The Finite Element Method and the Analysis of Systems with Uncertain Properties

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Contents of Today's Lecture

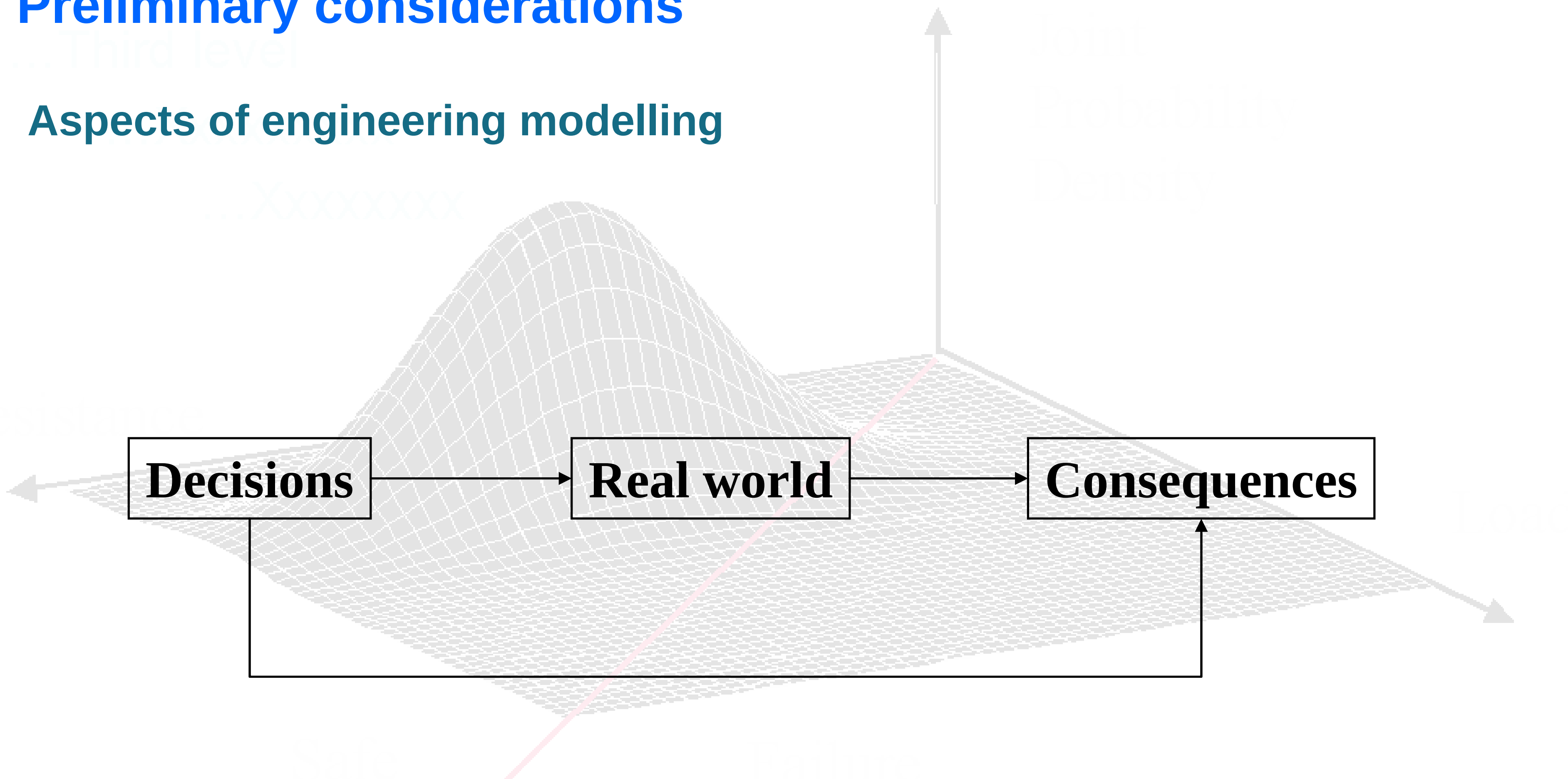
- **Some preliminary considerations**
- **An overview of the lecture**
- **Introduction to the use of finite element**

  - Physical problem, mathematical modeling and finite element solutions
  - Finite elements as a tool for computer supported design and assessment

