Chapter 3: Some Basic Concepts of Engineering Analysis and an Introduction to the Finite Element Method

<u>Objective:</u> Discuss some classical techniques used for the formulation and solution of mathematical models of engineering systems

2 mathematical models

Lumped-parameter Models	Continuum-mechanic-based Models
also: „discrete systems"	„continuous systems"
• the systems response is described by the solution of a finite number of <u>state variables</u>	• instead of a set of algebraic equations for the unknown state variables, differential equations govern the response

⇒ Engineer must decide wether the system should be represented by a lumped-parameter or a continuum-mechanic-based model

Lumped-parameter Models

A) Discrete Problems   Exp. 3.1, 3.2, 3.3, 3.4, 3.5

The solution requires the following steps:

1) System idealization
2) Element equilibrium
3) Element assemblage
4) Calculation of response

B) Variational formulation

$$\Pi = U - W$$

$\Pi :=$ total potential

$U :=$ strain Energy

$W :=$ total potential of the loads

for instance: Structural mechanics:

$$U = \frac{1}{2} \cdot \int_V \sigma_{ij} \cdot \varepsilon_{ij} \, dV$$

$$W = -\int_\Omega f_i \cdot u_i \cdot d\Omega$$

f_i: body forces

u_i: deformation field(s).

Note: The state variables can in many analysis also be obtained using an extremum of a given funtional

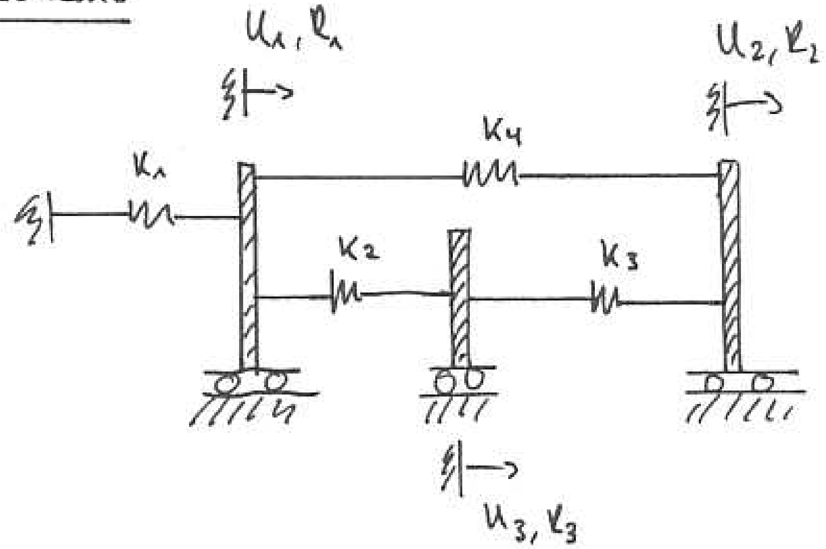
$$\Pi(u_1, u_2, u_3 \dots u_i)$$

$u_1, u_2, u_3 \dots u_i$:= state variables

$$\frac{\partial \Pi}{\partial u_i} \stackrel{!}{=} 0$$

Exp. Disrete Problems

Exp. 3.1



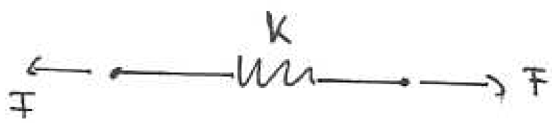
State variables : u_1, u_2, u_3

loads : R_1, R_2, R_3

spring stifness : K_1, K_2, K_3, K_4

- local equilibrium:

Spring law :



$$\boxed{F = \Delta X \cdot K}$$

- Element assemblage:

