FEM course – lecture 7

1. Element matrices in global coordinates
2. Displacement / pressure related elements for incompressible media

Repetition: General Solution of the Problem

Static solution of problem $\mathbf{KU} = \mathbf{R}$ (global coordinates)

Displacement ansatz $\mathbf{K} = \int_V \mathbf{B}^T\mathbf{C}\mathbf{B}\, dV$

Integration over natural vol. $dV = \det \mathbf{J}\, dr\, ds\, dt$ (natural coordinates)

Connection between natural and global coordinates $\frac{\partial}{\partial \mathbf{r}} = \mathbf{J} \frac{\partial}{\partial \mathbf{x}}$ e.g.

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Accordingly $\mathbf{K} = \int_V \mathbf{F}\, dr\, ds\, dt$ with $\mathbf{F} = \mathbf{B}^T\mathbf{C}\mathbf{B} \det \mathbf{J}$

Numerical integration $\mathbf{K} = \sum_{i,j,k} \alpha_{ijk}\mathbf{F}_{ijk}$ where $\mathbf{F}_{ijk}$ evaluation of $\mathbf{F}$ at point (r_i, s_j, t_k)

Repetition: General Solution of the Problem

How to get the strain-displacement matrix B?

→ basically from: $\boldsymbol{\epsilon} = \mathbf{B}\hat{\mathbf{u}}$ where: $\hat{u}^T = [u_1\, v_1\, w_1 \quad \dots \quad u_N\, v_N\, w_N]$

$$\varepsilon^T = [\varepsilon_{xx}\, \varepsilon_{yy}\, \varepsilon_{zz}\, \gamma_{xy}\, \gamma_{xz}\, \gamma_{yz}]$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial x}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial x},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

with $u = h_i u_i, \quad v = h_i v_i, \quad w = h_i w_i$

→ Problem: u, v, w in natural coordinates!

Relation via $\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial}\mathbf{x}} = \mathbf{J}^{-1}\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial}\mathbf{r}}$ where: $\boldsymbol{r}^T = [r\ s\ t]$

$$\boldsymbol{x}^T = [x\ y\ z]$$

→ since we have $\boldsymbol{\varepsilon}$ and $\hat{\mathbf{u}}$ we get **B**

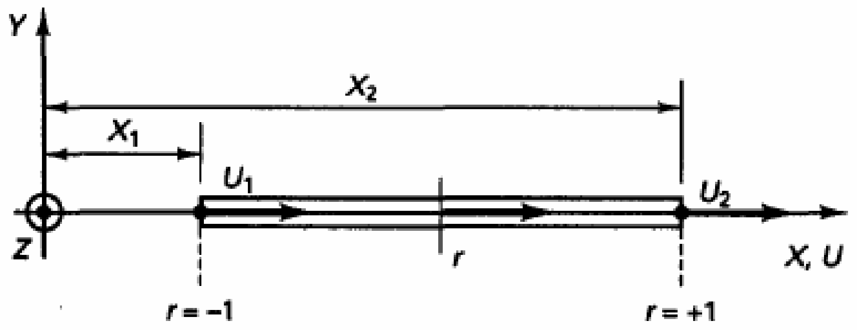
(if J^{-1} exists)

Integration of $K = \int_V B^T C\, B \det J\, dr\, ds\, dt$ with $\mathbf{C} = \boldsymbol{E}$

Natural vs. global coordinate system

- <u>Example: Derivation of stiffness matrix ***K*** for bar element</u>

Bar element in global and natural coordinate system



Natural coordinates: $-1 \leq r \leq 1$

Interpolation functions: $h_1 = \frac{1}{2}(r-1), \qquad h_2 = \frac{1}{2}(r+1)$

Displacement: $U = h_i U_i = \frac{1}{2}(r-1)U_1 + \frac{1}{2}(r+1)U_2$ (natural coordinates)

$$X = \frac{1}{2}(r-1)X_1 + \frac{1}{2}(r+1)X_2$$ (global coordinates)