- **Basic mathematical tools (vectors, tensors and matrix calculus)**

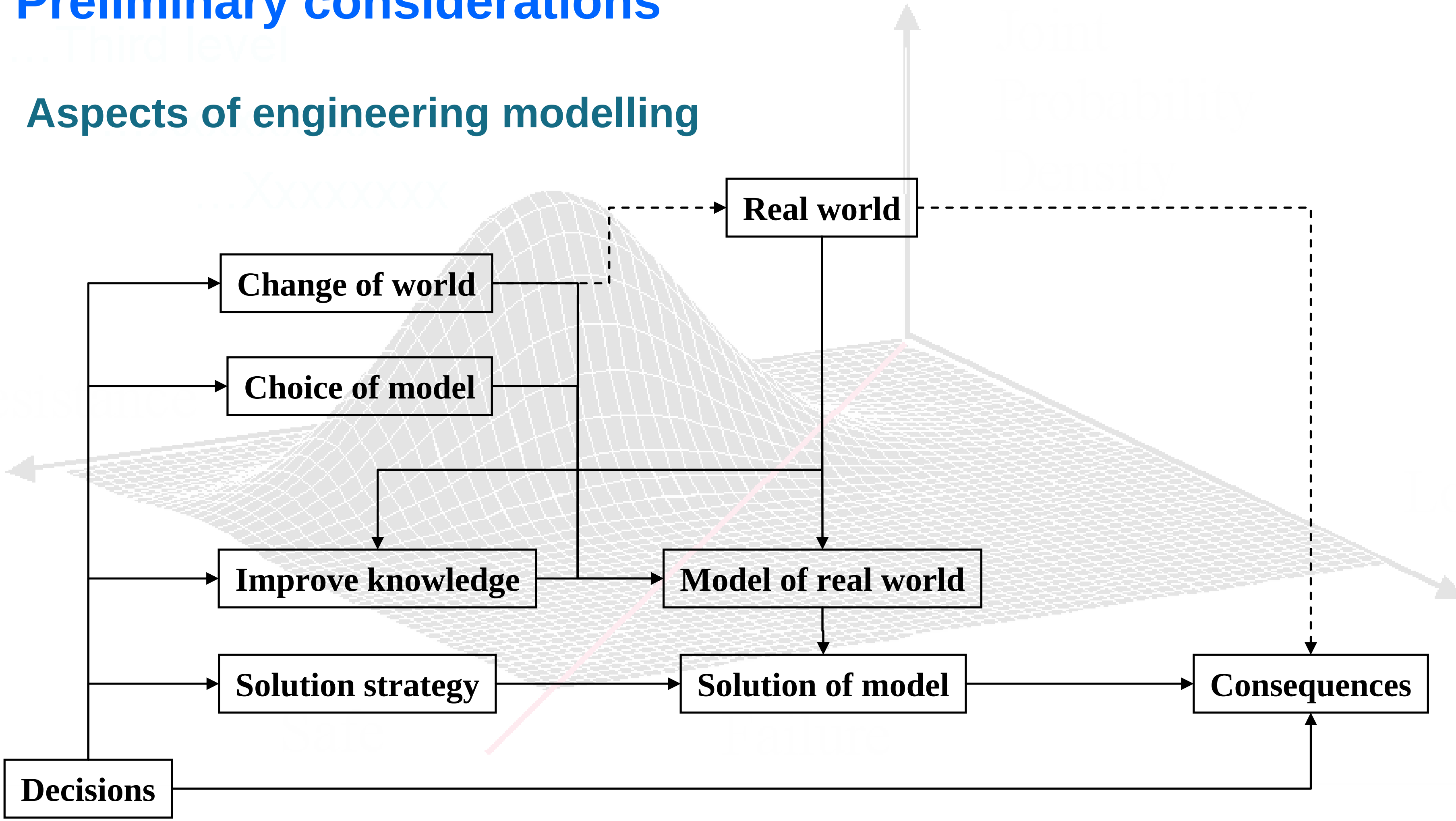
Preliminary considerations

- **Aspects of engineering modelling**



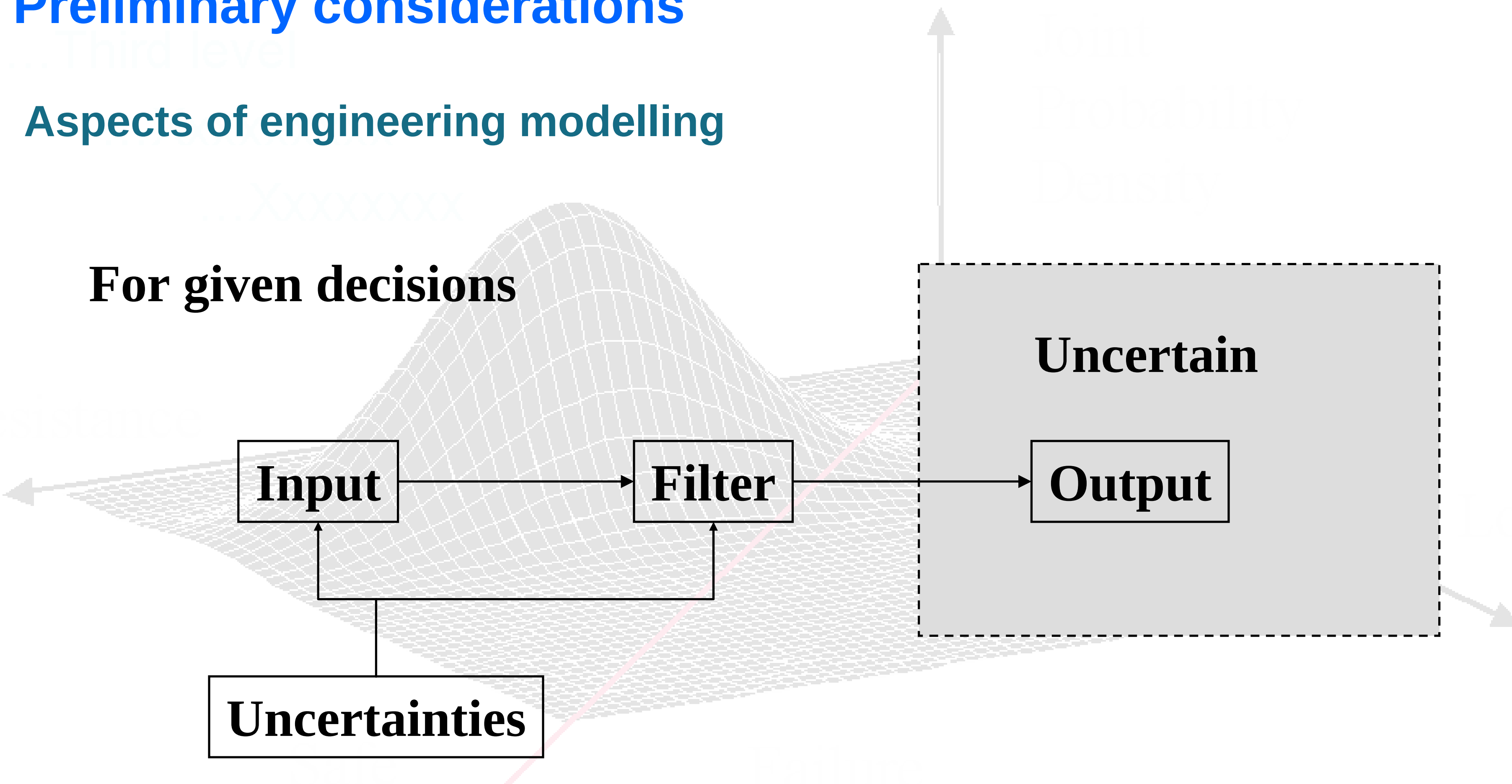