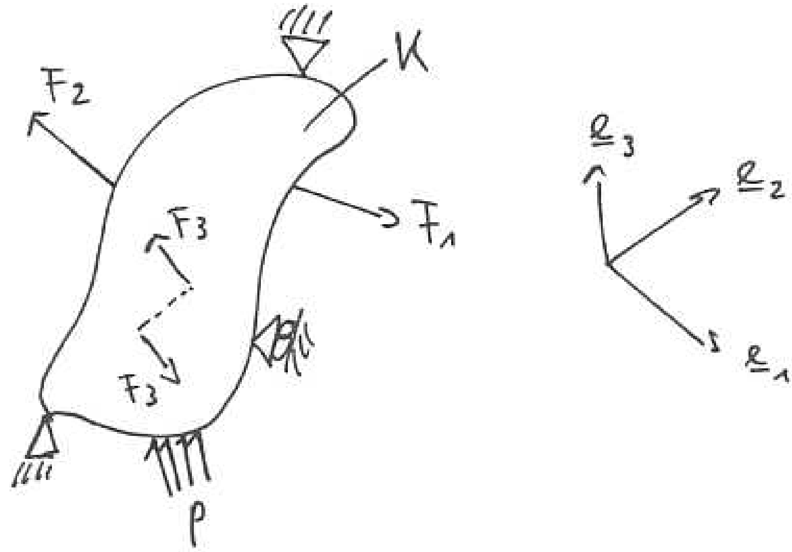
$$\boxed{\underline{\underline{K}} \cdot \underline{U} = \underline{R}}$$

Other Problems:

Heat transfer, electrical, fluid flow, structural

Continuum-mechanic-based models

Main problem of continuum-mechanic is:

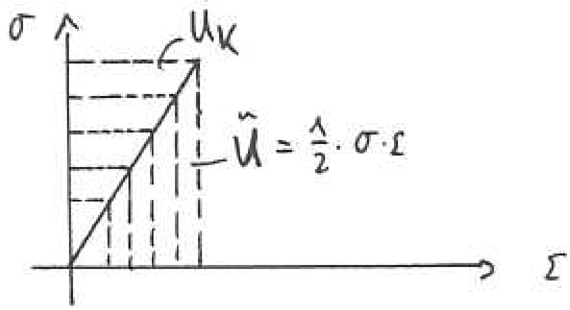


Given: Structure K with kinematic and statical boundary conditions

to seek:
- stress distribution in K
- deformation

Principle of Minimum complementary strain Energy

$$U_K = \frac{1}{2} \cdot \int_V \sigma_{ij} \cdot \varepsilon_{ij} \, dV$$: complementary strain Energy



Def. permissible stress field $\overset{**}{\sigma}_{ij}(x)$ satisfies:
- $\overset{**}{\sigma}_{ij,j} + f_i = 0$
- $\overset{**}{\sigma}_{ij}$ (bound.)

$\Downarrow$

constitutive law: $\overset{**}{\varepsilon}_{kl}(x) = C_{klij} \cdot \overset{**}{\sigma}_{ij}$

compatible distortion field:

$\overset{**}{\varepsilon}_{kl}$ satisfies not
- $\overset{**}{\varepsilon}_{ij} \neq \frac{1}{2}(\overset{**}{u}_{i,j} + \overset{**}{u}_{j,i})$
- Kinematic boundary conditions
- continuity

$$\overset{**}{U}_K = \frac{1}{2} \int_V \overset{**}{\sigma}_{ij} \overset{**}{\varepsilon}_{ij} \, dV$$

Def.: real stress field with permissible ε_{kl}

$$U_K = \frac{1}{2} \int_V \sigma_{ij} \cdot \varepsilon_{ij} \, dV$$

$$\boxed{\overset{**}{U}_K \geq U_K}$$ lower bound

=> Castigliano is based on the Principle of Min. complementary strain Energy.

Principle of Minimum potential Energy

Given Structure

1) σ_{ij} (boundary c.)
2) f_i : Body forces
3) u_i (boundary c.).

- permissible deformation field :

$$\overset{*}{u}_i(x) \quad \begin{cases} : \text{kinematic b.c} \\ : \text{continuity} \end{cases}$$

$$\Downarrow$$

$$\overset{*}{\varepsilon}_{ij} = \frac{1}{2}(\overset{*}{u}_{i,j} + u_{j,i})$$

$$\Downarrow$$

- compatible stress field:

$$\overset{*}{\sigma}_{ij} = C^{-1}_{ijkl} \cdot \overset{*}{\varepsilon}_{kl}$$

satisfies not!

$: \sigma_{ij,j} + f_i = 0$ $\qquad$ σ_{ij} (boundary) =

$$\boxed{\overset{*}{U} = \frac{1}{2}\int_V \overset{*}{\sigma}_{ij} \cdot \overset{*}{\varepsilon}_{ij}\, dV} \quad \boxed{\overset{*}{W} = -\int_\Omega f_i \cdot \overset{*}{u}_i \, d\Omega}$$

$\overset{*}{\Pi} = u^* + w^*$: total potential.

