

*compatibility equation* $$\{\varepsilon\} = [B]\{U\}$$

*constitutive equation* $$\{\sigma\} = [D]\{\varepsilon\}$$

*equilibrium equation* $$[k] = \sum_e [\int_{V^{(e)}} [B^{(e)}]^T [D^{(e)}][B^{(e)}] dV] = \sum_e [k^{(e)}]$$

$$[k]\{U\} = [F]$$

# Beam elements

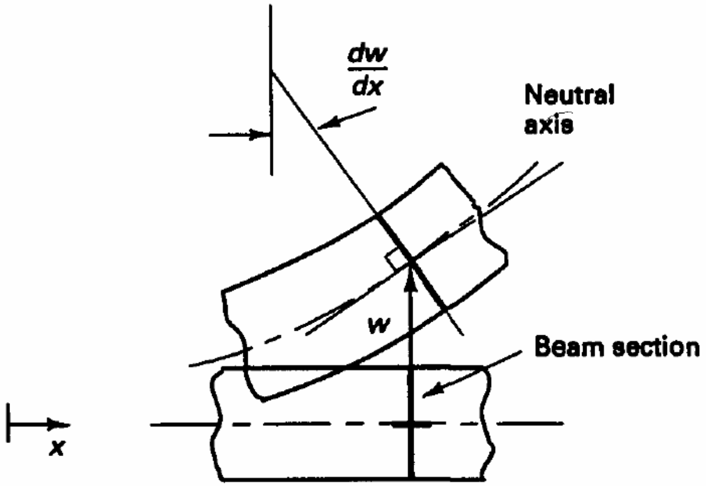
- Beam

  A beam is a structure that carries load primarily in bending (flexure).

- Beam elements

  In the finite element analysis, the element which has the similar condition with the beam is defined as beam element.

# Elements whose deformations exclude the shear effect



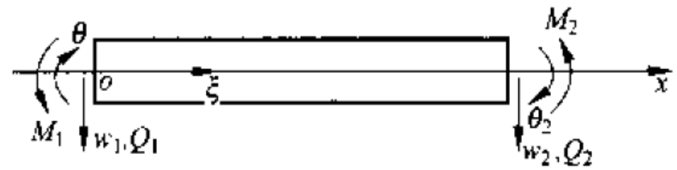
Deformation of cross section

Assumption: a normal to the midsurface (neutral axis) of the beam remains straight during deformation and its angular rotation is equal to the slope of the beam midsurface.

$$\text{shear strain: } \gamma = \frac{d\omega}{dx} - \theta = 0$$

$$\text{curvature: } \kappa = -\frac{d\theta}{dx} = -\frac{d^2\omega}{dx^2}$$

- Basic equations




$$\kappa = -\frac{d^2 w}{dx^2}$$

$$M = EI\kappa = -EI\frac{d^2 w}{dx^2}$$

$$Q = \frac{dM}{dx} = -EI\frac{d^3 w}{dx^3}$$

$$-\frac{dQ}{dx} = EI\frac{d^4 w}{dx^4} = q(x)$$

$$w(\xi) = \sum_{i=1}^{4} N_i(\xi) a_i = \boldsymbol{N} \boldsymbol{a}^e$$

$$\boldsymbol{N} = [N_1 \quad N_2 \quad N_3 \quad N_4]$$

$$\boldsymbol{a}^e = [w_1 \quad \theta_1 \quad w_2 \quad \theta_2]^T \quad \theta_i = \left(\frac{dw}{dx}\right)_i \quad (i = 1, 2)$$

$$N_1(\xi) = H_1^{(0)}(\xi) = 1 - 3\xi^2 + 2\xi^3$$

$$N_2(\xi) = H_1^{(1)}(\xi) = (\xi - 2\xi^2 + \xi^3) l$$

$$N_3(\xi) = H_2^{(0)}(\xi) = 3\xi^2 - 2\xi^3$$

$$N_4(\xi) = H_2^{(1)}(\xi) = (\xi^3 - \xi^2) l$$

$$\xi = \frac{x - x_1}{l} \quad (0 \leqslant \xi \leqslant 1)$$

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{P}$$

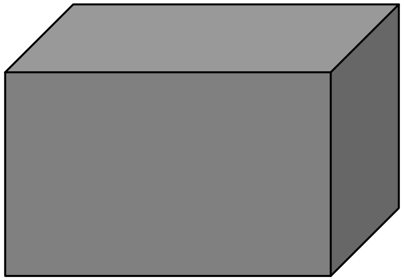
$$\boldsymbol{K} = \sum_{e}\boldsymbol{K}^{e} \qquad \boldsymbol{P} = \sum_{e}\boldsymbol{P}^{e} \qquad \boldsymbol{a} = \sum_{e}\boldsymbol{a}^{e}$$