Preliminary considerations

- Aspects of engineering modelling



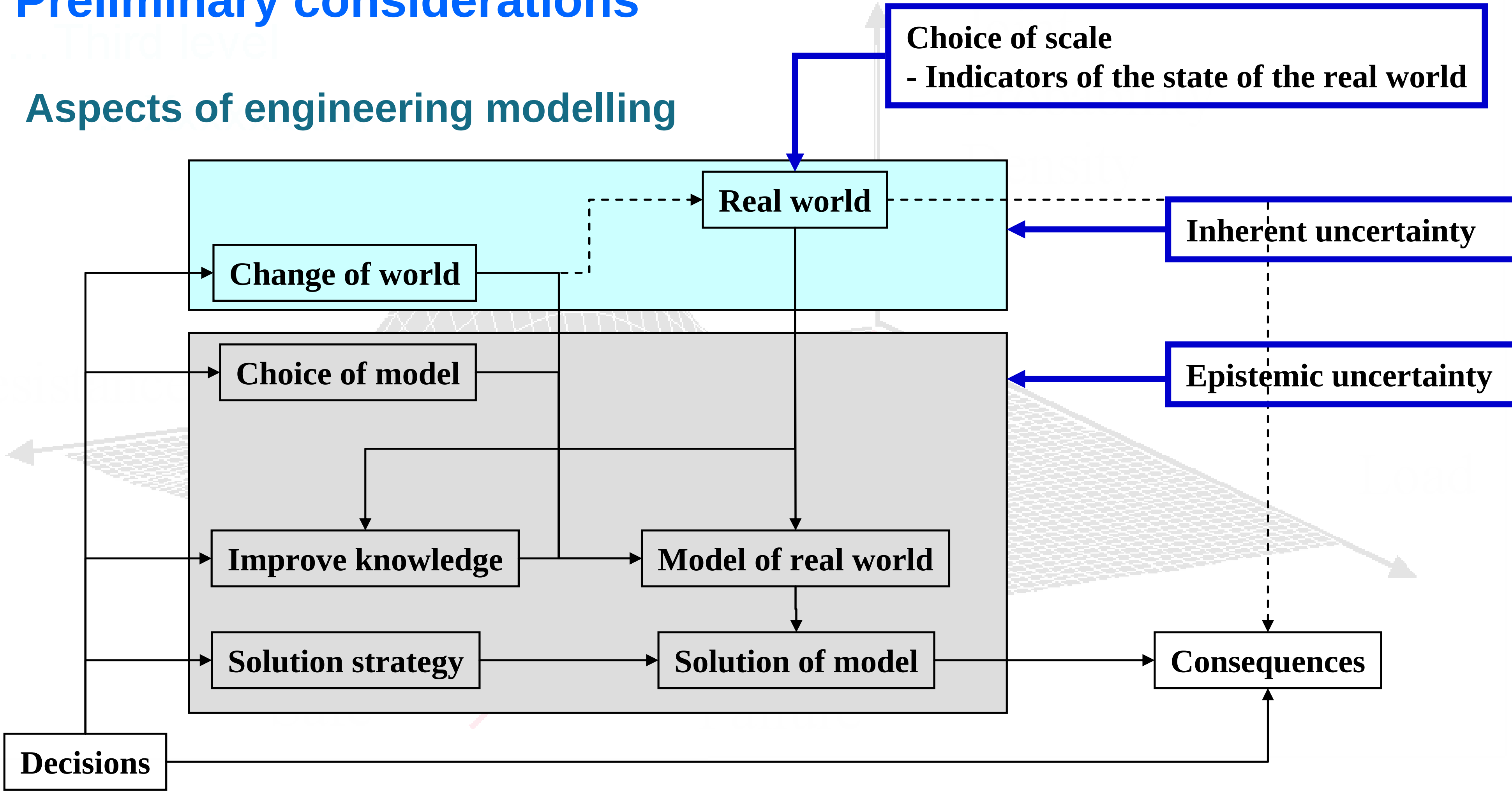
Preliminary considerations

- **Aspects of engineering modelling**



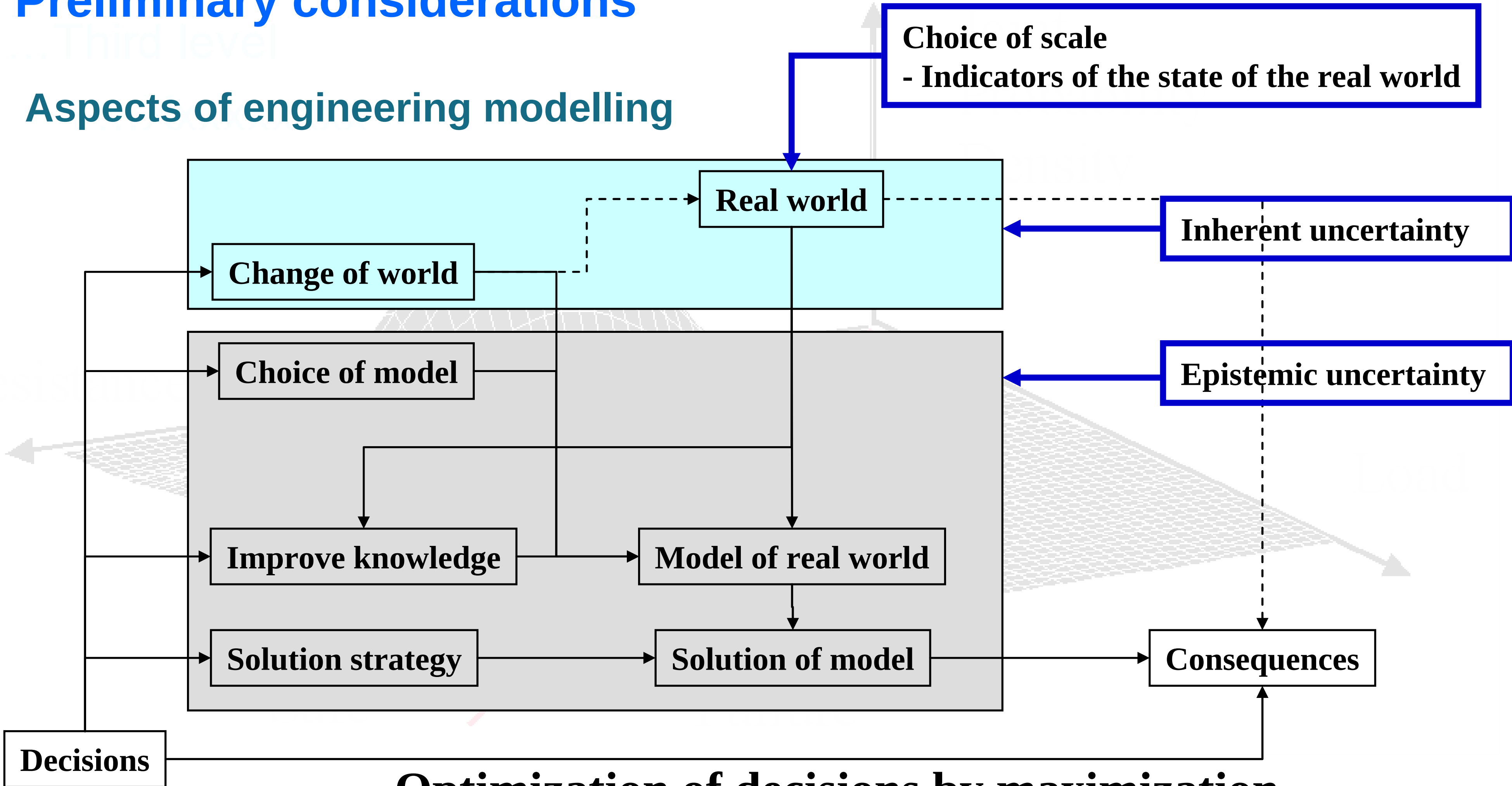
Preliminary considerations

- Aspects of engineering modelling



Preliminary considerations

- **Aspects of engineering modelling**



Optimization of decisions by maximization of expected utility – a function of risk

