Formulation and calculation of isoparametric finite element matrixes:
- Truss elements
- Continuum elements - triangular elements

Today‘ lesson:

- Short: properties of truss and triangular elements
- Coordinate systems
- Isoparametric derivation of bar element stiffness matrix
- Form functions and their properties
- Jacobian operator
- Triangular elements
- Illustration of use

Some terms

Truss element

- displacement only in longitudinal axis; no moments, nodes are flexible
- widely used in structural engineering for ex. bridges

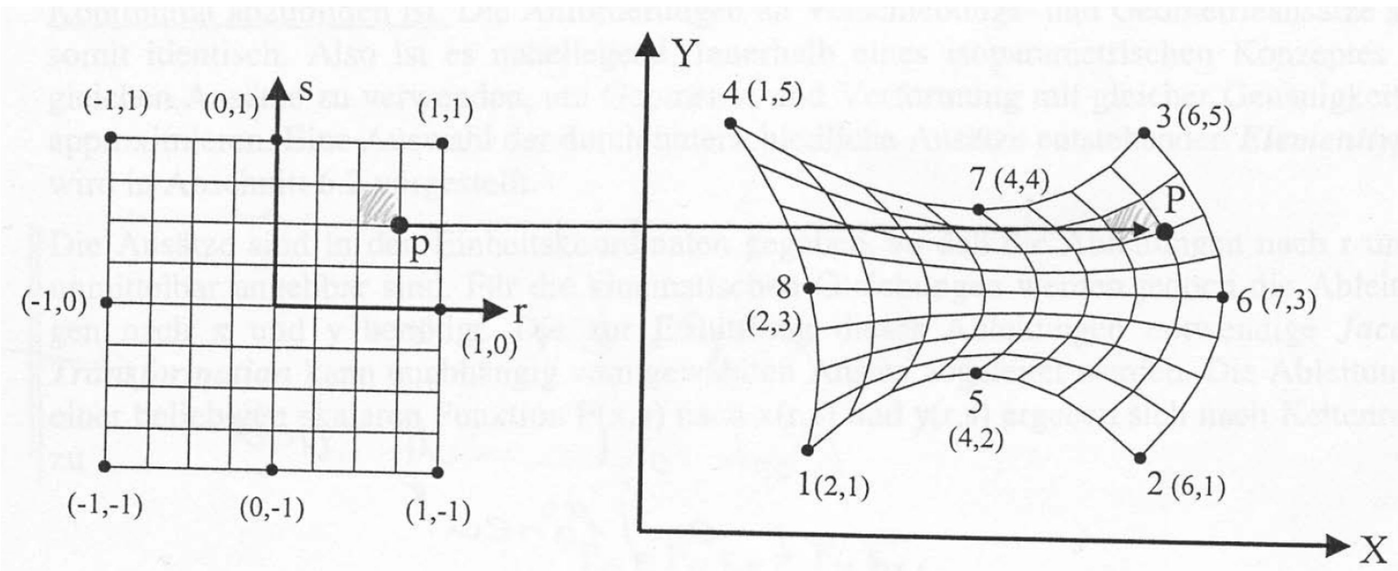
Triangular elements in plane strain elements

- two-dimensional stress in a xy plane with the stresses τ_{zz}, τ_{yz}, and τ_{zx} = 0