$$\boldsymbol{K}^{e} = \int_{0}^{1} \frac{EI}{l^{3}}\left(\frac{\mathrm{d}^{2}\boldsymbol{N}}{\mathrm{d}\xi^{2}}\right)^{\mathrm{T}}\left(\frac{\mathrm{d}^{2}\boldsymbol{N}}{\mathrm{d}\xi^{2}}\right)\mathrm{d}\xi = \frac{EI}{l^{3}}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^{2} & -6l & 2l^{2} \\ -12 & -6l & 12 & -6l \\ 6l & 2l^{2} & -6l & 4l^{2} \end{bmatrix}$$

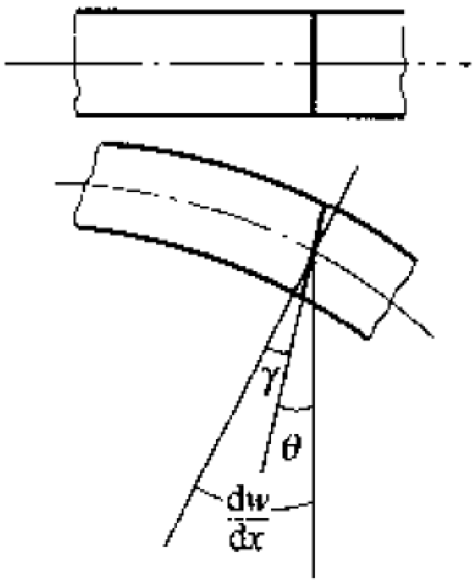
$$\boldsymbol{P}^{e} = \int_{0}^{1}\boldsymbol{N}^{\mathrm{T}} q l \,\mathrm{d}\xi + \sum_{j}\boldsymbol{N}^{\mathrm{T}}(\xi_{j})P_{j} - \sum_{k} \frac{\mathrm{d}\boldsymbol{N}^{\mathrm{T}}(\xi_{k})}{\mathrm{d}\xi}\frac{M_{k}}{l}$$

# Elements whose deformations include the shear effect

$h << l$



$h ! << l$




Assumption: a plane section originally normal to the neutral axis remains plane, but because of shear deformations this section does not remain normal to the neutral axis.

$$\text{shear strain: } \gamma = \frac{d\omega}{dx} - \theta$$

$$\text{curvature: } \kappa = -\frac{d\theta}{dx} \neq -\frac{d^2\omega}{dx^2}$$

- Timoshenko beam elements

$$w = \sum_{i=1}^{n} N_i w_i \qquad \boldsymbol{K}\boldsymbol{a} = \boldsymbol{P}$$

$$\theta = \sum_{i=1}^{n} N_i \theta_i \qquad \mathbf{K} = \sum_e \mathbf{K}^e \quad \boldsymbol{a} = \sum_e \boldsymbol{a}^e \quad \boldsymbol{P} = \sum_e \boldsymbol{P}^e$$

$$\boldsymbol{K}^e = \boldsymbol{K}_b^e + \boldsymbol{K}_s^e$$

$$\boldsymbol{K}_b^e = \frac{EIl}{2}\int_{-1}^{1} \boldsymbol{B}_b^{\mathrm{T}} \boldsymbol{B}_b \,\mathrm{d}\xi$$

$$\boldsymbol{K}_s^e = \frac{GAl}{2k}\int_{-1}^{1} \boldsymbol{B}_s^{\mathrm{T}} \boldsymbol{B}_s \,\mathrm{d}\xi$$

$$\boldsymbol{B}_b = [\boldsymbol{B}_{b1} \quad \boldsymbol{B}_{b2} \quad \cdots \quad \boldsymbol{B}_{bn}]$$

$$\boldsymbol{B}_s = [\boldsymbol{B}_{s1} \quad \boldsymbol{B}_{s2} \quad \cdots \quad \boldsymbol{B}_{sn}]$$