An overview of the lecture

Topic
Introduction and description of course material; Introduction to the use of finite elements - Physical problem, mathematical modelling and finite element solutions - Finite elements as a tool for computer supported design and assessment - Basic mathematical tools (vectors, tensors and matrix calculus)
Basic concepts of engineering analysis - Solution of mathematical models of discrete and continuous systems
Formulation of the method of finite elements - Formulation of the displacement based finite element method and (general) convergence of results
Formulation and calculation of isoparametric finite element matrixes - Truss element - Continuum elements – triangular elements
Formulation and calculation of isoparametric finite element matrixes - Continuum elements – square elements - (Detailed) Convergence considerations
Formulation and calculation of isoparametric finite element matrixes - Element matrixes in global coordinates - Displacement /pressure related elements for incompressible media
Formulation and calculation of isoparametric finite element matrixes - Formulation of structural elements (beam elements and axial-symmetrical shell elements)
Formulation and calculation of isoparametric finite element matrixes - Formulation of structural elements (plate and general shell elements)
Formulation and calculation of isoparametric finite element matrixes - Numerical integration and implementation of a finite element computer code
Solution of static finite element problems - The LDL- solution
Introduction to reliability analysis
Implicit performance functions and introduction to stochastic FEM (SFEM)
SFEM for linear static problems
SFEM for spatial variability problems

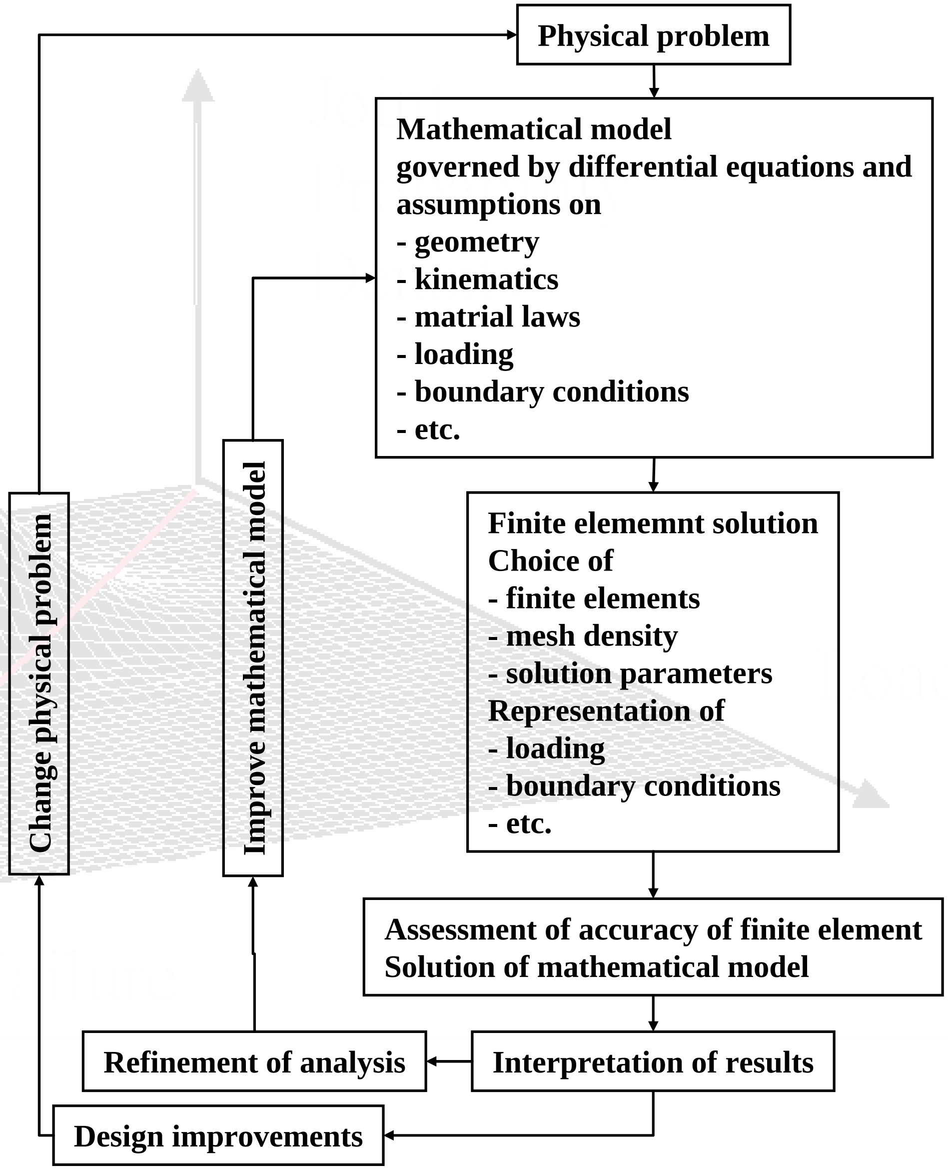
Introduction to the use of finite element

- **Physical problem, mathematical modeling and finite element solutions**

  - we are only working on the basis of mathematic models !

  - choice of mathematical model is crucial !