Natural vs. global coordinate system

Element strains $\varepsilon = \frac{dU}{dX} = J^{-1}\frac{dU}{dr}$ with $J = \frac{dX}{dr}$

resulting $\epsilon = \frac{U_2 - U_1}{L}$ since $\frac{dU}{dr} = \frac{U_2 - U_1}{2}$

$$\frac{dX}{dr} = \frac{X_2 - X_1}{2} = \frac{L}{2}$$

Strain-displacement matrix $\mathbf{B} = \frac{1}{L}[-1 \quad 1]$ from $\varepsilon = B\hat{U}$

$$\hat{U}^T = [U_1 \quad U_2]$$

Stiffness matrix of the bar element $\mathbf{K} = \int_V \mathbf{B}^T\mathbf{C}\mathbf{B}\, dV$

Replacing B, C and dV
one gets after integration

$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

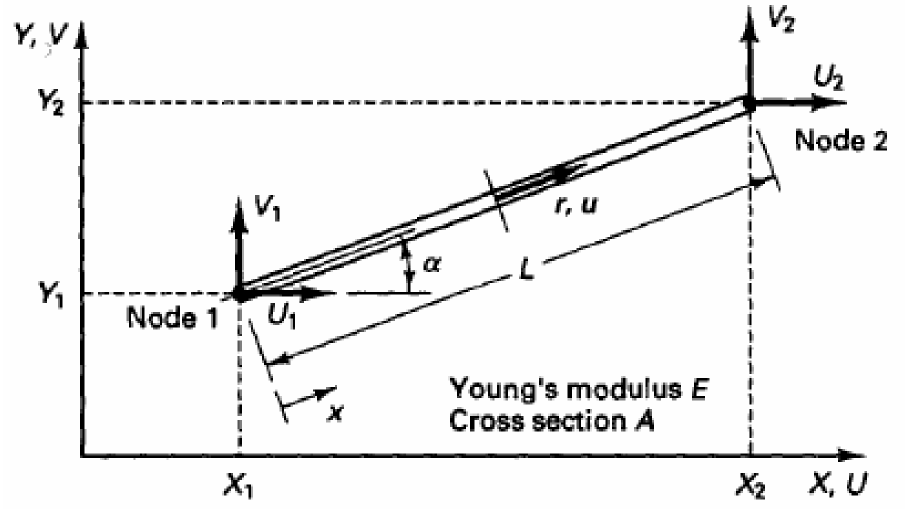
having $dV = \frac{AL}{2}\, dr$

$$\mathbf{C} = E$$

Natural vs. global coordinate system

- Arbitrary position in X, Y - plane

Element in global coordinate system



Displacement: $u = [\cos\alpha \quad \sin\alpha]\begin{bmatrix} \frac{1}{2}(1-r)U_1 + \frac{1}{2}(1+r)U_2 \\ \frac{1}{2}(1-r)V_1 + \frac{1}{2}(1+r)V_2 \end{bmatrix}$

Natural vs. global coordinate system

with $$\varepsilon = \frac{du}{dX} = J^{-1}\frac{du}{dr} = \frac{2}{L}\frac{du}{dr}$$

and $$\mathbf{U}^T = [U_1 \quad V_1 \quad U_2 \quad V_2]$$

Strain-displacement matrix

$$\mathbf{B} = \frac{1}{L}[\cos\alpha \quad \sin\alpha \quad \cos\alpha \quad \sin\alpha]\begin{bmatrix} -1 & & & \\ & -1 & \text{zeros} & \\ & & 1 & \\ \text{zeros} & & & 1 \end{bmatrix}$$

Replacing B, C and dV one gets after integration

$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha & -\sin\alpha\cos\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \cos\alpha\sin\alpha \\ -\sin\alpha\cos\alpha & -\sin^2\alpha & \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}$$