- real / exacte deformation field $u_i(x)$ satisfies,

$\cdot$ kinematic b.c
$\cdot$ continuity

its stress-field σ_{ij} : $\begin{cases} \sigma_{ij,i} + f_i = 0 & \text{ok} \checkmark \\ \sigma_{ij}(\text{b.c}) & \text{ok} \checkmark \end{cases}$

$$\boxed{\Pi = \frac{1}{2}\int_V \sigma_{ij}\,\varepsilon_{ij}\, dV - \int_\Omega f_i \cdot u_i \, d\Omega}$$

$\boxed{\overset{*}{\Pi} \geq \Pi}$ upper bound.

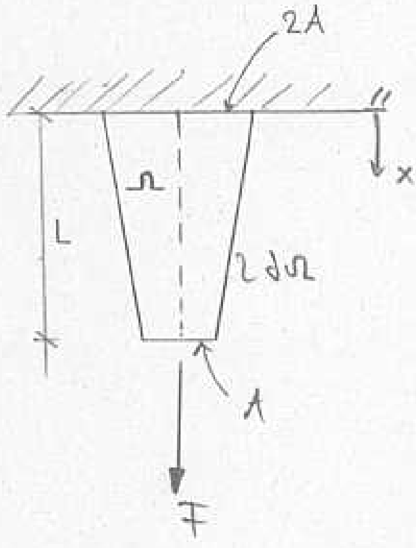
Comment: The majority of the FEM-Software Programs are dealing with the principle of Minimum potential energy.

- Π := functional of the problem
- Two classes of boundary conditions:
  - i) essential {b.c} (geometric)
  - ii) natural b.c (force b.c)
- δ := Variational symbol.

The essence of the variational approach is to calculate the total potential (Π) of the system and to invoke the stationarity of $\Pi \rightarrow \delta\Pi = 0$, with respect to the state variables.

In structural mechanics problems, we employ the Principle of Minimum potential Energy for the variational formulations.

Exp. 1 Principle of the Minimum potential Energy (P.o.t.M.p.E.)



given: Structure with:

- $\sigma_{ij}(x)$: stress boundary conditions
- $u_i(x)$: geometric boundary conditions

A: Surface [m]

F: Force [N]

→ displacement field: arbitrary, permissible: $u^*(x) = u_0^* \cdot \frac{x}{L}$ (willkürlich zulässiges Verschiebungsfeld)

strain Energy: $U_{tot}^* = \int_\Omega u^* \, dV \qquad u^* = \frac{1}{2} \cdot \sigma_{ij}^* \cdot \varepsilon_{ij}^* \qquad \sigma_{ij}^* = C_{ijkl} \cdot \varepsilon_{kl}^*$

$$= \int_0^L \frac{1}{2} \cdot E \cdot \varepsilon^{*2} \cdot \left(2A - A\frac{x}{L}\right) dx \qquad \varepsilon^* = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{u_0^*}{L}$$

$$U_{tot}^* = \frac{3}{4} \frac{A \cdot E}{L} \cdot u_0^{*2}$$

Potential of the external Forces: $V_{tot}^* = -\int_{\partial\Omega} \underline{u}^* \cdot f^0 \, dA \qquad f^0$: body force $= \frac{F}{A}$

$$= -u_0^* \cdot \left(\frac{F}{A}\right) \cdot A = -F \cdot u_0^*$$

$\Pi^* = U_{tot}^* + V_{tot}^* = \frac{3}{4} \frac{AE}{L} \cdot u_0^{*2} - F \cdot u_0^*$: Total Potential of the System.

minimaise Π^*

$$\frac{\partial \Pi^*}{\partial u_0^*} \overset{!}{=} 0 \quad \frac{3}{2} \cdot \frac{AE}{L} \cdot u_0^* = F \quad \Rightarrow \quad \underline{\underline{u_0^* = 0{,}667 \cdot \frac{FL}{AE}}}$$

More accurante solution:

$$\sigma(x) = \frac{F}{A} = \frac{F}{A(2-\frac{x}{L})} \quad : \text{ permissible stress field.}$$

$$\varepsilon(x) = \frac{\sigma(x)}{E} = \frac{F}{AE(2-\frac{x}{L})} \quad : \text{ compatible distortion field}$$

$$\Rightarrow u_0\Big|_{x=L} = \int_0^L \varepsilon(x)\cdot dx = \frac{F}{AE}\cdot\int_0^L \frac{dx}{2-\frac{x}{L}} = \log(2)\cdot\frac{FL}{AE} \cong \underline{\underline{0{,}693\,\frac{FL}{AE}}}$$

∇

- Ansatzlösung von $u^*(x)$ führt auf $u_0^* < u_0$, als wäre Material steifer.

