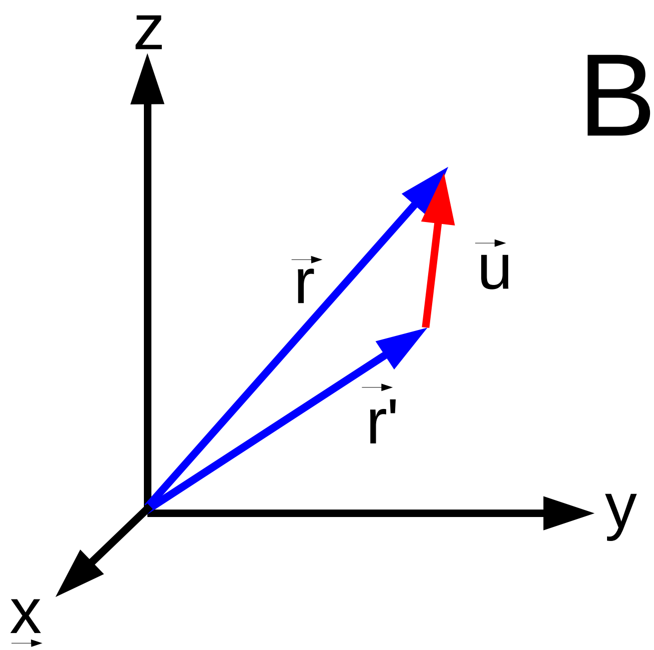
Formulation of the displacement-based Finite Element Method and General Convergence Results

Basics of Elasticity Theory



$\vec{r}$: before deformation
$\vec{r}'$: after deformation
$\vec{u}$: displacement

strain ε: measure of relative distortions

$$\varepsilon^T = (\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{zz}\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx})$$

for small displacements $\vec{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

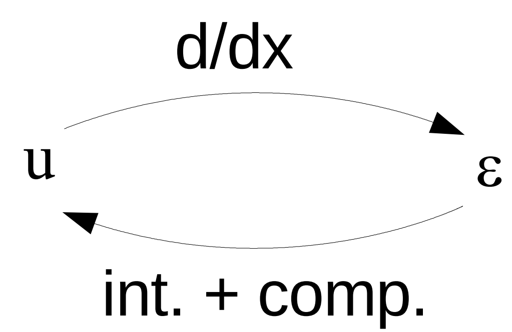
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

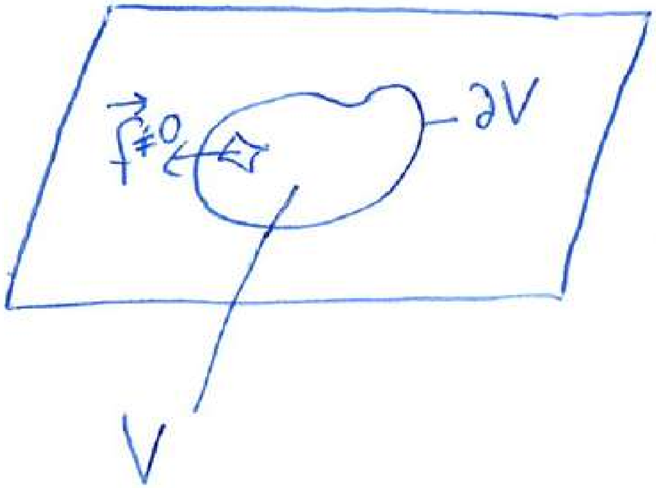
compatibility of the strain field:

$$\frac{\partial^2 \varepsilon_{ii}}{\partial x_k^2} + \frac{\partial^2 \varepsilon_{kk}}{\partial x_i^2} - 2\frac{\partial^2 \varepsilon_{ik}}{\partial x_i \partial x_k} = 0$$

$$\frac{\partial^2 \varepsilon_{ik}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{ii}}{\partial x_k^2} - \frac{\partial^2 \varepsilon_{kk}}{\partial x_i \partial x_l} = 0$$



Basics of Elasticity Theory



deformed solid

deformation creates internal forces

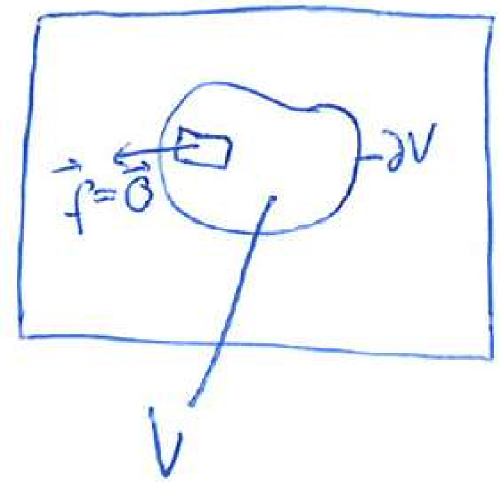
$$\tau = C\varepsilon + \tau^I$$

$$\tau^T = [\tau_{xx} \; \tau_{yy} \; \tau_{zz} \; \tau_{xy} \; \tau_{yz} \; \tau_{zx}]$$

C: material matrix (symmetric)

ε : strain

τ^I : stress b.c.



solid in equilibrium

$$\nabla \cdot \vec{\tau} = 0$$

$$\tau = C\varepsilon + \tau^I$$

$$\nabla \cdot \vec{\tau} + \vec{f} = 0$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

the problem of elasticity:

$$G \cdot \left\{ \nabla \cdot \nabla \vec{u} - \frac{1}{1-2\nu} \nabla(\nabla \cdot \vec{u}) \right\} + \rho \vec{f} = 0$$

find u in the presence of boundary conditions:

$u \big|_{S_u}$ (essential) $\qquad f \big|_{S_f}$ (natural)

The principle of virtual displacements

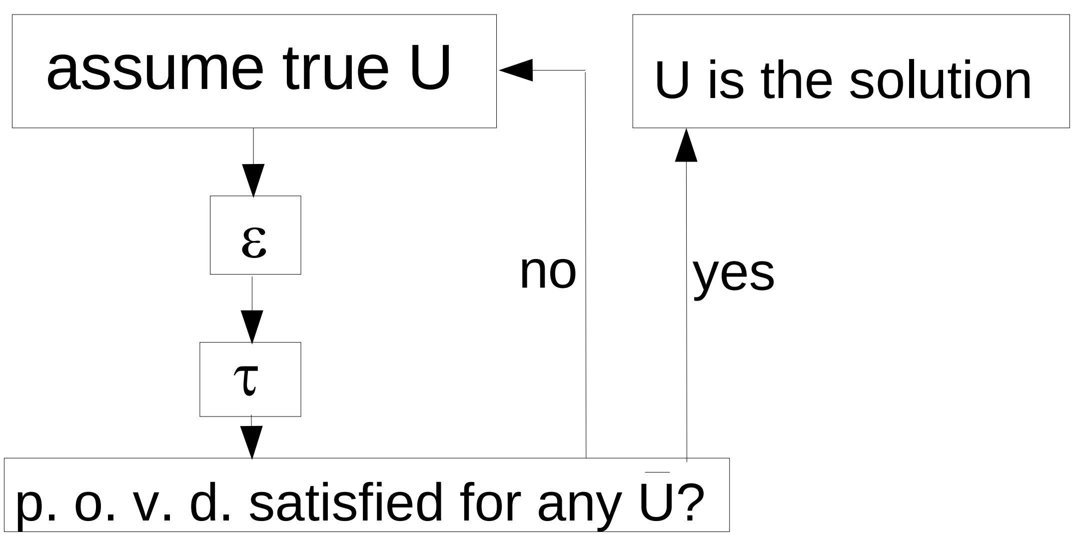
for any $\bar{U}$ that obey the b.c. and the true equilibrium stress field:

internal virtual work — virtual work by external forces

$$\int_V \bar{\boldsymbol{\varepsilon}}^{T} \boldsymbol{\tau} \, dV = \int_V \bar{\mathbf{U}}^{T} \mathbf{f}^{B} \, dV + \int_{S_f} \bar{\mathbf{U}}^{S_f\,T} \mathbf{f}^{S_f} \, dS + \sum_i \bar{\mathbf{U}}^{i\,T} \mathbf{R}_C^{i}$$

stress field in equilibrium with

virt. strain derived from virt. displacements $\bar{U}$

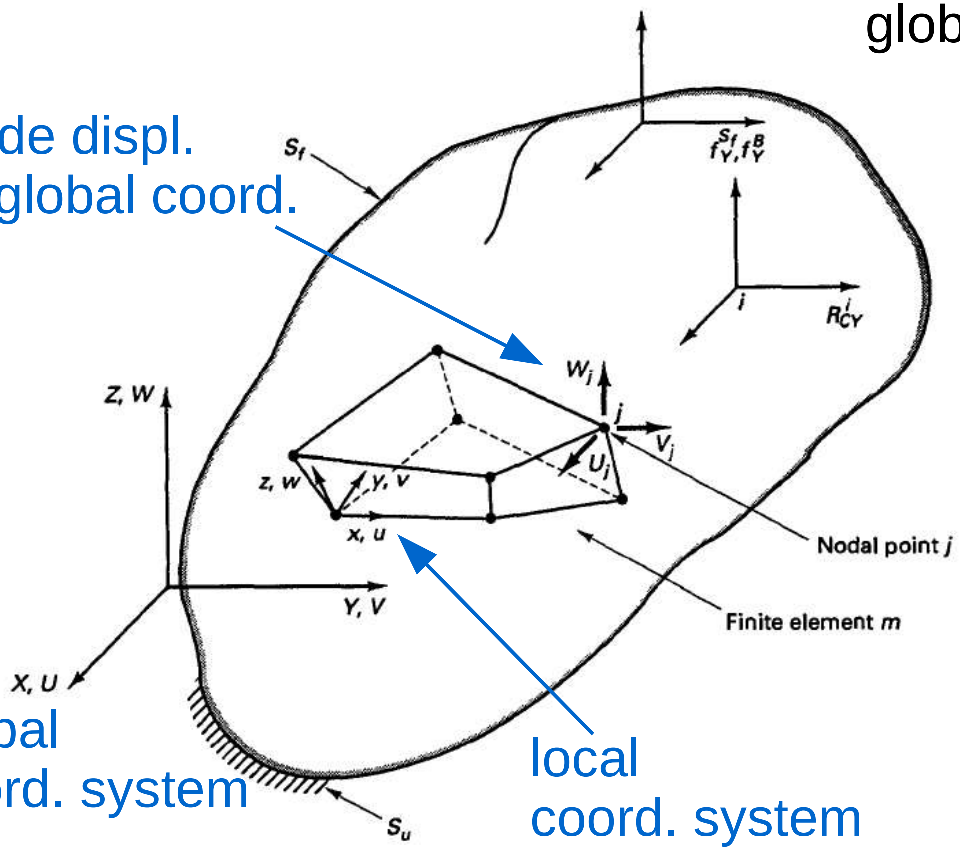


implicitly contains:

- force boundary conditions
- direct stiffness method
  (in FEM formulation)

FEM: domain discretization

node displ.
in global coord.



global
coord. system

local
coord. system
of m-th element

global, nodal displacements:

$$\hat{\mathbf{U}}^T = [U_1\ V_1\ W_1 \quad U_2\ V_2\ W_2 \quad \ldots \quad U_N\ V_N\ W_N]$$

interpolation matrix $H^{(m)}$:

$$\mathbf{u}^{(m)}(x, y, z) = \mathbf{H}^{(m)}(x, y, z)\ \hat{\mathbf{U}}$$

strain-displacement matrix $B^{(m)}$:

$$\boldsymbol{\epsilon}^{(m)}(x, y, z) = \mathbf{B}^{(m)}(x, y, z)\hat{\mathbf{U}}$$

it holds: $\boldsymbol{\tau}^{(m)} = \mathbf{C}^{(m)}\boldsymbol{\epsilon}^{(m)} + \boldsymbol{\tau}^{I(m)}$

solid decomposed into
assemly of finite elements:
no overlap no holes
nodal points coincide

principle of virtual displacements
for the discretized body:

$$\sum_m \int_{V^{(m)}} \bar{\boldsymbol{\epsilon}}^{(m)T} \boldsymbol{\tau}^{(m)}\, dV^{(m)} = \sum_m \int_{V^{(m)}} \bar{\mathbf{u}}^{(m)T} \mathbf{f}^{B(m)}\, dV^{(m)}$$
$$+ \sum_m \int_{S_1^{(m)}, \ldots, S_q^{(m)}} \bar{\mathbf{u}}^{S(m)T} \mathbf{f}^{S(m)}\, dS^{(m)} + \sum_i \bar{\mathbf{u}}^{iT} \mathbf{R}_C^i$$

FEM: matrix equations

ansatz for displacement field

$$\boldsymbol{\epsilon}^{(m)}(x, y, z) = \mathbf{B}^{(m)}(x, y, z)\hat{\mathbf{U}}$$
$$\mathbf{u}^{(m)}(x, y, z) = \mathbf{H}^{(m)}(x, y, z)\,\hat{\mathbf{U}}$$

ansatz for virt. displacement field

$$\overline{\mathbf{u}}^{(m)}(x, y, z) = \mathbf{H}^{(m)}(x, y, z)\hat{\overline{\mathbf{U}}}$$
$$\overline{\boldsymbol{\epsilon}}^{(m)}(x, y, z) = \mathbf{B}^{(m)}(x, y, z)\hat{\overline{\mathbf{U}}}$$

discretized principle of virt. displ.