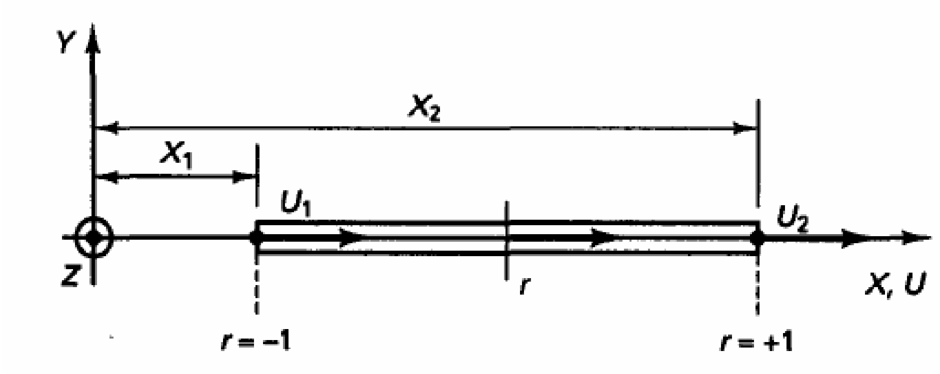
Coordinate systems

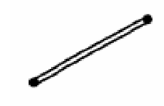
Global coordinate system = **Cartesian** coordinates: **xyz**

Local element coordinate sytem (**natural** coordinate system): **rst**



Isoparametric derivation of bar element stiffness matrix: Transformation





Transformation: how to express CSnatural in CSglobal

$$X = \tfrac{1}{2}(1 - r)X_1 + \tfrac{1}{2}(1 + r)X_2$$

$$h_1 = \tfrac{1}{2}(1 - r) \text{ and } h_2 = \tfrac{1}{2}(1 + r)$$

Shape (or interpolation) functions
At free nodal points = value 1
At other nodal points = value 0

$$X = \sum_{i=1}^{2} h_i X_i$$

$$U = \sum_{i=1}^{2} h_i U_i$$

Bar global displacement, expressed in same way as global coordinates = Basis of isoparametric finite element formulation

Isoparametric derivation of bar element stiffness matrix: Matrix

1. Element strains $\epsilon = dU/dX$.

$$\epsilon = \frac{dU}{dr}\frac{dr}{dX}$$

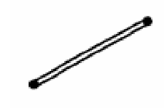
$$U = \sum_{i=1}^{2} h_i U_i \longrightarrow \frac{dU}{dr} = \frac{U_2 - U_1}{2}$$

$$X = \sum_{i=1}^{2} h_i X_i \longrightarrow \frac{dX}{dr} = \frac{X_2 - X_1}{2} = \frac{L}{2}$$

$$\epsilon = \frac{U_2 - U_1}{L}$$

Isoparametric derivation of bar element stiffness matrix: Matrix

2. Strain displacement transformation matrix



$$\boxed{\boldsymbol{\epsilon} = \mathbf{B}\hat{\mathbf{u}}}$$

$$\mathbf{B} = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

3. Obtain *K* with volume integral (not necessary in this case, but for demo)

$$\mathbf{K} = \frac{AE}{L^2}\int_{-1}^{1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} J\, dr$$

Where A = bar area and E = Modulus of elasticity are taken as constant;
J = Jacobian Operator

J is the Jacobian Operator relating an element length in CS_{global} to an element length in $CS_{natural}$

Here: $dX = J\,dr$ → we know it already: $J = L / 2$ ⟶ $\mathbf{K} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Isoparametric derivation of bar element stiffness matrix: Excursion on Jacobian operator

J = Jacobian Operator

relating element length in CS_{global} to element length in $CS_{natural}$

For stiffness matrix, *strain-displacement transformation matrix* has to be calculated

Element strains are obtained by derivates of element displacements, with respect to local coordinates

It is: $x = f_1(r, s, t); \quad y = f_2(r, s, t); \quad z = f_3(r, s, t)$

And: $r = f_4(x, y, z); \quad s = f_5(x, y, z); \quad t = f_6(x, y, z)$

We need: $\partial/\partial x, \partial/\partial y, \text{ and } \partial/\partial z$ → chain rule $\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x}$

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\mathbf{r}} = \mathbf{J}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\mathbf{x}}$$