  - mathematical models must be *reliable* and *effective*



Introduction to the use of finite element

- ***Reliability* of a mathematical model**

  The chosen mathematical model is reliable if the required response is known to be predicted within a selected level of accuracy measured on the response of a very comprehensive mathematical model

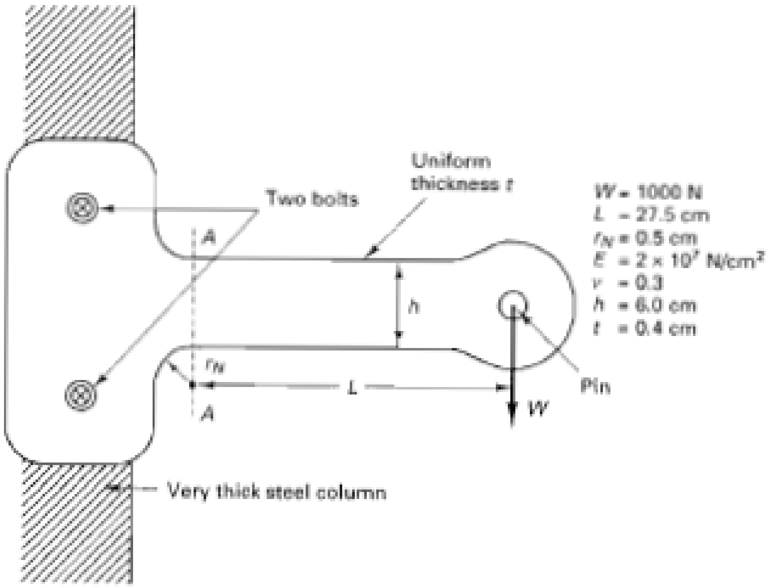
- ***Effectiveness* of a mathematical model**

  The most effective mathematical model for the analysis is surely that one which yields the required response to a sufficient accuracy and at least costs

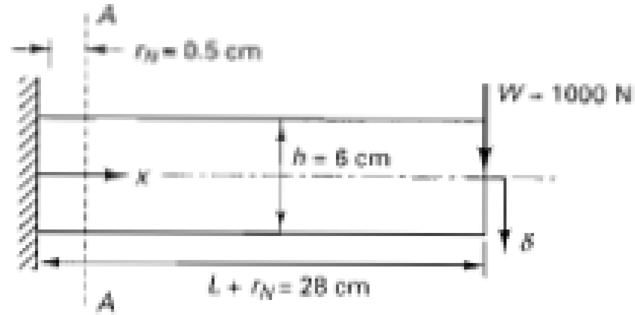
Introduction to the use of finite element

- **Example**

  Complex physical problem modeled by a simple mathematical model



(a) Physical problem of steel bracket



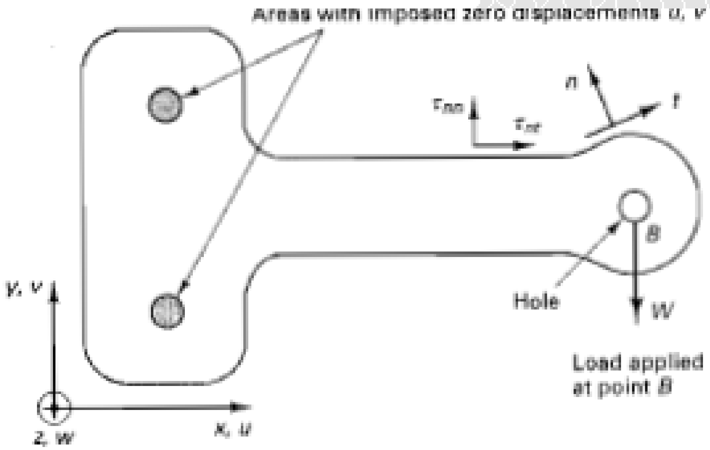
(b) Beam model

$$M = WL$$
$$= 27{,}500 \text{ N cm}$$
$$\delta\big|_{\text{at load } W} = \frac{1}{3}\frac{W(L + r_N)^3}{EI} + \frac{W(L + r_N)}{\frac{5}{6}AG}$$
$$= 0.053 \text{ cm}$$

Introduction to the use of finite element

- **Example**

Detailed reference model – 2-D plane stress model – for FEM analysis



$$\left.\begin{aligned}\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0\\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0\end{aligned}\right\} \text{ in domain of bracket}$$

$\tau_{nn} = 0,\ \tau_{nt} = 0$ on surfaces except at point B and at imposed zero displacements

Stress-strain relation (see Table 4.3):

$$\begin{bmatrix}\tau_{xx}\\ \tau_{yy}\\ \tau_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{bmatrix}$$

E = Young's modulus, ν = Poisson's ratio

Strain-displacement relations (see Section 4.2):

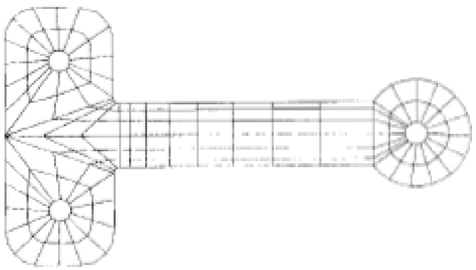
$$\epsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}; \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

(c) Plane stress model

Introduction to the use of finite element

- **Example**

Comparison between simple and more refined model results



$$M = WL$$
$$= 27{,}500 \text{ N cm}$$
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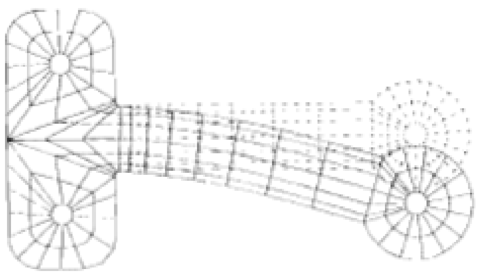
$$= 0.053 \text{ cm}$$



$$\delta\big|_{\text{at load } W} = 0.064 \text{ cm}$$

$$M\big|_{x=0} = 27{,}500 \text{ N cm}$$

Reliability and efficiency may be quantified !