Natural vs. global coordinate system

- Take home messages:

  - Expression of displacements easier in natural coordinates (integration can be as well)
  - Connection between natural an global coordinates by

$$\frac{\partial}{\partial \mathbf{r}} = \mathbf{J} \frac{\partial}{\partial \mathbf{x}}$$

Displacement / pressure related elements for incompressible media

Problem: displacement-based finite elements not sufficiently effective for analysis of (almost) incompressible media and plates, shells

Solution: mixed interpolation e.g. displacement and pressure

Principle of virtual work in terms of **u** and *p*

$$\int_V \bar{\boldsymbol{\epsilon}}'^T \mathbf{S}\, dV - \int_V \bar{\epsilon}_V p\, dV = \mathcal{R}$$

Deviatoric stress $\mathbf{S} = \boldsymbol{\tau} + p\boldsymbol{\delta}$

Deviatoric strain $\boldsymbol{\epsilon}' = \boldsymbol{\epsilon} - \frac{1}{3}\epsilon_V \boldsymbol{\delta}$

Volumetric strain $\epsilon_V = \epsilon_{kk} = \frac{\Delta V}{V}$

$(= \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ in Cartesian coordinates$)$

Displacement / pressure related elements for incompressible media

For almost incompressible materials

$$\tau_{ij} = \kappa \epsilon_V \delta_{ij} + 2G\epsilon'_{ij}$$

where $\kappa = \dfrac{E}{3(1 - 2\nu)}$ (bulk modulus)

$G = \dfrac{E}{2(1 + \nu)}$ (shear modulus)

$\delta_{ij} \begin{cases} = 1; & i = j \\ = 0; & i \neq j \end{cases}$ (Kronecker delta)

for the pressure in the body holds $p = -\kappa \epsilon_V$

$$\tau_{ij} = -p\delta_{ij} + 2G\epsilon'_{ij}$$

$$p = -\frac{\tau_{kk}}{3}$$

$$\left(= -\frac{\tau_{xx} + \tau_{yy} + \tau_{zz}}{3} \text{ in Cartesian coordinates} \right)$$

For totally incompressible materials

$\nu \to 0.5,$

$\varepsilon_V \to 0$

$$\Longrightarrow \quad \int_V \epsilon_V \overline{p}\, dV = 0$$

Displacement / pressure related elements for incompressible media

Solution for almost incompressible conditions:

already known: $\mathbf{u} = \mathbf{H}\hat{\mathbf{u}}$

allows determination of: $\boldsymbol{\epsilon}' = \mathbf{B}_D\hat{\mathbf{u}}$, $\epsilon_V = \mathbf{B}_V\hat{\mathbf{u}}$

additional $p = \mathbf{H}_p\hat{\mathbf{p}}$

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}$$

$$\mathbf{K}_{uu} = \int_V \mathbf{B}_D^T \mathbf{C}' \mathbf{B}_D \, dV$$

$$\mathbf{K}_{up} = \mathbf{K}_{pu}^T = -\int_V \mathbf{B}_V^T \mathbf{H}_p \, dV$$

$$\mathbf{K}_{pp} = -\int_V \mathbf{H}_p^T \frac{1}{\kappa} \mathbf{H}_p \, dV$$

Solution for totally incompressible conditions:

$\kappa \longrightarrow \infty$

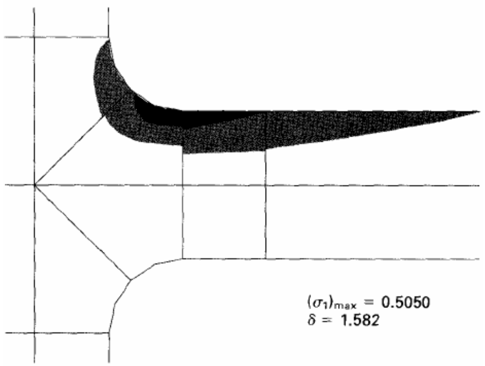
$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}$$