Isoparametric derivation of bar element stiffness matrix: Matrix

4. In generalized coordinates

$$r = \frac{X - [(X_1 + X_2)/2]}{L/2}$$

$$U = \alpha_0 + \alpha_1 X$$

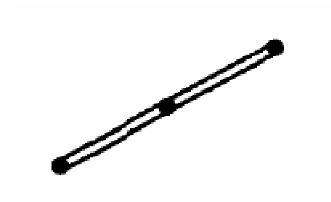
$$\left.\begin{aligned} \alpha_0 &= \tfrac{1}{2}(U_1 + U_2) - \frac{X_1 + X_2}{2L}(U_2 - U_1) \\ \alpha_1 &= \frac{1}{L}(U_2 - U_1) \end{aligned}\right\}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \dfrac{1}{2} + \dfrac{X_1 + X_2}{2L} & \dfrac{1}{2} - \dfrac{X_1 + X_2}{2L} \\ -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} \mathbf{U}$$

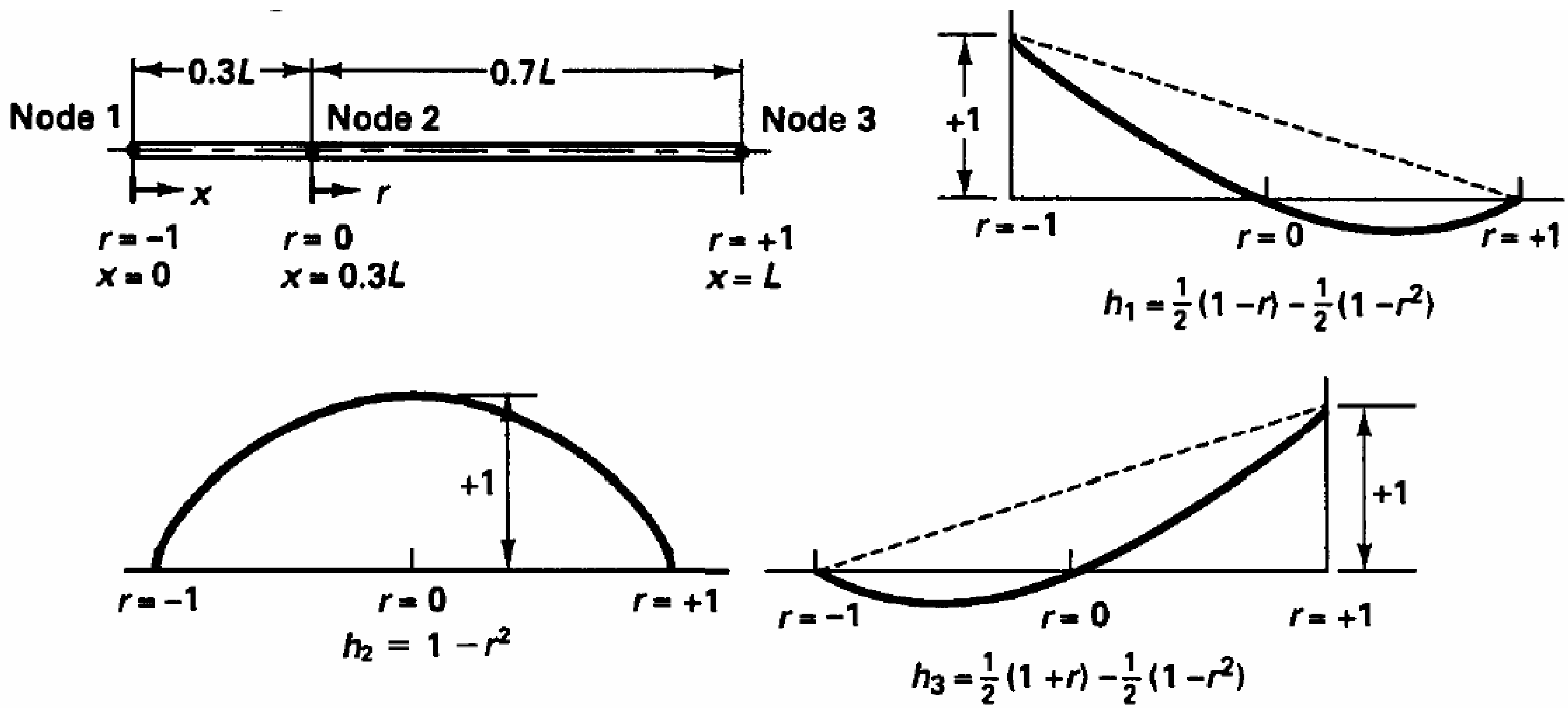
$$\boldsymbol{\alpha}^T = [\alpha_0 \quad \alpha_1]; \qquad \mathbf{U}^T = [U_1 \quad U_2]$$