Introduction to the use of finite element

- Observations

  Choice of mathematical model must correspond to desired response measures

  The most effective mathematical model delivers reliable answers with the least amount of efforts

  Any solution (also FEM) of a mathematical model is limited to information contained in the model – bad input – bad output

  Assessment of accuracy is based on comparisons with results from very comprehensive models – however, in practice often based on experience

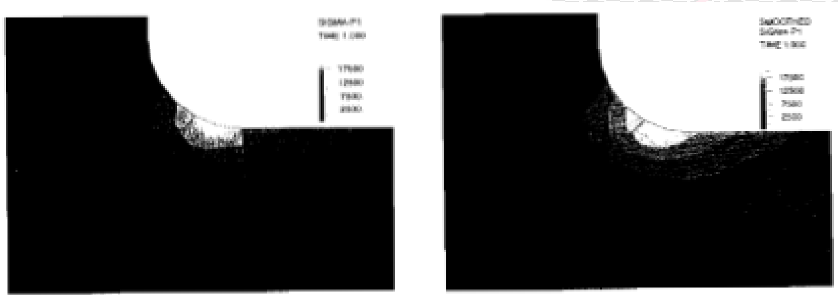
Introduction to the use of finite element

- **Observations**

  Some times the chosen mathematical model results in problems such as singularities in stress distributions

  The reason for this is that simplifications have been made in the mathematical modeling of the physical problem

  Depending on the response which is really desired from the analysis this may be fine – however, typically refinements of the mathematical model will solve the problem

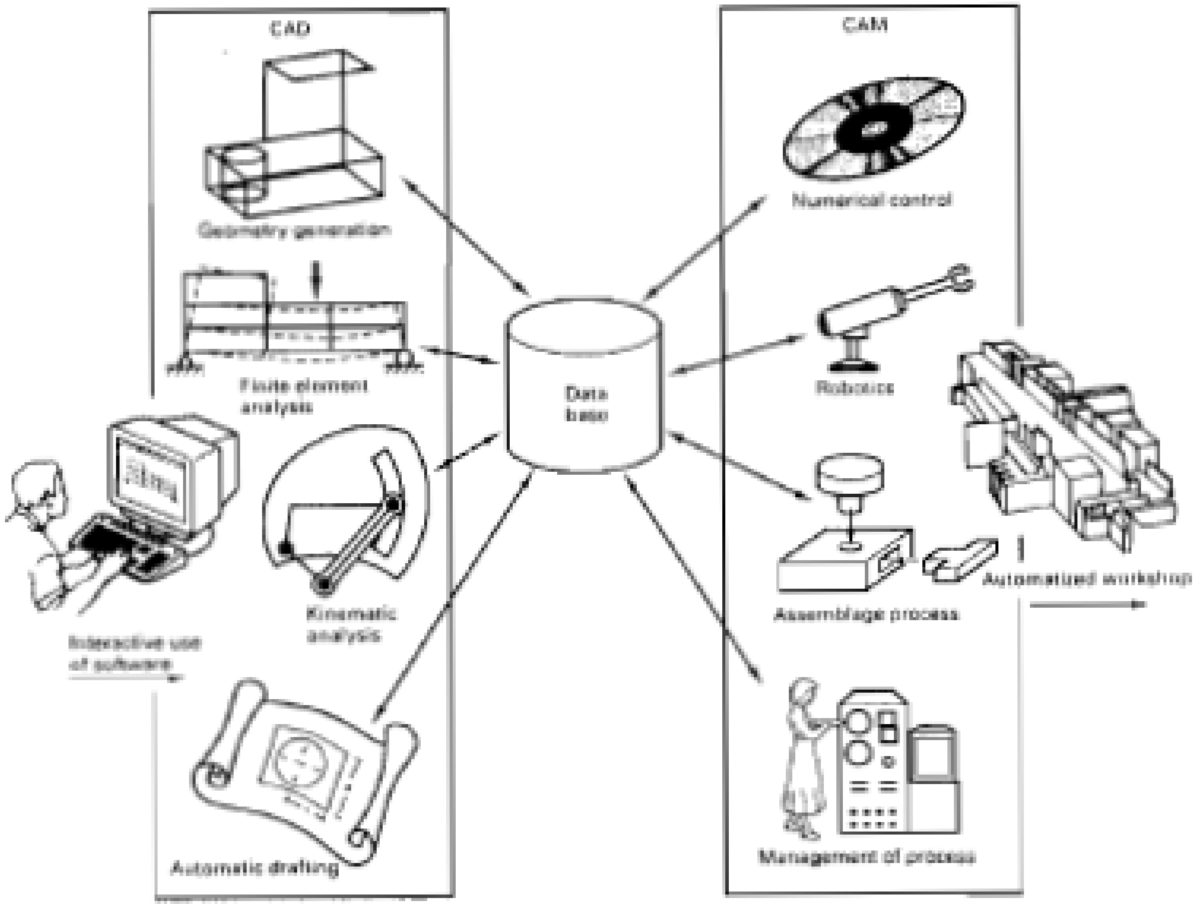


Introduction to the use of finite element

- **Finite elements as a tool for computer supported design and assessment**

FEM forms a basic tool framework in research and applications covering many different areas

- Fluid dynamics
- Structural engineering
- Aeronautics
- Electrical engineering
- etc.



