Isoparametric Quadrilateral Finite Elements and Convergence Considerations

Motivation

- Achieve the relationship between displacement and nodal points by means of *shape functions*
- Therefore, $\mathbf{A}^{-1}$ need not be evaluated

$$\hat{\mathbf{u}} = \mathbf{A}\boldsymbol{\alpha}$$

$$\boldsymbol{\alpha} = \mathbf{A}^{-1}\hat{\mathbf{u}}$$

$$\boldsymbol{\varepsilon} = \mathbf{E}\boldsymbol{\alpha}$$

Formulation

- Coordinate interpolations

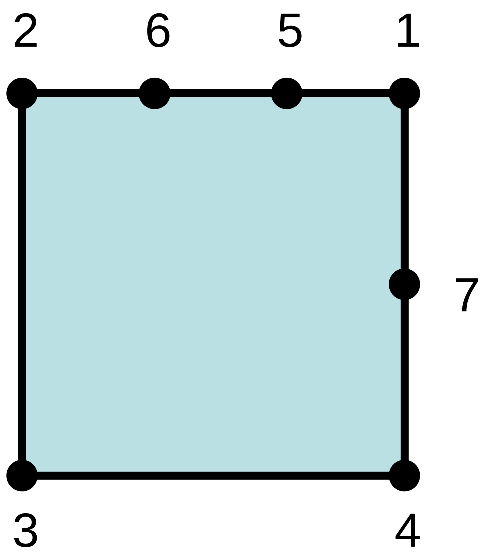
$$x = \sum_{i=1}^{q} h_i x_i; \qquad y = \sum_{i=1}^{q} h_i y_i; \qquad z = \sum_{i=1}^{q} h_i z_i$$

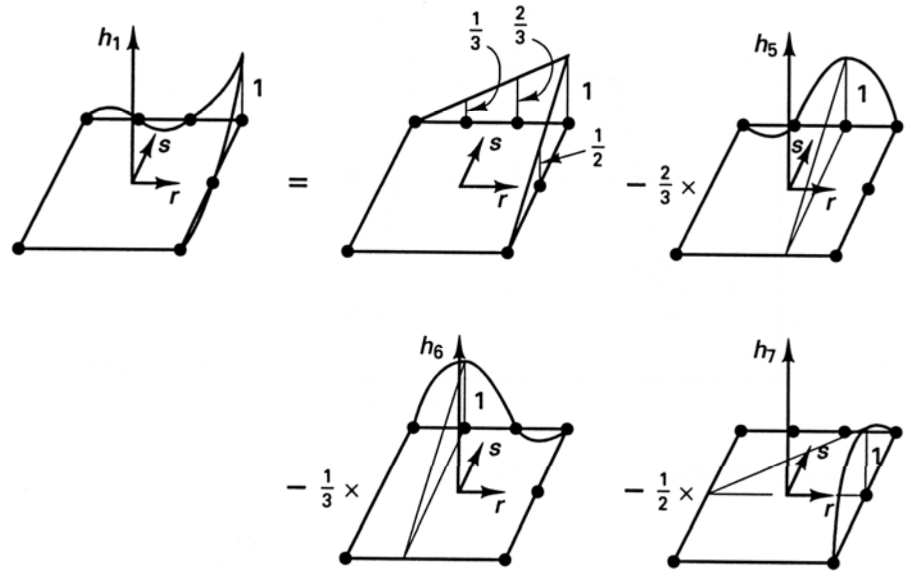
  - h_i is equal 1 at node *i*, 0 otherwise

- Displacement interpolations

$$u = \sum_{i=1}^{q} h_i u_i; \qquad v = \sum_{i=1}^{q} h_i v_i; \qquad w = \sum_{i=1}^{q} h_i w_i$$

Example 1





$$h_5 = [\tfrac{1}{16}(-27r^3 - 9r^2 + 27r + 9)][\tfrac{1}{2}(1 + s)]$$

$$h_6 = [(1 - r^2) + \tfrac{1}{16}(27r^3 + 7r^2 - 27r - 7)][\tfrac{1}{2}(1 + s)]$$

$$h_2 = [\tfrac{1}{2}(1 - r) - \tfrac{1}{2}(1 - r^2) + \tfrac{1}{16}(-9r^3 + r^2 + 9r - 1)][\tfrac{1}{2}(1 + s)]$$

$$h_3 = \tfrac{1}{4}(1 - r)(1 - s)$$

$$h_7 = \tfrac{1}{2}(1 - s^2)(1 + r)$$

$$h_4 = \tfrac{1}{4}(1 + r)(1 - s) - \tfrac{1}{2}h_7$$

$$h_1 = \tfrac{1}{4}(1 + r)(1 + s) - \tfrac{2}{3}h_5 - \tfrac{1}{3}h_6 - \tfrac{1}{2}h_7$$