Formulation of continuum elements: Straight truss and cable elements

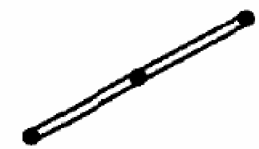
Unknown in $x = \sum_{i=1}^{q} h_i x_i$ is the interpolation function h_i

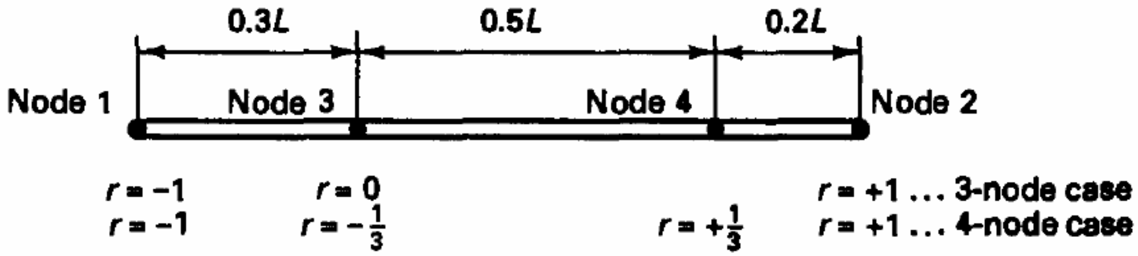


Remember: node i = 1, other nodes = 0



Formulation of continuum elements: Straight truss and cable elements





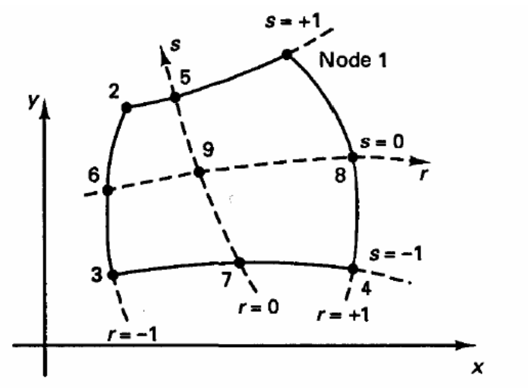
(a) 2 to 4 variable-number-nodes truss element

	Include only if node 3 is present	Include only if nodes 3 and 4 are present
$h_1 = \frac{1}{2}(1 - r)$	$-\frac{1}{2}(1 - r^2)$	$+\frac{1}{16}(-9r^3 + r^2 + 9r - 1)$
$h_2 = \frac{1}{2}(1 + r)$	$-\frac{1}{2}(1 - r^2)$	$+\frac{1}{16}(9r^3 + r^2 - 9r - 1)$
$h_3 = (1 - r^2)$		$+\frac{1}{16}(27r^3 + 7r^2 - 27r - 7)$
$h_4 = \frac{1}{16}(-27r^3 - 9r^2 + 27r + 9)$		

(b) Interpolation functions

Formulation of continuum elements: Triangular elements by *collapsing quadrilateral elements*

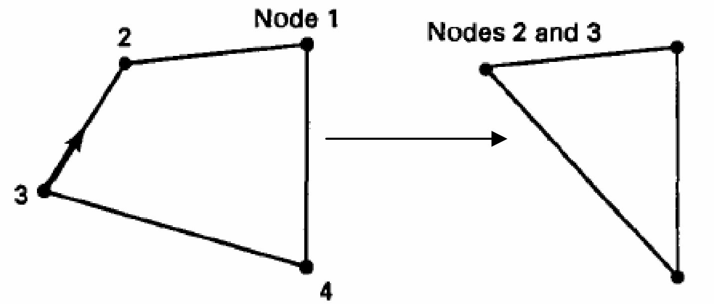
Collapsing any one side of a four-node element will always result in a constant strain triangle (Dreieck konstanter Verzerrung)