$$\boldsymbol{B}_{bi} = \begin{bmatrix} 0 & -\frac{\mathrm{d}N_i}{\mathrm{d}x} \end{bmatrix}$$

$$\boldsymbol{B}_{si} = \begin{bmatrix} \frac{\mathrm{d}N_i}{\mathrm{d}x} & -N_i \end{bmatrix} \quad (i = 1, 2, \cdots, n)$$

$$\boldsymbol{P}^e = \frac{l}{2}\int_{-1}^{1} \boldsymbol{N}^{\mathrm{T}} \begin{bmatrix} q \\ 0 \end{bmatrix} \mathrm{d}\xi + \sum_j \boldsymbol{N}^{\mathrm{T}}(\xi_j) \begin{bmatrix} p_j \\ 0 \end{bmatrix} - \sum_k \boldsymbol{N}(\xi_k) \begin{bmatrix} 0 \\ M_k \end{bmatrix}$$

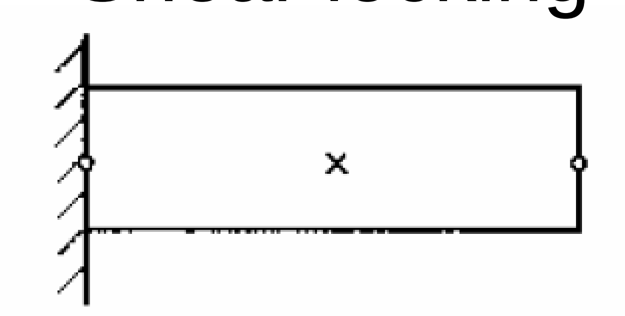
$$\boldsymbol{N} = [\boldsymbol{N}_1 \quad \boldsymbol{N}_2 \quad \cdots \quad \boldsymbol{N}_n]$$

$$\boldsymbol{N}_i = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \quad (i = 1, 2, \cdots, n)$$

$$\boldsymbol{a}^e = [\boldsymbol{a}_1^{\mathrm{T}} \quad \boldsymbol{a}_2^{\mathrm{T}} \quad \cdots \quad \boldsymbol{a}_n^{\mathrm{T}}]^{\mathrm{T}}$$

$$\boldsymbol{a}_i = \begin{pmatrix} w_i \\ \theta_i \end{pmatrix} \quad (i = 1, 2, \cdots, n)$$

- Shear locking



$$w = \sum_{i=1}^{2} N_i w_i \qquad \theta = \sum_{i=1}^{2} N_i \theta_i$$

$$N_1 = \frac{1}{2}(1-\xi) \qquad N_2 = \frac{1}{2}(1+\xi)$$

$$\xi = \frac{2(x-x_c)}{l} \qquad x_c = \frac{x_1+x_2}{2} \quad (-1 \leqslant \xi \leqslant 1)$$

$$\gamma = \frac{\mathrm{d}w}{\mathrm{d}x} - \theta = \frac{1}{l}(w_2 - w_1) - \frac{1}{2}(\theta_1 + \theta_2) - \frac{1}{2}(\theta_2 - \theta_1)\xi$$

$$h \to 0 \Rightarrow \gamma \to 0$$

$$\frac{1}{l}(w_2 - w_1) = \frac{1}{2}(\theta_1 + \theta_2)$$

$$\frac{1}{2}(\theta_2 - \theta_1) = 0$$

$$\theta_1 = \theta_2$$

If we assume a constant shear strain

$$\gamma = \frac{1}{l}(\omega_2 - \omega_1) - \frac{1}{2}(\theta_1 + \theta_2)$$

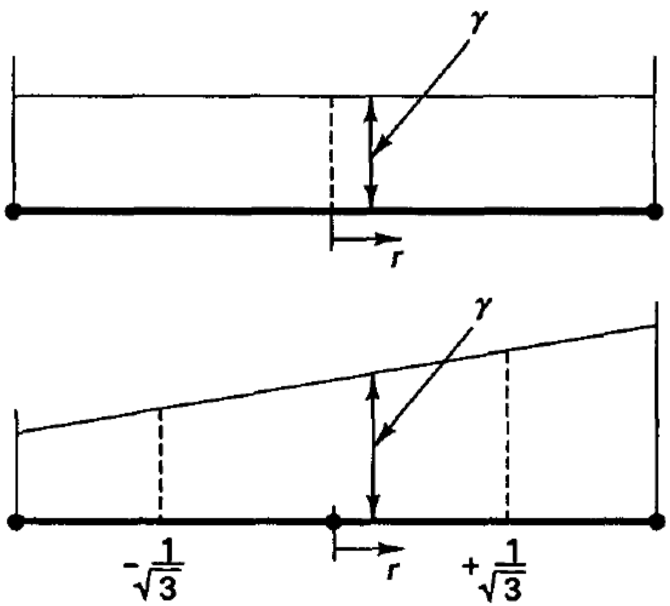
a more attractive element might be obtained.