Stiffness matrix evaluation

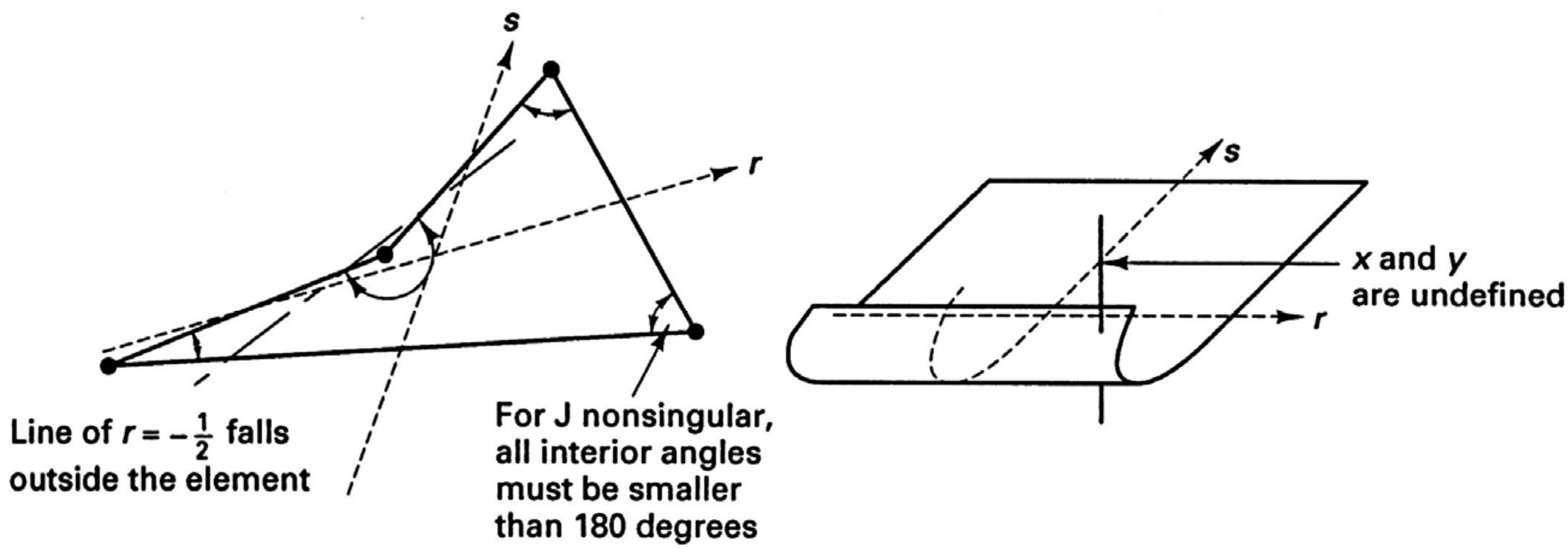
- *Element strains:* derivatives of element displacements with respect to local coordinates

$$x = f_1(r, s, t); \qquad y = f_2(r, s, t); \qquad z = f_3(r, s, t)$$

$$r = f_4(x, y, z); \qquad s = f_5(x, y, z); \qquad t = f_6(x, y, z)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x}$$

Stiffness matrix evaluation



(a) Distorted element

(b) Element folding upon itself

$$\frac{\partial}{\partial \mathbf{x}} = \mathbf{J}^{-1} \frac{\partial}{\partial \mathbf{r}}$$

Stiffness matrix evaluation

$$\boldsymbol{\epsilon} = \mathbf{B}\hat{\mathbf{u}}$$

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} \, dV \qquad dV = \det \mathbf{J} \, dr \, ds \, dt$$