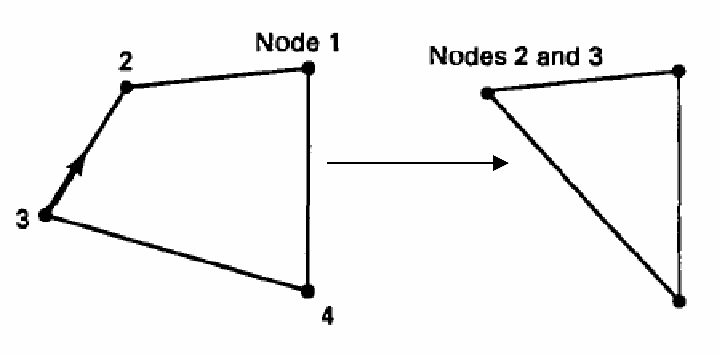
(a) 4 to 9 variable-number-nodes two-dimensional element

Include only if node i is defined

		$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$					$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1-s^2)(1-r)$					$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$					$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1-s^2)(1+r)$					$-\frac{1}{2}h_9$
$h_9 =$	$(1-r^2)(1-s^2)$					



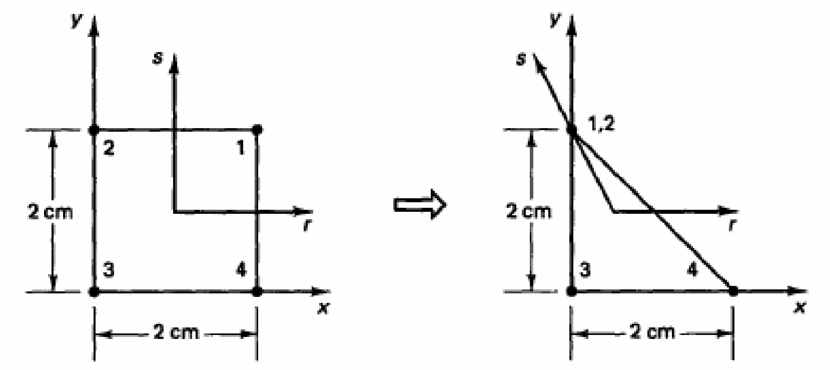
Formulation of continuum elements: Triangular elements by *collapsing quadrilateral elements*



-use interpolation function

-collapse the 2 nodes: $x_2=x_3$ and $y_2=x_3$

- express *x* and *y* with *r* and *s*:



-calculate J and J^{-1} for r and s

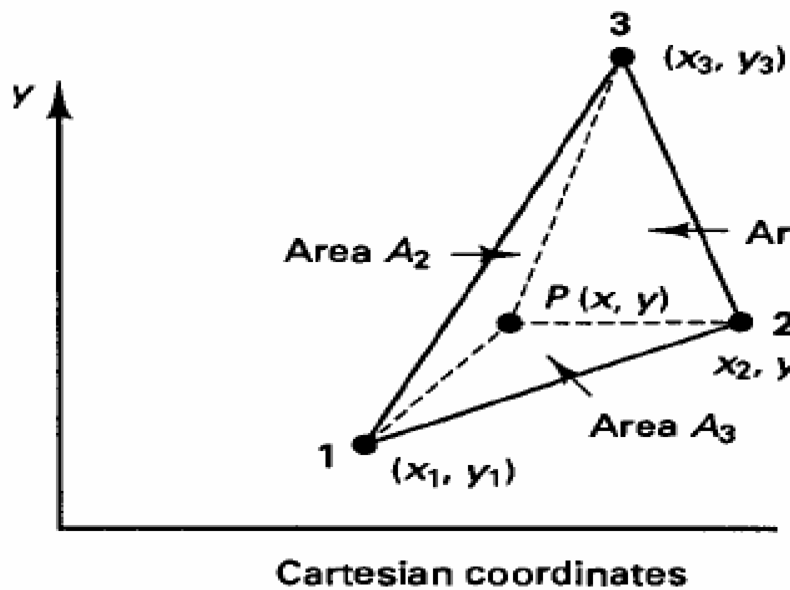
-use the same interpolation functions for *u* and *v*