discrete displacement field to be determined

$$\hat{\overline{\mathbf{U}}}^T\left[\sum_m \int_{V^{(m)}} \mathbf{B}^{(m)T}\mathbf{C}^{(m)}\mathbf{B}^{(m)}\,dV^{(m)}\right]\hat{\mathbf{U}} = \hat{\overline{\mathbf{U}}}^T\left[\left\{\sum_m \int_{V^{(m)}} \mathbf{H}^{(m)T}\mathbf{f}^{B(m)}\,dV^{(m)}\right\} + \left\{\sum_m \int_{S_1^{(m)},\ldots,S_q^{(m)}} \mathbf{H}^{S(m)T}\mathbf{f}^{S(m)}\,dS^{(m)}\right\} - \left\{\sum_m \int_{V^{(m)}} \mathbf{B}^{(m)T}\boldsymbol{\tau}^{I(m)}\,dV^{(m)}\right\} + \mathbf{R}_C\right]$$

$\hat{\mathbf{U}}$ solution when p.o.v.d. fullfilled $\forall\ \hat{\overline{\mathbf{U}}}$ especially: $\hat{\overline{\mathbf{U}}} = e_i$

with $\mathbf{U} \equiv \hat{\mathbf{U}}$ one obtains: $\mathbf{KU} = \mathbf{R}$, where $\mathbf{R} = \mathbf{R}_B + \mathbf{R}_S - \mathbf{R}_I + \mathbf{R}_C$

$$\mathbf{K} = \sum_m \underbrace{\int_{V^{(m)}} \mathbf{B}^{(m)T}\mathbf{C}^{(m)}\mathbf{B}^{(m)}\,dV^{(m)}}_{=\,\mathbf{K}^{(m)}}$$

$$\mathbf{R}_B = \sum_m \underbrace{\int_{V^{(m)}} \mathbf{H}^{(m)T}\mathbf{f}^{B(m)}\,dV^{(m)}}_{=\,\mathbf{R}_B^{(m)}} \qquad \mathbf{R}_S = \sum_m \underbrace{\int_{S_1^{(m)},\ldots,S_q^{(m)}} \mathbf{H}^{S(m)T}\mathbf{f}^{S(m)}\,dS^{(m)}}_{=\,\mathbf{R}_S^{(m)}}$$

$$\mathbf{R}_I = \sum_m \underbrace{\int_{V^{(m)}} \mathbf{B}^{(m)T}\boldsymbol{\tau}^{I(m)}\,dV^{(m)}}_{=\,\mathbf{R}_I^{(m)}}$$

implicit direct stiffness

boundary forces contained

FEM: matrix equations

dynamic case with dissipative forces ~ dU/dt:

modified body forces: $\mathbf{R}_B = \sum_m \int_{V^{(m)}} \mathbf{H}^{(m)T}[\mathbf{f}^{B(m)} - \rho^{(m)}\mathbf{H}^{(m)}\ddot{\mathbf{U}} - \kappa^{(m)}\mathbf{H}^{(m)}\dot{\mathbf{U}}]\, dV^{(m)}$

inertial forces damping forces

define:

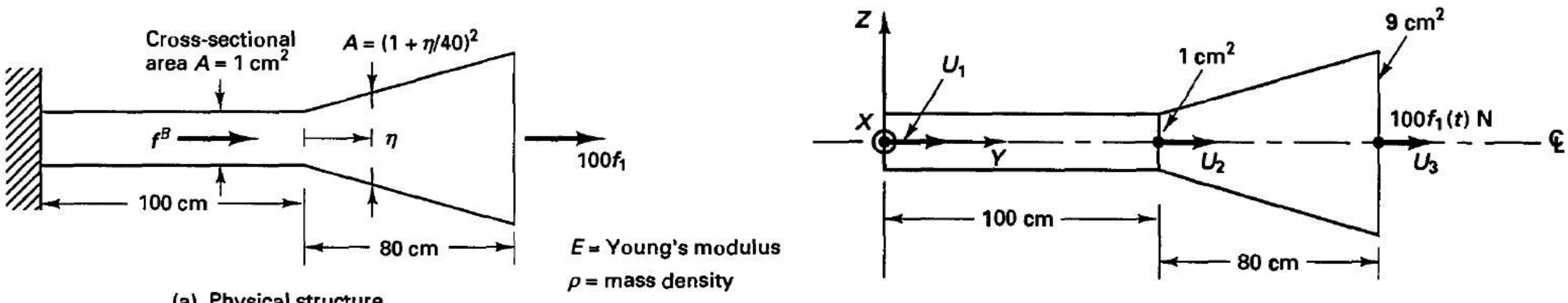
$$\mathbf{C} = \sum_m \underbrace{\int_{V^{(m)}} \kappa^{(m)}\mathbf{H}^{(m)T}\mathbf{H}^{(m)}\, dV^{(m)}}_{= \mathbf{C}^{(m)}} \qquad \mathbf{M} = \sum_m \underbrace{\int_{V^{(m)}} \rho^{(m)}\mathbf{H}^{(m)T}\mathbf{H}^{(m)}\, dV^{(m)}}_{= \mathbf{M}^{(m)}}$$

$$\rightarrow \quad \mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$$

$$\boldsymbol{\epsilon}^{(m)}(x, y, z) = \mathbf{B}^{(m)}(x, y, z)\hat{\mathbf{U}}$$
$$\mathbf{u}^{(m)}(x, y, z) = \mathbf{H}^{(m)}(x, y, z)\,\hat{\mathbf{U}}$$
$$\overline{\mathbf{u}}^{(m)}(x, y, z) = \mathbf{H}^{(m)}(x, y, z)\hat{\overline{\mathbf{U}}}$$
$$\overline{\boldsymbol{\epsilon}}^{(m)}(x, y, z) = \mathbf{B}^{(m)}(x, y, z)\hat{\overline{\mathbf{U}}}$$

→ K and M are symmetric

FEM: Two-element bar assemblage



(a) Physical structure

(b) Element assemblage in global system

$$u^{(m)}(x) = H^{(m)}(x)\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} H_1^{(m)}(x) & H_2^{(m)}(x) & H_3^{(m)}(x) \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

two 2-point elements, polynomial ansatz

→ $u^{(m)}(x)$ linear: $H_i^{(m)}(x) = \alpha_i^{(m)} + \beta_i^{(m)} x$

interpolation condition: $u^{(1)}(0) = U_1 \;,\; \forall U_2$ $\quad H_3^{(1)}(x) = 0$ $\quad$ m = 1

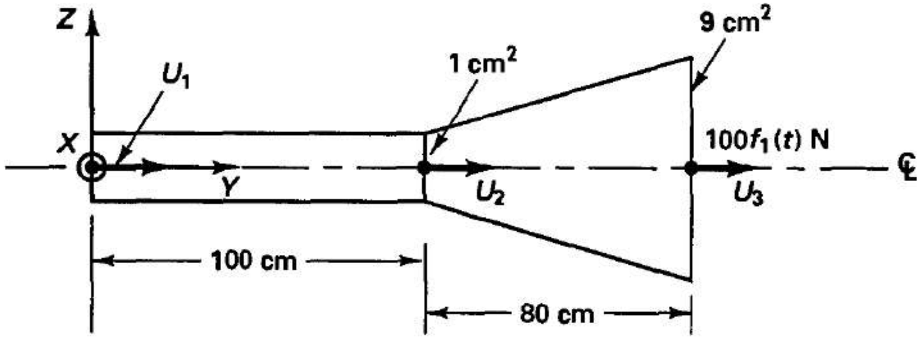
$$u^{(1)}(100) = U_2 \;,\; \forall U_1$$