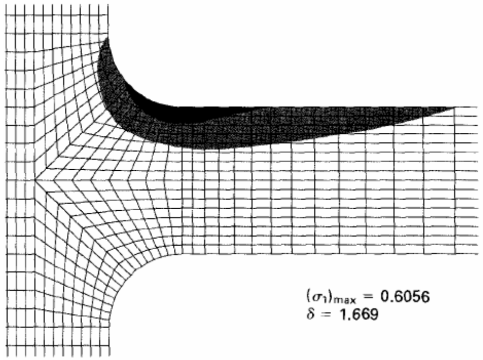
→ evaluation needs special considerations because of “0” in main diagonal

Examples for almost incompressible conditions

- pure displacement-based FEM analysis

$\nu = 0.3$

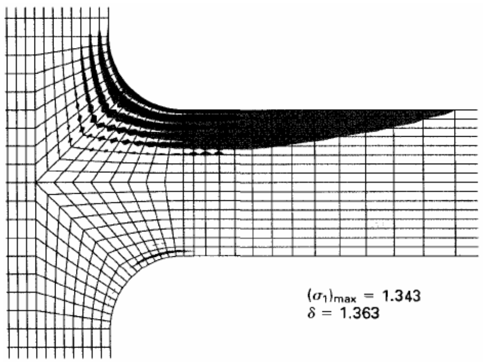




→ possible for not too much incompressible media

→ requires very fine discretization

$\nu = 0.499$



→ fails for highly incompressible media

→ “locking” behavior

“Non-locking” behavior due to interpolation of *p*

Displacement/pressure formulation u/p

- u/p formulation:

  pressure variables belong to the specific element only

  continuity of pressure is not enforced between elements but results from fine mesh

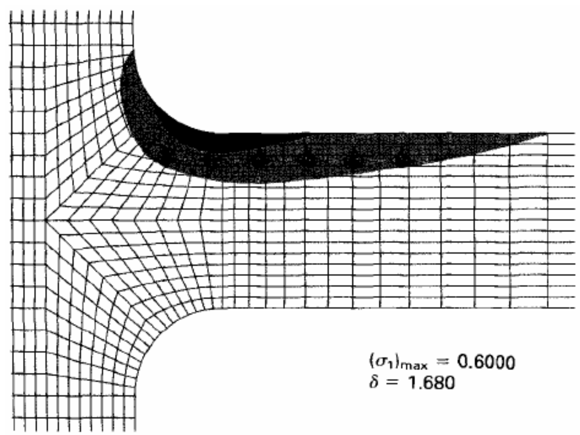
- u/p-c formulation

  element pressure is defined b nodal pressure variables

  continuity of pressure between elements is directly enforced

Example:

9/3 element, $\nu = 0.499$



Typical pressure interpolations

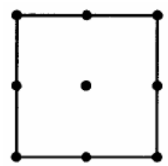
- constant: $p = p_0$, (9/1 element)
- linear: $p = p_0 + p_1 x + p_2 y$, (9/3 element)
- bilinear: $p = p_0 + p_1 x + p_2 y + p_3 xy$, (9/4 element)

Typical elements

a) 9/3 element

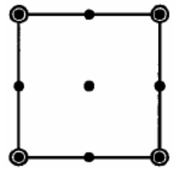
b) 9/4-c element

a)



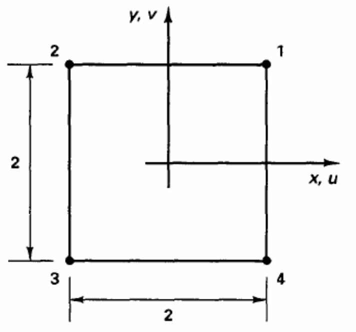
$p = p_0 + p_1 x + p_2 y$

b)