-(isoparametric concept)

-evaluate derivates fo the displacements *u* and *v* to *r* and

$$\boldsymbol{\epsilon} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

Formulation of continuum elements: Triangular elements by *area coordinates*



$$L_1 = \frac{A_1}{A}; \qquad L_2 = \frac{A_2}{A}; \qquad L_3 = \frac{A_3}{A}$$

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

$$L_i = \frac{1}{2A}(a_i + b_i x + c_i y); \qquad i = 1, 2, 3$$

expressing area in terms of x and y:

$$2A = x_1 y_2 + x_2 y_3 + x_3 y_1 - y_1 x_2 - y_2 x_3 - y_3 x_1$$

$$x = L_1 x_1 + L_2 x_2 + L_3 x_3$$
$$y = L_1 y_1 + L_2 y_2 + L_3 y_3$$

$$a_1 = x_2 y_3 - x_3 y_2; \qquad a_2 = x_3 y_1 - x_1 y_3; \qquad a_3 = x_1 y_2 - x_2 y_1$$
$$b_1 = y_2 - y_3; \qquad b_2 = y_3 - y_1; \qquad b_3 = y_1 - y_2$$
$$c_1 = x_3 - x_2; \qquad c_2 = x_1 - x_3; \qquad c_3 = x_2 - x_1$$

Formulation of continuum elements: Triangular elements by *area coordinates*

Following procedure is the same: with

$$u = \sum_{i=1}^{3} h_i u_i; \qquad x = \sum_{i=1}^{3} h_i x_i$$

$$v = \sum_{i=1}^{3} h_i v_i; \qquad y = \sum_{i=1}^{3} h_i y_i$$

The finite element matrixes can be evaluated

For natural CS we use Jacobian operator

Integrations are made over natural coordinates

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} \, dV$$

$$dV = \det \mathbf{J} \, dr \, ds \, dt$$

$$\mathbf{K} = \int_V \mathbf{F} \, dr \, ds \, dt$$

$$\mathbf{K} = \sum_{i,j,k} \alpha_{ijk} \mathbf{F}_{ijk}$$

$$\mathbf{u}(r, s, t) = \mathbf{H}\hat{\mathbf{u}}$$

$$\mathbf{M} = \int_V \rho \mathbf{H}^T \mathbf{H} \, dV$$

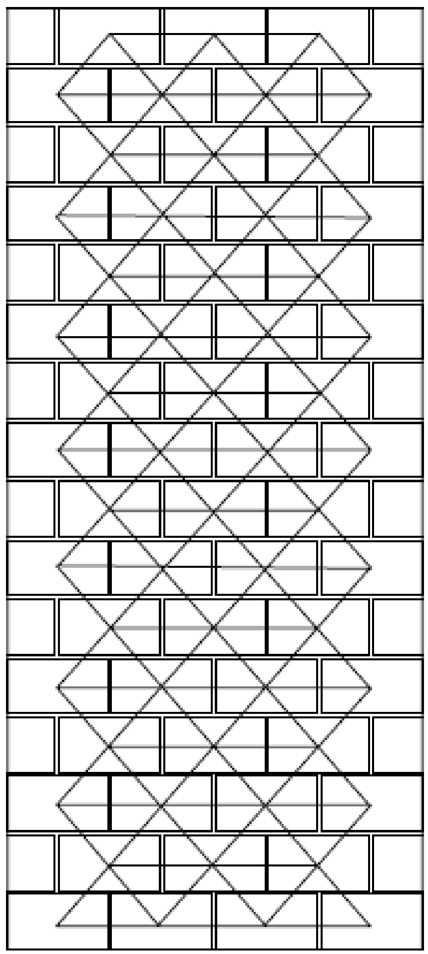
$$\mathbf{R}_B = \int_V \mathbf{H}^T \mathbf{f}^B \, dV$$