⟷ $u^{(1)}(0) = \alpha_1^{(1)} U_1 + \alpha_2^{(1)} U_2 = U_1 \;,\; \forall U_2$ $\qquad u^{(1)}(100) = (1 + 100 \cdot \beta_1^{(1)}) U_1 + \beta_2^{(1)} U_2 = U_2 \;,\; \forall U_1$

$\Rightarrow \alpha_1^{(1)} = 1 \;,\; \alpha_2^{(1)} = 0$ $\qquad \Rightarrow \beta_2^{(1)} = 1 \;,\; \beta_1^{(1)} = -\frac{1}{100}$

$$\Rightarrow u^{(1)}(x) = \left(1 - \frac{x}{100}\right) U_1 + \frac{x}{100} U_2 \quad \Longrightarrow \quad H^{(1)}(x) = \begin{bmatrix} \left(1 - \frac{x}{100}\right) & \frac{x}{100} & 0 \end{bmatrix}$$

FEM: Two-element bar assemblage



(b) Element assemblage in global system

calculate $B^{(1)}(x)$ for given $u^{(1)}(x)$:

$$\varepsilon^{(m)}(x) = \begin{bmatrix} B_1^{(m)}(x) & B_2^{(m)}(x) & B_3^{(m)}(x) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$= B_1^{(m)}(x) U_1 + B_2^{(m)}(x) U_2 + B_3^{(m)}(x) U_3$$

m = 1:

$$\varepsilon^{(1)}(x) = \frac{d}{dx} u^{(1)}(x) = -\frac{1}{100} U_1 + \frac{1}{100} U_2$$

$$\Rightarrow B^{(1)}(x) = \begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix}$$

analogous:

$$\mathbf{H}^{(2)} = \begin{bmatrix} 0 & \left(1 - \frac{x}{80}\right) & \frac{x}{80} \end{bmatrix}$$

$$\mathbf{B}^{(2)} = \begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix}$$

material matrix:

$$\mathbf{C}^{(1)} = E; \qquad \mathbf{C}^{(2)} = E$$

FEM: Two-element bar assemblage

slow load application: $\mathbf{KU} = \mathbf{R}$, $\mathbf{R} = \mathbf{R}_B + \mathbf{R}_S - \mathbf{R}_I + \mathbf{R}_C$

$$\mathbf{K} = \sum_m \underbrace{\int_{V^{(m)}} \mathbf{B}^{(m)T}\mathbf{C}^{(m)}\mathbf{B}^{(m)}\, dV^{(m)}}_{= \mathbf{K}^{(m)}}$$

$$K = \int_0^{100} A B^{(1)T}(x)\, C^{(1)} B^{(1)}(x)\, dx + \int_0^{80} A_2 B^{(2)T}(x)\, C^{(2)} B^{(2)}(x)\, dx$$

$$= C^{(1)}\int_0^{100} A_1\, dx \begin{bmatrix} -1/100 \\ 1/100 \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix} dx + C^{(2)}\int_0^{80} A_1\left(1+\frac{x}{40}\right)^2 dx \begin{bmatrix} 0 \\ -1/80 \\ 1/80 \end{bmatrix}\begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix}$$

$$= C^{(1)} A_1 \cdot 100 \cdot \begin{bmatrix} 1/10^4 & -1/10^4 & 0 \\ -1/10^4 & 1/10^4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + C^{(2)} A_1 \int_0^{80} \left(1+\frac{x}{40}\right)^2 dx \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/80^2 & -1/80^2 \\ 0 & -1/80^2 & 1/80^2 \end{bmatrix}$$

$$= A_1 E \cdot \left(\frac{1}{100} \cdot \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{13}{240} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right)$$

$$\mathbf{R}_B = \sum_m \underbrace{\int_{V^{(m)}} \mathbf{H}^{(m)T}\mathbf{f}^{B(m)}\, dV^{(m)}}$$

$$\mathbf{R}_B = \left\{ (1)\int_0^{100} \begin{bmatrix} 1 - \frac{x}{100} \\ \frac{x}{100} \\ 0 \end{bmatrix} (1)\, dx + \int_0^{80} \left(1 + \frac{x}{40}\right)^2 \begin{bmatrix} 0 \\ 1 - \frac{x}{80} \\ \frac{x}{80} \end{bmatrix} \left(\frac{1}{10}\right) dx \right\} f_2(t)$$

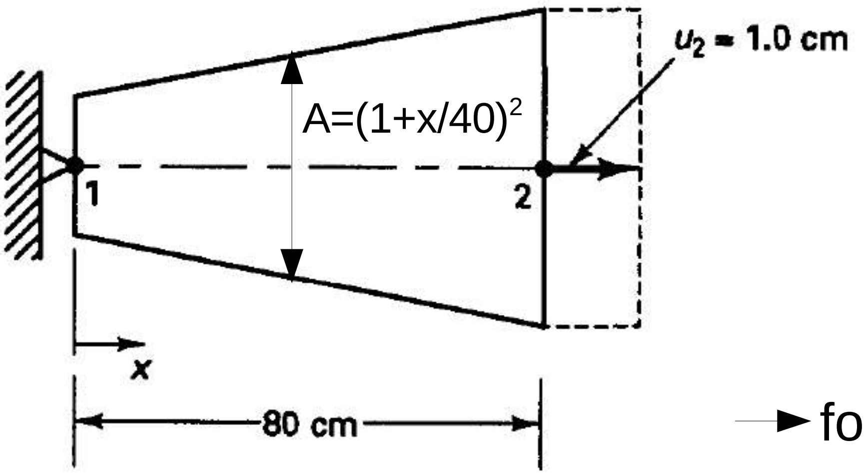
$$= \frac{1}{3}\begin{bmatrix} 150 \\ 186 \\ 68 \end{bmatrix} f_2(t)$$

$$\mathbf{R}_C = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} f_1(t)$$

$R_S = R_I = 0$

obtain U(t*) by solving: $\mathbf{KU}\big|_{t=t^*} = \mathbf{R}_B\big|_{t=t^*} + \mathbf{R}_C\big|_{t=t^*}$

FEM: Exact stiffness vs. FEM approx.