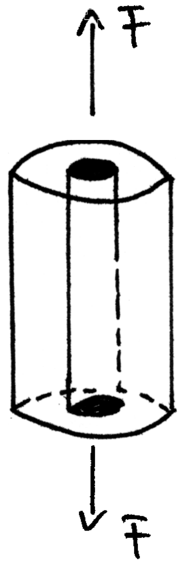
Exp. 2 Principle of Minimum complementary strain Energy

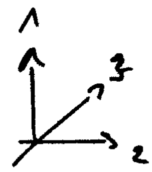
Fiber composite

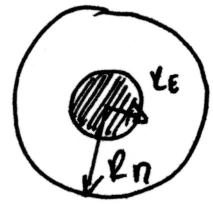
$$= \frac{N}{A}$$

$$= \frac{\sigma_{11}}{E_1}$$

$$\varepsilon_{22} = \varepsilon_{33} = -\nu \cdot \varepsilon_{11}$$





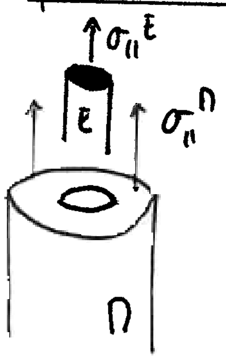


r_E: Radius of the Embedding

r_M: Radius " " Matrix

$$c := \frac{r_E^2}{r_M^2} \quad : \text{Fiber concentration}$$

a.1) permissible stress field :



equilibrium:

$$\sigma_{11}^M (\pi \cdot r_M^2 - \pi \cdot r_E^2) + \sigma_{11}^E \cdot \pi \cdot r_E^2 = \sigma_{11} \cdot \pi \cdot r_M^2$$

$$\Rightarrow \quad (1-c)\,\sigma_{11}^M + c \cdot \sigma_{11}^E = \sigma_{11}$$

compatible distortion field

$$\varepsilon_{11}^M = \varepsilon_{11} = \varepsilon_{11}^E$$

$$\left.\begin{aligned} \sigma_{11}^M &= E_M \cdot \varepsilon_{11} \\ \sigma_{11}^E &= E_E \cdot \varepsilon_{11} \end{aligned}\right\} \boxed{E_1^{**} = (1-c)\,E_M + c \cdot E_E}$$

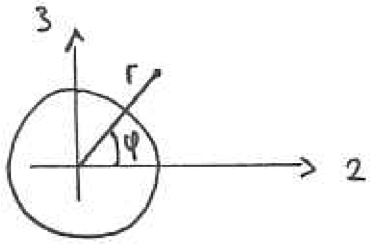
$$\sigma_{11} = E_1 \cdot \varepsilon_{11}$$

E_1^{**} : lower bound.

a.2) Check if the displacement-field is permissible?

$\varepsilon_{11}^{**}, \varepsilon_{22}^{**}, \varepsilon_{33}^{**} \neq 0$ cst. in x_1, x_2, x_3

$\varepsilon_{11}^{**} = u_{1,1}^{**} \quad \rightarrow \quad u_1^{**} = \varepsilon_{11}^{**} \cdot x_1.$



$x_2, x_3 \rightarrow r, \varphi$

lateral strain:

- in the embedding:

$$\varepsilon_\varphi^{**} = \frac{u_r^{**}}{r} = -\nu_E \cdot \varepsilon_{11}^{**}$$

$$\Rightarrow \underline{u_r^{**} = -\nu_E \cdot \varepsilon_{11}^{**} \cdot r} \quad \text{d.f}$$

$$\varepsilon_r^{**} = u_{r,r}^{**} = -\nu_E \cdot \varepsilon_{11}^{**} \quad \text{ok!}$$