$$\mathbf{K} = \int_V \mathbf{F} \, dr \, ds \, dt$$

$$\mathbf{K} = \sum_{i, j, k} \alpha_{ijk} \mathbf{F}_{ijk}$$

Mass Matrix and Loads

$$\mathbf{u}(r, s, t) = \mathbf{H}\hat{\mathbf{u}}$$

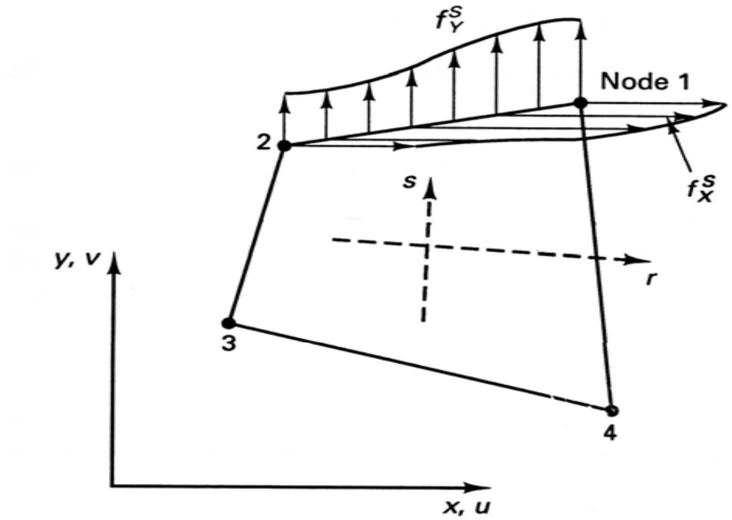
$$\mathbf{M} = \int_V \rho \mathbf{H}^T \mathbf{H} \, dV$$

$$\mathbf{R}_B = \int_V \mathbf{H}^T \mathbf{f}^B \, dV$$

$$\mathbf{R}_S = \int_S \mathbf{H}^{S^T} \mathbf{f}^S \, dS$$

$$\mathbf{R}_I = \int_V \mathbf{B}^T \boldsymbol{\tau}^I \, dV$$

Example 2



$$u = \tfrac{1}{4}(1 + r)(1 + s)u_1 + \tfrac{1}{4}(1 - r)(1 + s)u_2 + \tfrac{1}{4}(1 - r)(1 - s)u_3 + \tfrac{1}{4}(1 + r)(1 - s)u_4$$

$$v = \tfrac{1}{4}(1 + r)(1 + s)v_1 + \tfrac{1}{4}(1 - r)(1 + s)v_2 + \tfrac{1}{4}(1 - r)(1 - s)v_3 + \tfrac{1}{4}(1 + r)(1 - s)v_4$$

$$x = \tfrac{1}{4}(1 + r)(1 + s)x_1 + \tfrac{1}{4}(1 - r)(1 + s)x_2 + \tfrac{1}{4}(1 - r)(1 - s)x_3 + \tfrac{1}{4}(1 + r)(1 - s)x_4$$

$$y = \tfrac{1}{4}(1 + r)(1 + s)y_1 + \tfrac{1}{4}(1 - r)(1 + s)y_2 + \tfrac{1}{4}(1 - r)(1 - s)y_3 + \tfrac{1}{4}(1 + r)(1 - s)y_4$$

$$\frac{\partial x}{\partial r} = \frac{1}{4}(1 + s)x_1 - \frac{1}{4}(1 + s)x_2 - \frac{1}{4}(1 - s)x_3 + \frac{1}{4}(1 - s)x_4$$

$$\frac{\partial y}{\partial r} = \frac{1}{4}(1 + s)y_1 - \frac{1}{4}(1 + s)y_2 - \frac{1}{4}(1 - s)y_3 + \frac{1}{4}(1 - s)y_4$$

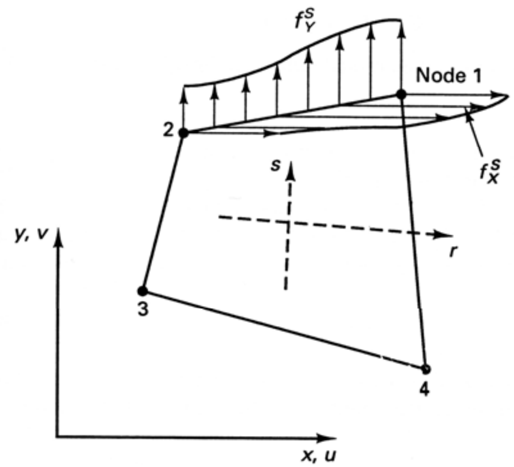
$$u^S = \tfrac{1}{2}(1 + r)u_1 + \tfrac{1}{2}(1 - r)u_2$$

$$v^S = \tfrac{1}{2}(1 + r)v_1 + \tfrac{1}{2}(1 - r)v_2$$

Example 2

$$\frac{\partial x}{\partial r} = \frac{1}{4}(1 + s)x_1 - \frac{1}{4}(1 + s)x_2 - \frac{1}{4}(1 - s)x_3 + \frac{1}{4}(1 - s)x_4$$