from variational principle: $E\frac{d}{dx}\left(A\frac{du}{dx}\right) = 0$

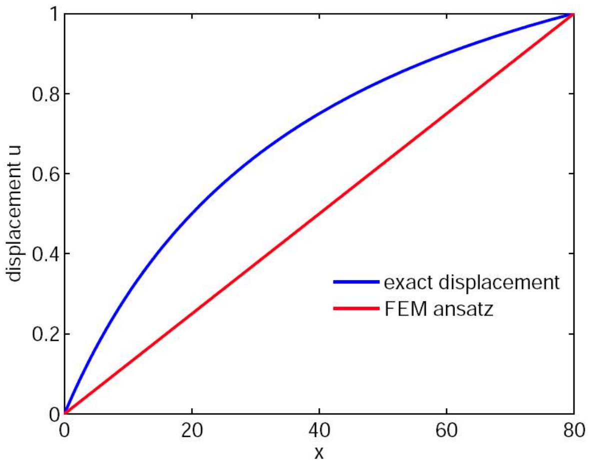
exact solution w. b.c. u(0) = 0 and u(80) = 1:

$$u = \frac{3}{2}\left(1 - \frac{1}{1 + x/40}\right)$$

→ forces at the ends of the bar:

$$k_{12} = -EA\left.\frac{du}{dx}\right|_{x=0}$$

$$k_{22} = EA\left.\frac{du}{dx}\right|_{x=L}$$



→ $$\mathbf{K} = \frac{3}{80}E\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(exact stiffness matrix)

$$\mathbf{K}_{FEM} = \frac{13E}{240}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(FEM approx., prev. example)

displacement ansatz too rigid → overestimate of stiffness

FEM: Displacement constraints

decompose U = $\begin{bmatrix} \mathbf{U}_a \\ \mathbf{U}_b \end{bmatrix}$

$\mathbf{U}_a$: unconstrained

$\mathbf{U}_b$: constrained

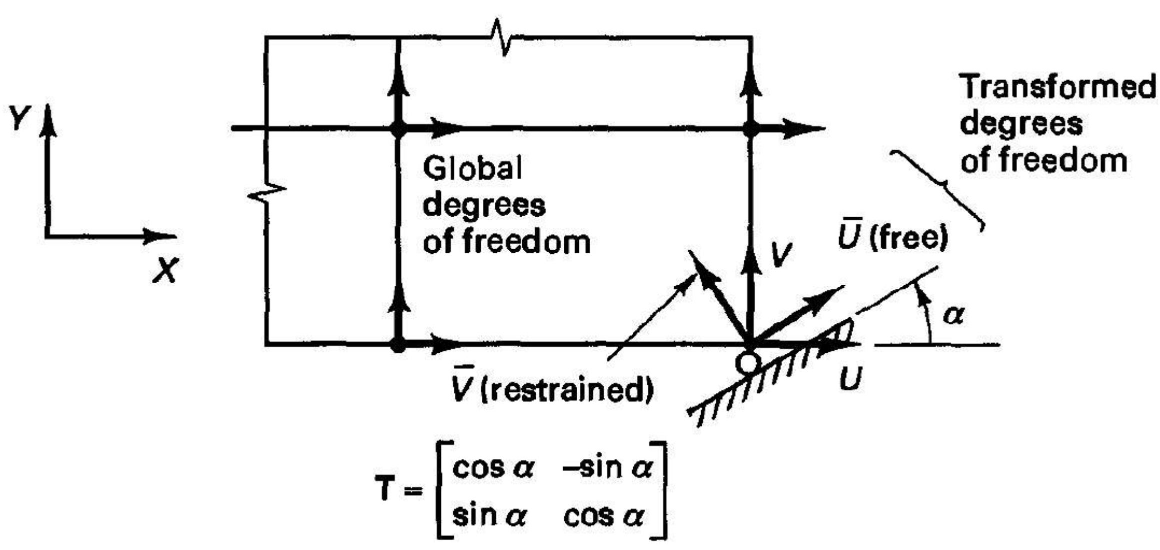
FEM equations:

$$\begin{bmatrix} \mathbf{M}_{aa} & \mathbf{M}_{ab} \\ \mathbf{M}_{ba} & \mathbf{M}_{bb} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}}_a \\ \ddot{\mathbf{U}}_b \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{U}_a \\ \mathbf{U}_b \end{bmatrix} = \begin{bmatrix} \mathbf{R}_a \\ \mathbf{R}_b \end{bmatrix}$$

$$\mathbf{M}_{aa}\ddot{\mathbf{U}}_a + \mathbf{K}_{aa}\mathbf{U}_a = \underbrace{\mathbf{R}_a - \mathbf{K}_{ab}\mathbf{U}_b - \mathbf{M}_{ab}\ddot{\mathbf{U}}_b}_{\text{modified force vector}}$$

modified force vector

if constraints don't coincide with U_b, try: $\mathbf{U} = \mathbf{T}\overline{\mathbf{U}}$ where $\overline{U}_b$ can be constrained



need to transform FEM matrices accordingly:

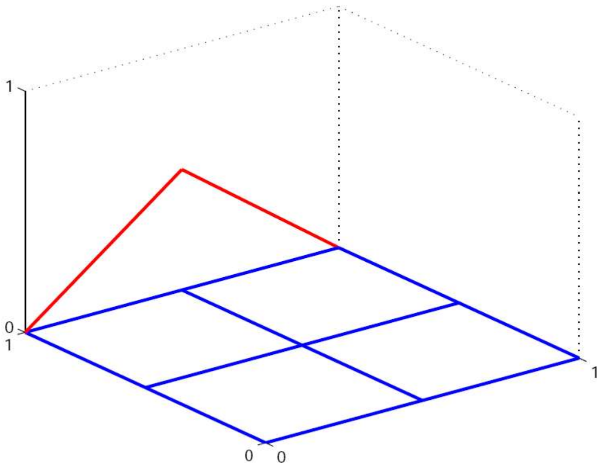
$$\overline{\mathbf{M}} = \mathbf{T}^T\mathbf{M}\mathbf{T}$$

$$\overline{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\mathbf{T}$$

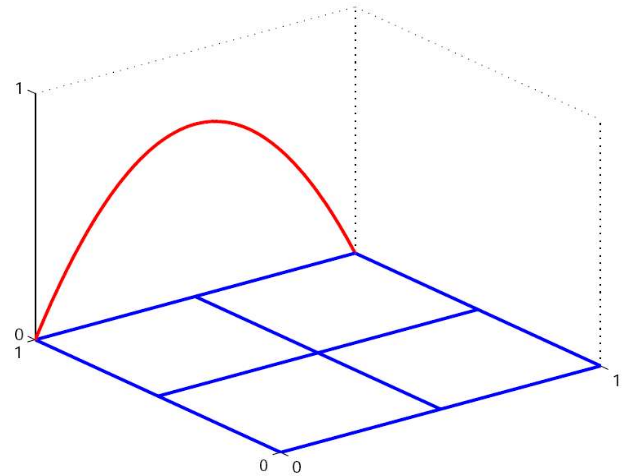
$$\overline{\mathbf{R}} = \mathbf{T}^T\mathbf{R}$$

$$\overline{\mathbf{M}}\ddot{\overline{\mathbf{U}}} + \overline{\mathbf{K}}\overline{\mathbf{U}} = \overline{\mathbf{R}}$$