- Solution: assumed shear strains

$$\cancel{\gamma = \frac{\mathrm{d}w}{\mathrm{d}x} - \theta}$$

$$\bar{\gamma} = \sum_{j=1}^{m} \overline{N}_j(\xi)\bar{\gamma}_j$$

$$\bar{\gamma}_j = \gamma(\xi_j) = \left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right)\bigg|_{\xi=\xi_j} = \sum_{i=1}^{n}\left(\frac{\mathrm{d}N_i(\xi)}{\mathrm{d}x} w_i - N_i(\xi)\theta_i\right)\bigg|_{\xi=\xi_j}$$




$$m = 1 \qquad \xi_1 = 0 \qquad \overline{N}_1 = 1$$

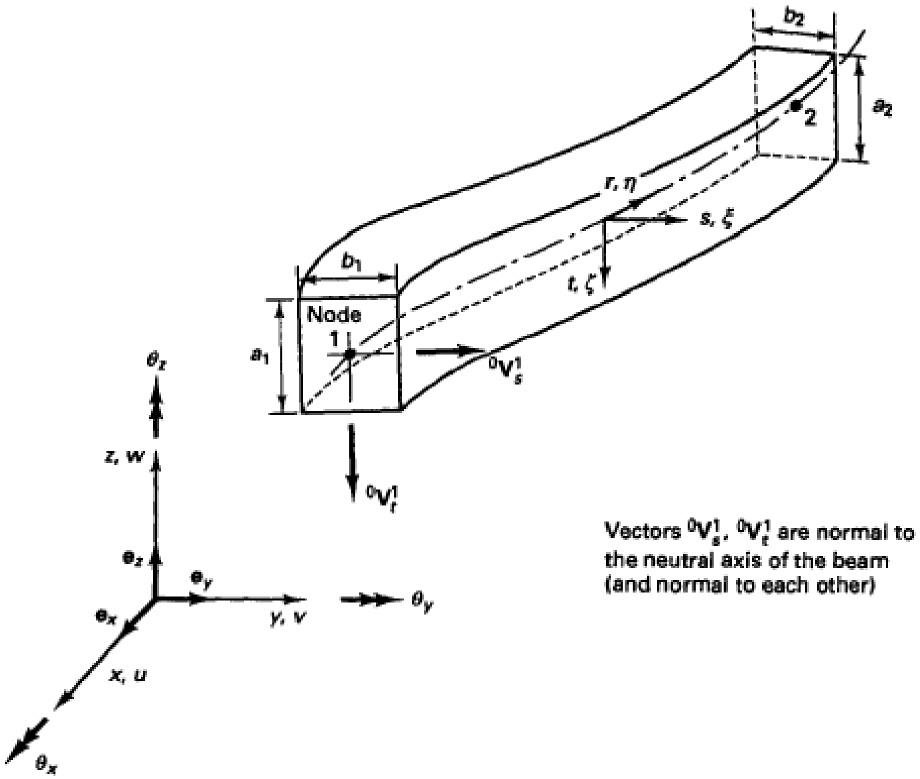
$$\bar{\gamma} = \gamma(\xi_1) = \left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right)\Big|_{\xi=0} = \frac{1}{l}(w_2 - w_1) - \frac{1}{2}(\theta_1 + \theta_2)$$

$$m = 2 \qquad \xi_1 = -\frac{1}{\sqrt{3}} \qquad \xi_2 = \frac{1}{\sqrt{3}}$$

$$\overline{N}_1 = \frac{1}{2}(1 - \sqrt{3}\xi) \qquad N_2 = \frac{1}{2}(1 + \sqrt{3}\xi)$$

$$\bar{\gamma} = \overline{N}_1\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right)\Big|_{\xi_1 = -\frac{1}{\sqrt{3}}} + \overline{N}_2\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right)\Big|_{\xi_2 = \frac{1}{\sqrt{3}}}$$