$$\frac{\partial y}{\partial r} = \frac{1}{4}(1 + s)y_1 - \frac{1}{4}(1 + s)y_2 - \frac{1}{4}(1 - s)y_3 + \frac{1}{4}(1 - s)y_4$$



$$u^S = \tfrac{1}{2}(1 + r)u_1 + \tfrac{1}{2}(1 - r)u_2$$

$$v^S = \tfrac{1}{2}(1 + r)v_1 + \tfrac{1}{2}(1 - r)v_2$$

$$\mathbf{H}^S = \begin{bmatrix} \frac{1}{2}(1 + r) & 0 & \frac{1}{2}(1 - r) & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1 + r) & 0 & \frac{1}{2}(1 - r) & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{f}^S = \begin{bmatrix} f_x^S \\ f_y^S \end{bmatrix}$$

$$\frac{\partial x}{\partial r} = \frac{x_1 - x_2}{2}; \qquad \frac{\partial y}{\partial r} = \frac{y_1 - y_2}{2}$$

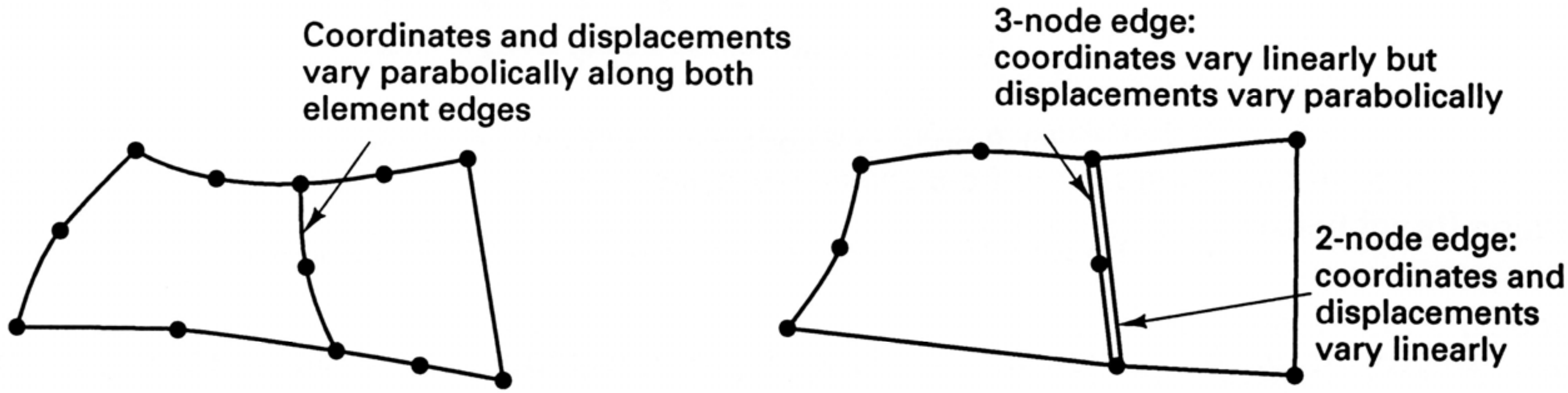
$$dl = \det \mathbf{J}^S \, dr; \qquad \det \mathbf{J}^S = \left[\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2\right]^{1/2}$$