Refining the FEM solution

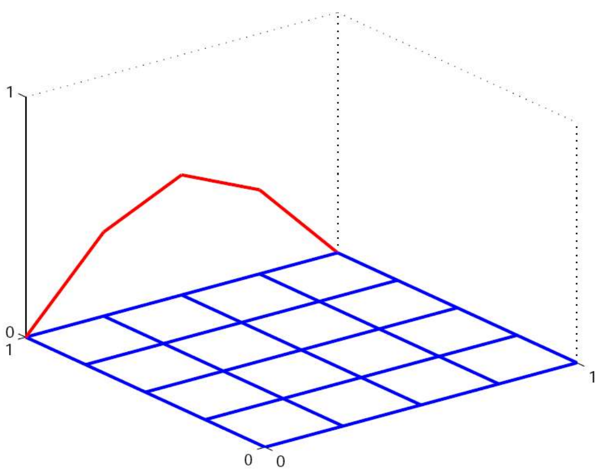


keep element size



p refinement

keep order
of interpolation



h refinement

h/p refinement:
both simultaneously

Convergence of a FEM solution

$$\int_V \bar{\boldsymbol{\epsilon}}^T \boldsymbol{\tau}\, dV = \int_{S_f} \bar{\mathbf{u}}^{S_f^T} \mathbf{f}^{S_f}\, dS + \int_V \bar{\mathbf{u}}^T \mathbf{f}^B\, dV \quad \longleftrightarrow \quad a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

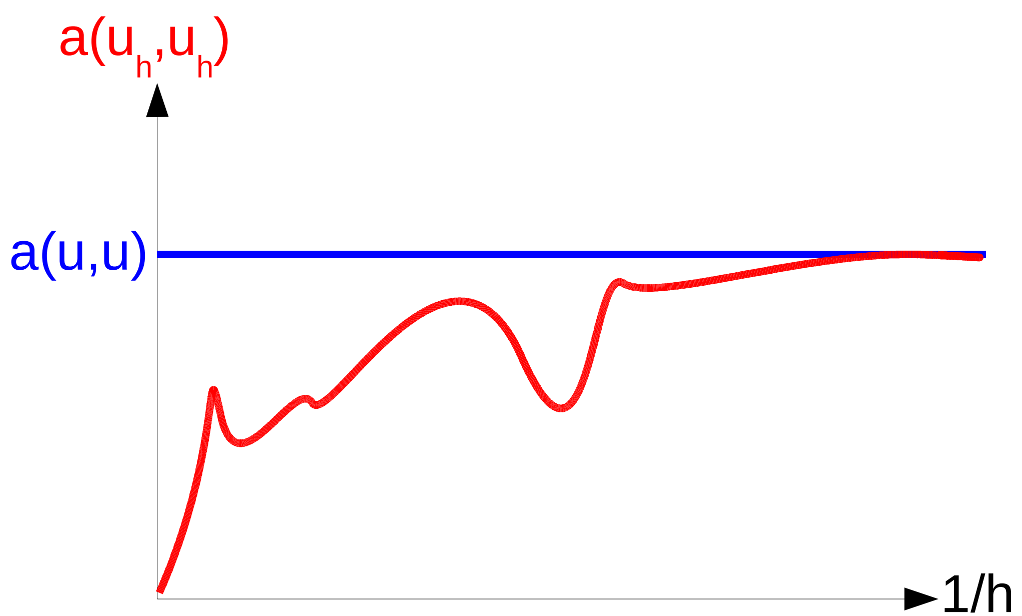
strain energy $a(\mathbf{u}, \mathbf{v})$ pos. def. bilinear form

Convergence in the norm induced by a:

$$a(\mathbf{u}_h, \mathbf{u}_h) \rightarrow a(\mathbf{u}, \mathbf{u}) \qquad \text{as } h \rightarrow 0$$

measures discretization and interpolation errors, only

monotonic convergence:



necessary:

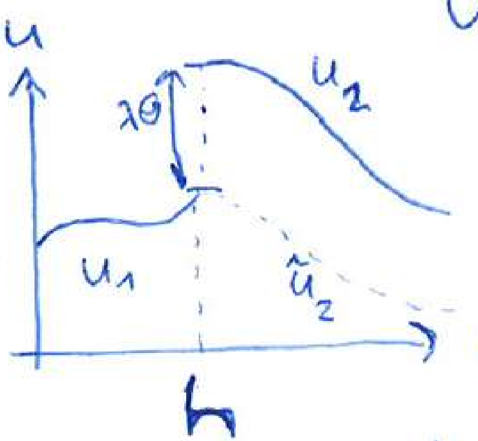
- compatibility of mesh and elements
- completeness of elements

mon. Conv.: Compatibility

Compatibility:

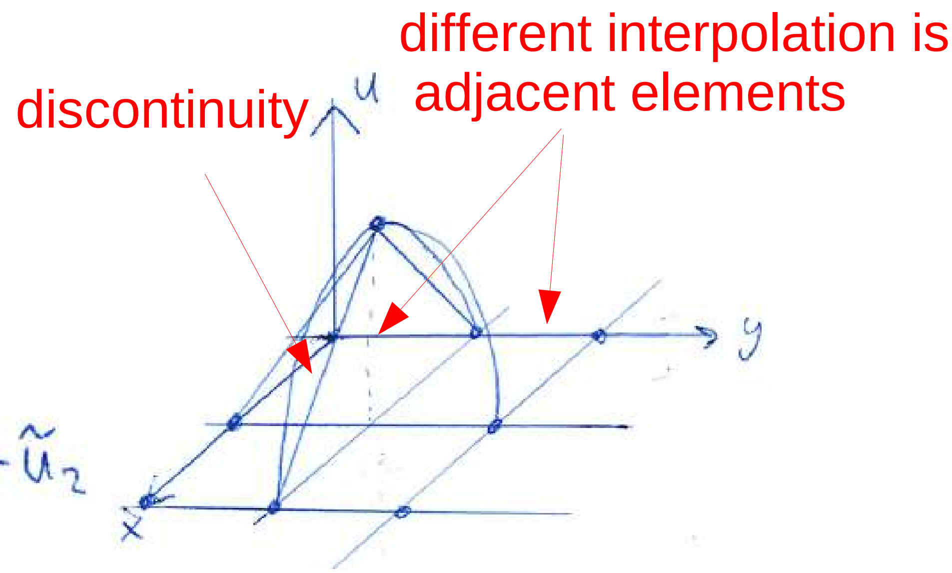
continuity of u within and
across element boundaries