# General Curved Beam Elements




Assumption: plane sections originally normal to the centerline axis remain plane and undistorted under deformation but not necessarily normal to this axis.

$$ {}^{\ell}x(r,s,t) = \sum_{k=1}^{q} h_k\, {}^{\ell}x_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k\, {}^{\ell}V_{tx}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k\, {}^{\ell}V_{sx}^k $$

$$ {}^{\ell}y(r,s,t) = \sum_{k=1}^{q} h_k\, {}^{\ell}y_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k\, {}^{\ell}V_{ty}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k\, {}^{\ell}V_{sy}^k $$

$$ {}^{\ell}z(r,s,t) = \sum_{k=1}^{q} h_k\, {}^{\ell}z_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k\, {}^{\ell}V_{tz}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k\, {}^{\ell}V_{sz}^k $$

$$ u(r,s,t) = \sum_{k=1}^{q} h_k u_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k\, V_{tx}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k\, V_{sx}^k $$

$$ v(r,s,t) = \sum_{k=1}^{q} h_k v_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k\, V_{ty}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k\, V_{sy}^k $$

$$ w(r,s,t) = \sum_{k=1}^{q} h_k w_k + \frac{t}{2}\sum_{k=1}^{q} a_k h_k\, V_{tz}^k + \frac{s}{2}\sum_{k=1}^{q} b_k h_k\, V_{sz}^k $$

$$ \mathbf{V}_t^k = {}^{1}\mathbf{V}_t^k - {}^{0}\mathbf{V}_t^k; \qquad \mathbf{V}_s^k = {}^{1}\mathbf{V}_s^k - {}^{0}\mathbf{V}_s^k $$

# Axial-symmetrical Shell Elements

The word *shell* is an old one and is commonly used to describe the hard covering of eggs, crustacea, tortoises, etc.

In structural engineering, shell structures can be defined as curved structures capable of transmitting loads in more than two directions to supports. Loads applied to shell surfaces are carried to the ground by the development of compressive, tensile, and shear stresses acting in the in-plane direction of the surface.

Thickness is much shorter than other two dimensions.

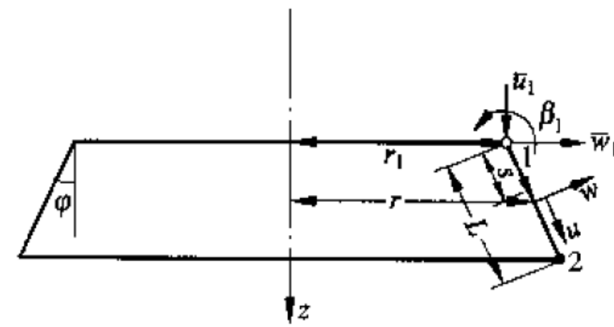
The midsurface is not plane but convex.

Swiss Re "Gherkin", with a lattice shell by Norman Foster and Ken Shuttleworth, London.

$$\bar{u} = \sum_{i=1}^{2} N_i \bar{u}_i \qquad \bar{w} = \sum_{i=1}^{2} N_i \bar{w}_i \qquad \beta = \sum_{i=1}^{2} N_i \beta_i$$

$$N_1 = 1 - \xi \quad N_2 = \xi \quad \xi = s/L$$

$$\begin{pmatrix} u \\ w \end{pmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} \bar{u} \\ \bar{w} \end{pmatrix}$$




$$\begin{Bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \kappa_s \\ \kappa_\theta \\ \gamma \end{Bmatrix} = \begin{Bmatrix} \dfrac{du}{ds} + \dfrac{w}{R_s} \\ \dfrac{1}{r}(u\sin\phi + w\cos\phi) \\ -\dfrac{d\beta}{ds} \\ -\dfrac{\sin\phi}{r}\beta \\ \dfrac{dw}{ds} - \dfrac{u}{R_s} - \beta \end{Bmatrix} = \begin{bmatrix} \cos\phi\dfrac{d}{ds} & \sin\phi\dfrac{d}{ds} & 0 \\ 0 & \dfrac{1}{r} & 0 \\ 0 & 0 & -\dfrac{d}{ds} \\ 0 & 0 & -\dfrac{\sin\phi}{r} \\ -\sin\phi\dfrac{d}{ds} & \cos\phi\dfrac{d}{ds} & -1 \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{w} \\ \beta \end{Bmatrix}$$

# Reference: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Thank you all!!