$$\mathbf{R}_S = \sum_i \alpha_i t_{ri} \mathbf{F}_i$$

$$\mathbf{F}_i = \mathbf{H}_i^{S^T} \mathbf{f}_i^S \det \mathbf{J}_i^S$$

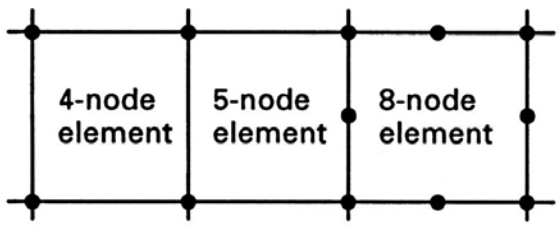
Requirements for Convergence

- Elements must be:
  - compatible
  - complete

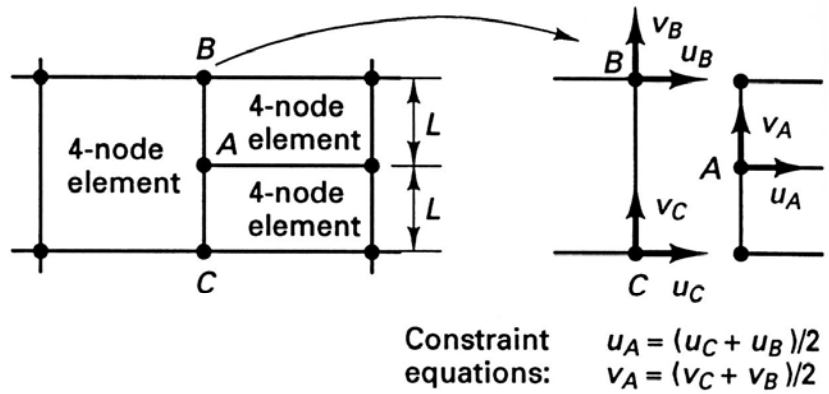


(a) Compatible element assemblage (b) Incompatible element assemblage

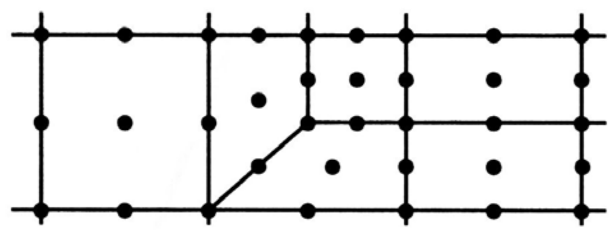
Compatibility



(a) 4-node to 8-node element transition region



(b) 4-node to 4-node element transition; from one to two layers



(c) 9-node to 9-node element transition region; from one to two layers

Completeness

- Rigid body displacement
- Constant strain states

$$\left.\begin{aligned} u &= a_1 + b_1 x + c_1 y + d_1 z \\ v &= a_2 + b_2 x + c_2 y + d_2 z \\ w &= a_3 + b_3 x + c_3 y + d_3 z \end{aligned}\right\}$$

$$\left.\begin{aligned} u_i &= a_1 + b_1 x_i + c_1 y_i + d_1 z_i \\ v_i &= a_2 + b_2 x_i + c_2 y_i + d_2 z_i \\ w_i &= a_3 + b_3 x_i + c_3 y_i + d_3 z_i \end{aligned}\right\}$$

$$u = \sum_{i=1}^{q} h_i u_i; \qquad v = \sum_{i=1}^{q} h_i v_i; \qquad w = \sum_{i=1}^{q} h_i w_i$$

Completeness

$$\left.\begin{aligned}
u &= a_1 \sum_{i=1}^{q} h_i + b_1 \sum_{i=1}^{q} h_i x_i + c_1 \sum_{i=1}^{q} h_i y_i + d_1 \sum_{i=1}^{q} h_i z_i \\
v &= a_2 \sum_{i=1}^{q} h_i + b_2 \sum_{i=1}^{q} h_i x_i + c_2 \sum_{i=1}^{q} h_i y_i + d_2 \sum_{i=1}^{q} h_i z_i \\
w &= a_3 \sum_{i=1}^{q} h_i + b_3 \sum_{i=1}^{q} h_i x_i + c_3 \sum_{i=1}^{q} h_i y_i + d_3 \sum_{i=1}^{q} h_i z_i
\end{aligned}\right\}$$

$$\left.\begin{aligned}
u &= a_1 \sum_{i=1}^{q} h_i + b_1 x + c_1 y + d_1 z \\
v &= a_2 \sum_{i=1}^{q} h_i + b_2 x + c_2 y + d_2 z \\
w &= a_3 \sum_{i=1}^{q} h_i + b_3 x + c_3 y + d_3 z
\end{aligned}\right\}$$

$$\sum_{i=1}^{q} h_i = 1$$

Order of Convergence

- Isoparametric elements always have capability to represent rigid body modes
- Assumptions
  - Elements based on polynomial expantions
  - Uniform meshes with charakteristic dim. *H*
  - Exact solution is smooth

$$\| \mathbf{u} - \mathbf{u}_h \|_1 \leq c\ h^k$$

Order of Convergence

- Normally exact solution is not smooth
- Mesh grading must be employed in nonsmooth stress distributions
- With *regular* meshes
  - Density of solution error is nearly constant

$$\| \mathbf{u} - \mathbf{u}_h \|_1^2 \leq c \sum_m h_m^{2k} \| \mathbf{u} \|_{k+1,m}^2$$

Take home message

- With isoparametric finite elements
  - Stiffness, mass and load matrices can be calculated straightforward
  - Simple formulation $\rightarrow$ computation efficency
- For better convergence
  - Regular mesh
  - Appropriate node number elements
- Know what you are doing!

Thank you
for your attention!