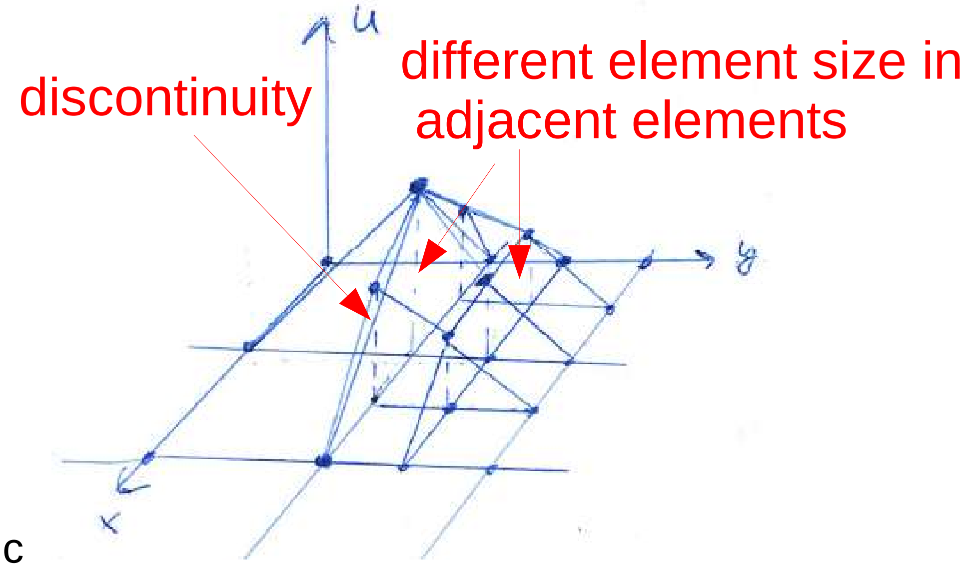
Example:



$$u = \begin{cases} u_1, & x<h \\ u_2 = \lambda\Theta(x-h)+\tilde{u}_2 \end{cases}$$

$$\frac{du_2}{dx} \propto \lambda\delta(x-h)+\frac{d\tilde{u}_2}{dx}$$





calculation of $a(u_h,u_h)$:

$u \xrightarrow{d/dx} \varepsilon$

$\varepsilon \xrightarrow{C} \tau$

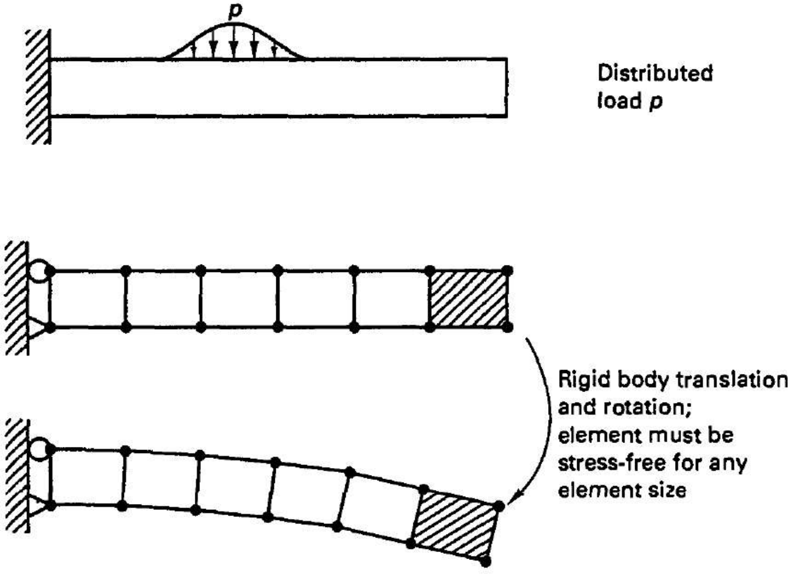
delta function gives unphysical contributions to $a(u_h,u_h)$ and destroys monotonic convergence

mon. Conv.: Completeness

Completeness:

element must be able to represent all rigid-body modes and states of constant strain

Reason:



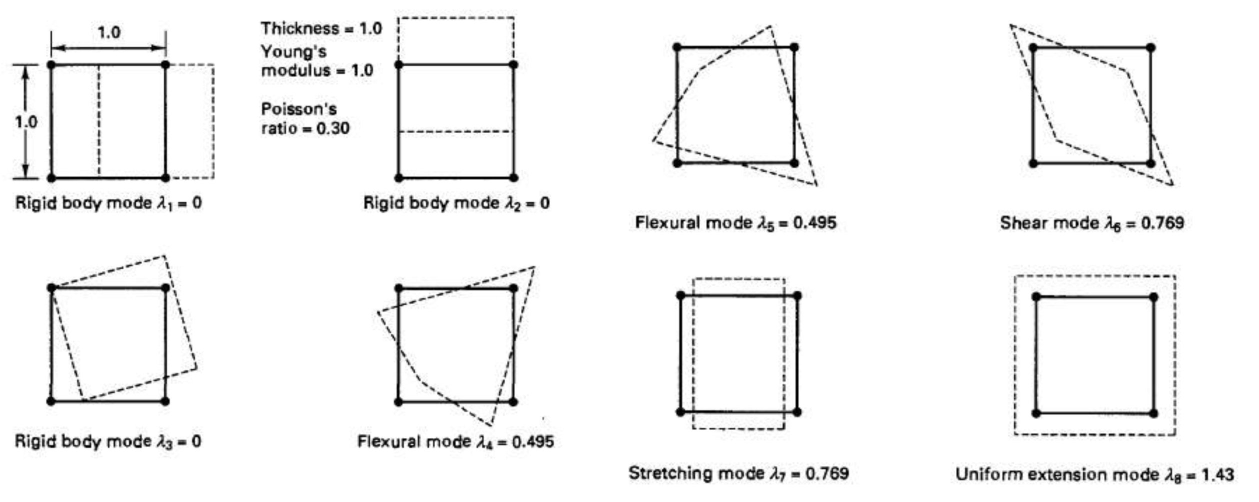
Number of rigid-body modes:

$$\mathbf{K}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}$$

$\rightarrow \lambda_1, \ldots \lambda_n$

#rbm = dim(Ker(K))

rbm determined by the eigenvectors of K:



Convergence rates

Lack of smoothness	→	slower convergence

Assumptions:

- interpolation w. complete polynomials up to order k

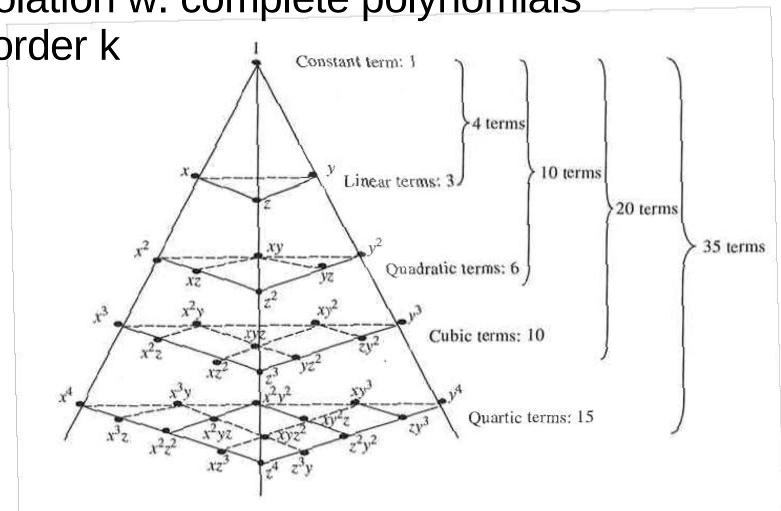


Figure 3.3. Pascal pyramid of monomials (three-dimensional case).