- in the matrix:

$$\varepsilon_\varphi^{**} = \frac{u_r^{**}}{r} = -\nu_M \cdot \varepsilon_{11}^{**}$$

$$\underline{u_r^{**} = -\nu_M \cdot \varepsilon_{11}^{**} \cdot r}$$

$$\varepsilon_r^{**} = u_{r,r}^{**} = -\nu_M \cdot \varepsilon_{11}^{**} \quad \text{ok.}$$

- transition condition at $r = R_E$

$$u_r(R_E^+) \overset{?}{=} u_r(R_E^-)$$

$$-\nu_E \cdot \varepsilon_{11}^{**} \cdot R_E^+ = -\nu_M \cdot \varepsilon_{11}^{**} \cdot R_E^- \quad \Rightarrow \varepsilon_{11}^{**} = \varepsilon_{11}^{**}$$

not satisfied if $\nu_E \neq \nu_M$

$\rightarrow$ No continuity !!!

Approach for the displacement field:

$$u_r^E = a_E \cdot r + b_E/r$$

$$u_r^M = a_M \cdot r + b_M/r$$

i) essential b. conditions $\quad u(r=0) = 0$: embedding

$u_r(R_E^+) = u_r(R_E^-)$: transition cond.

ii) natural b. conditions $\quad \sigma_r(R_E^+) = \sigma_r(R_E^-)$: transition cond.

$\Rightarrow$ find a_E, b_E, a_M, b_M

solution for E_1-Modul

$$E_1 = c \cdot E_E + (1-c) E_M + \frac{2 \cdot c(1-c)(\nu_M - \nu_E)^2 \cdot E_M}{1 + \nu_M + (1+\nu_M)(1-2\nu_M) \cdot c + \frac{E_M}{E_E}(1+\nu_E)(1-2\nu_E)(1-c)}$$

$\hookrightarrow$ correction-term in the area of ‰ !!!

R. Hill, J. Mech. Phys Solids 1964

Z. Hashin. AIAA J, 1966

For more complex systems, approximate procedures of solution must be employed.

Weighted Residual Methods, Ritz Method are classical techniques which are used to obtain an approximate solution.

Weighted Residual Method

$$L_{2m}[\phi] = r \quad \Leftrightarrow \quad \sigma_{ij,j} + f_i = 0$$

$L_{2m}[\phi]$:= linear differential operator

ϕ := state variables

r := forcing function

Basic step in the weighted residual and Ritz analysis is, to assume:

$$\boxed{\bar{\phi} = \sum_{i=1}^{n} a_i \cdot q_i}$$

q_i = linearly independent trial functions, shape functions

a_i = multipliers, coefficients (to be determined)

$$\boxed{R = r - L_{2m}\left[\sum_{i=1}^{n} a_i \cdot q_i\right]}$$

Exact solution : $R = 0$

Aim: calculate a_i such that R is small (minimum)

=> the trial functions need only satisfy the essential boundary conditions.

=> We know that the exact solution also satisfies the natural boundary conditions

Ritz-Analysis-Method

The fundamental diffrence from the weighted residual method is that in the Ritz-method we operate on the functional Π

$$\boxed{\Pi = \frac{1}{2} \cdot \int_V \sigma_{ij} \cdot \varepsilon_{ij} \, dV - \int_\Omega f_i \cdot u_i \, d\Omega}$$

stationarity condition : $\frac{\delta \Pi}{\delta a_i} = 0 \qquad i = 1..n$

the trial functions q_i only need to satisfy the essential boundary conditions, not the natural b.c.

$\frac{\delta \Pi}{\delta a_i} = 0 \quad \Rightarrow$ sense: We minimaize the violation of the internal equilibrium requirements and the vialation of the natural boundary conditions.

Steps of a Finite-Element Analysis

1) Discretization of a given domain into a collection of preselected finite elements

   i) Construct the finite-element mesh
   ii) Number of nodes and elements
   iii) generate the geometric properties

2) Derivation of element equations

   i) Construct the variational formulation
   ii) Assume that a typical dependent variable u:

$$u = \sum_{i=1}^{n} u_i \cdot \phi_i \qquad \phi_i : \text{shape funktion}$$

   iii) Subsititute ii) in i) to obtain element equations in the form:

$$\boxed{K \cdot u = f}$$

   iv) Derive element interpolation functions N_i and compute element matrices

3) Assembly of element equations

4) Imposition of the boundary conditions

5) Solution of the assembled equations.

The End

Thank you for your

Attention :-))