Example for almost incompressible case with u/p = 4/1 element

four-node strain plane element



we are looking for $\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \Longrightarrow \mathbf{K} = \mathbf{K}_{uu} - \mathbf{K}_{up}\mathbf{K}_{pp}^{-1}\mathbf{K}_{pu}$

thus, we need $\mathbf{B}_D$ and $\mathbf{B}_V$

we know $\boldsymbol{\epsilon}' = \mathbf{B}_D \hat{\mathbf{u}}$

$\epsilon_V = \mathbf{B}_V \hat{\mathbf{u}}$

$$\boldsymbol{\epsilon}' = \begin{bmatrix} \epsilon_{xx} - \frac{1}{3}(\epsilon_{xx} + \epsilon_{yy}) \\ \epsilon_{yy} - \frac{1}{3}(\epsilon_{xx} + \epsilon_{yy}) \\ \gamma_{xy} \\ -\frac{1}{3}(\epsilon_{xx} + \epsilon_{yy}) \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\frac{\partial u}{\partial x} - \frac{1}{3}\frac{\partial v}{\partial y} \\ \frac{2}{3}\frac{\partial v}{\partial y} - \frac{1}{3}\frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ -\frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \end{bmatrix} ; \qquad \epsilon_V = \epsilon_{xx} + \epsilon_{yy}$$

- Example for almost incompressible case with u/p = 4/1 element

we get $\mathbf{u} = \mathbf{H}\hat{\mathbf{u}}$ from

$$\hat{\mathbf{u}}^T = [u_1 \quad u_2 \quad u_3 \quad u_4 \; \vdots \; v_1 \quad v_2 \quad v_3 \quad v_4]$$

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & \vdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & h_1 & h_2 & h_3 & h_4 \end{bmatrix}$$

$$h_1 = \tfrac{1}{4}(1 + x)(1 + y); \qquad h_2 = \tfrac{1}{4}(1 - x)(1 + y)$$

$$h_3 = \tfrac{1}{4}(1 - x)(1 - y); \qquad h_4 = \tfrac{1}{4}(1 + x)(1 - y)$$

for assumption of const. pressure

we get $\mathbf{H}_p = [1]; \qquad \hat{\mathbf{p}} = [p_0]$

strain-displacement matices

$$\mathbf{B}_D = \begin{bmatrix} \frac{2}{3}h_{1,x} & \frac{2}{3}h_{2,x} & \cdots & \vdots & -\frac{1}{3}h_{1,y} & -\frac{1}{3}h_{2,y} & \cdots \\ -\frac{1}{3}h_{1,x} & -\frac{1}{3}h_{2,x} & \cdots & \vdots & \frac{2}{3}h_{1,y} & \frac{2}{3}h_{2,y} & \cdots \\ h_{1,y} & h_{2,y} & \cdots & \vdots & h_{1,x} & h_{2,x} & \cdots \\ -\frac{1}{3}h_{1,x} & -\frac{1}{3}h_{2,x} & \cdots & \vdots & -\frac{1}{3}h_{1,y} & -\frac{1}{3}h_{2,y} & \cdots \end{bmatrix}$$

$$\mathbf{B}_V = [h_{1,x} \quad h_{2,x} \quad \cdots \; \vdots \; h_{1,y} \quad h_{2,y} \quad \cdots]$$

material matrix

$$\mathbf{C}' = \begin{bmatrix} 2G & 0 & 0 & 0 \\ 0 & 2G & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & 2G \end{bmatrix}; \qquad G = \frac{E}{2(1 + \nu)}$$

Displacement / pressure related elements for incompressible media

- <u>Take home messages:</u>

  - for little compressible media ($\nu \rightarrow 0.5$) pure displacement interpolation is not sufficient to obtain correct stress distribution
  - among the displacement/pressure formulations, the 9/3 element is a good example (in 2D) for non-locking elements (has high order of convergence and no “over-interpolation” of *p*)