$$\mathbf{R}_S = \int_S \mathbf{H}^{S^T} \mathbf{f}^S \, dS$$

$$\mathbf{R}_I = \int_V \mathbf{B}^T \boldsymbol{\tau}^I \, dV$$

Illustration of use: example

Load-strain relationships obtained from the displacements of the measurement net.



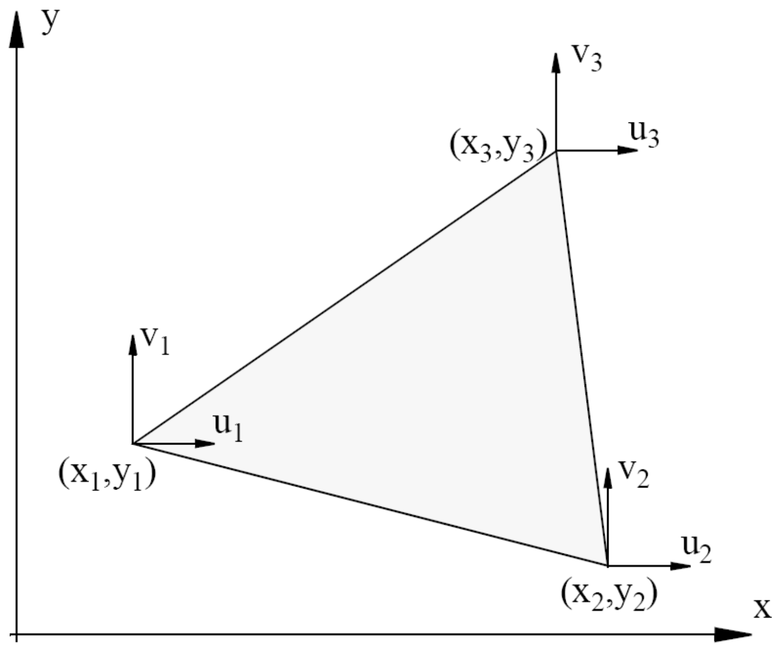


Illustration of use: example

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

$$\mathbf{u} = \mathbf{N}\boldsymbol{\alpha}$$

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}.$$

$$\mathbf{a} = \mathbf{T}\boldsymbol{\alpha}$$

$$\mathbf{a} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \qquad \mathbf{T} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix}$$

$$\boldsymbol{\alpha} = \mathbf{T}^{-1}\mathbf{a}$$

$$\mathbf{u} = \mathbf{N}\mathbf{T}^{-1}\mathbf{a} = \mathbf{H}\mathbf{a}.$$

Illustration of use: example

$$\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u} = \mathbf{L}\mathbf{H}\mathbf{a} = \mathbf{B}\mathbf{a}$$

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}.$$

$$\mathbf{B} = \frac{1}{\Delta}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix},$$

$$\Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\beta_i = y_j - y_k \qquad \gamma_i = x_k - x_j \qquad i,j,k = 1,2,3$$

Thanks!

Literature:

Bathe, K.-J. (1990): Finite-Elemente-Methoden. Springer Verlag

Fröhlicher, P. (2005): FEM Anwendungspraxis. Deutsch-englische Ausgabe. Vieweg Verlag

Nebojsa Mojsilovic (1994): Versuche an kombiniert beanspruchten

Mauerwerkswänden. IBK, ETH Zürich

Zahlten, W.: Lineare Finite-Elemente-Methoden. Skript RWTH Aachen