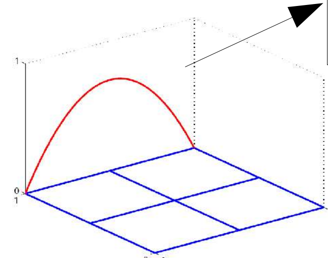
- exact solution is smooth enough, such that Sobolev-Norm of order k+1 is finite:

$$\|\mathbf{u}\|_{k+1} = \left\{ \int_{\mathrm{Vol}} \left[\sum_{i=1}^{3} (u_i)^2 + \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} \right)^2 + \sum_{n=2}^{k+1} \sum_{i=1}^{3} \sum_{r+s+t=n} \left(\frac{\partial^n u_i}{\partial x_1^r \, \partial x_2^s \, \partial x_3^t} \right)^2 \right] d\mathrm{Vol} \right\}^{1/2} < \infty$$

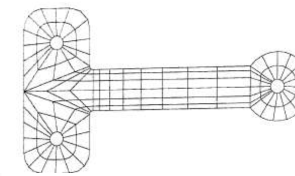
- uniform mesh

Estimate (p-refinement):

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le c \, h^{k+1} \, \|\mathbf{u}\|_{k+1}$$



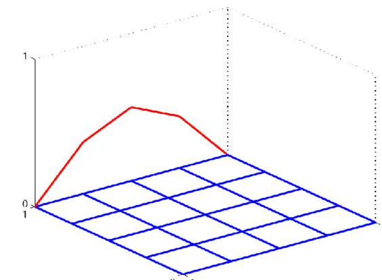
non-uniform grid:



$$\|\mathbf{u} - \mathbf{u}_h\|_1^2 \le c \sum_m h_m^{2k} \, \|\mathbf{u}\|_{k+1,m}^2$$

h refinement:

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \le \frac{c}{(N)^{k/d}}$$



h/p refinement:

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \le \frac{c}{\exp\left[\beta (N)^{\gamma}\right]}$$

Def.: (Sobolev-Norm of order 0)

$$(\|\mathbf{v}\|_0)^2 = \int_{\mathrm{Vol}} \left(\sum_{i=1}^{3} (v_i)^2 \right) d\mathrm{Vol}$$

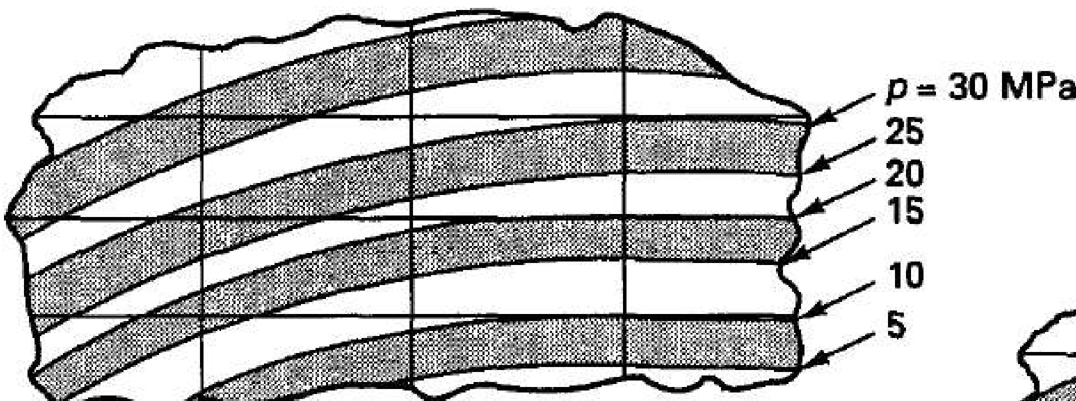
Error assessment

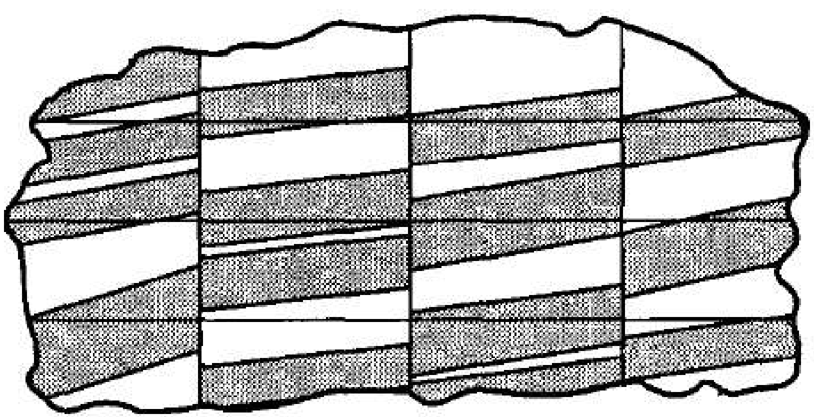
FEM solution violates differential equilibrium:

$$\nabla \cdot \vec{\tau} + \vec{f} \neq 0$$

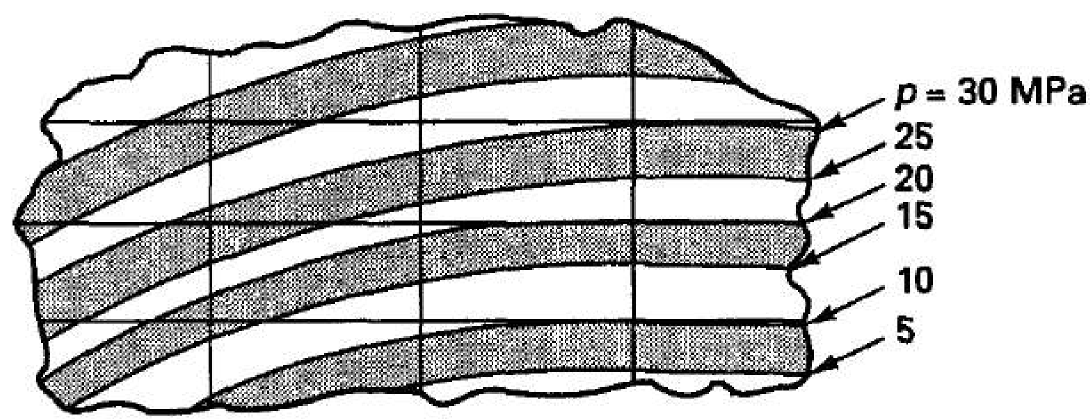
→ non-continous stress possible

Graphical assement: Isoband-plots





$\Delta p > 5$ MPa.



Thank you for your attention!