Introduction to the use of finite element

- **Finite elements as a tool for computer supported design and assessment**

  The practical application necessitates that solutions obtained by FEM are reliable and efficient

  however

  also it is necessary that the use of FEM is *robust* – this implies that minor changes in any input to a FEM analysis should not change the response quantity significantly

  Robustness has to be understood as directly related to the desired type of result – response

Basic mathematical tools

- **Vectors and matrixes**

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1i} & & a_{1n} \\ \vdots & \ddots & & & \\ a_{i1} & & a_{ii} & & \vdots \\ \vdots & & & \ddots & \\ a_{m1} & & \cdots & & a_{mn} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\mathbf{A}^T$ is the transpose of $\mathbf{A}$

if $\mathbf{A} = \mathbf{A}^\mathbf{T}$ there is

$m = n$ (square matrix)

and $a_{ij} = a_{ji}$ (symmetrical matrix)

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$ is a unit matrix

Basic mathematical tools

- **Banded matrixes**

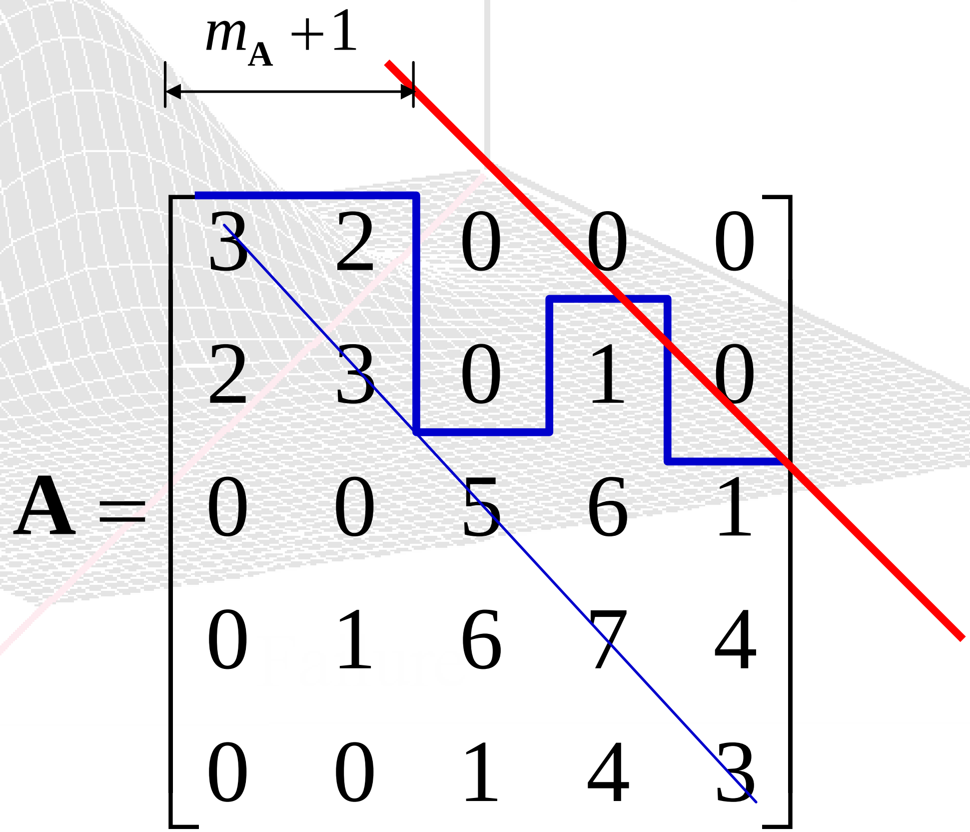
Symmetric banded matrixes

$$a_{ij} = 0 \text{ for } j > i + m_{\mathrm{A}},\ \ 2m_{\mathrm{A}} + 1 \text{ is the bandwidth}$$

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 2 & 3 & 4 & 1 & 0 \\ 1 & 4 & 5 & 6 & 1 \\ 0 & 1 & 6 & 7 & 4 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix} \qquad m_{\mathrm{A}} = 2$$

Basic mathematical tools

- **Banded matrixes and skylines**



$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 6 & 1 \\ 0 & 1 & 6 & 7 & 4 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix}$$

Basic mathematical tools

- **Matrix equality**

$$\mathbf{A}(m \times p) = \mathbf{B}(n \times q) \text{ if and only if}$$

$$m = n \text{ and } p = q \text{ and}$$

$$a_{ij} = b_{ij}$$

Basic mathematical tools

- **Matrix addition**

$\mathbf{A}(n \times q)$ and $\mathbf{B}(n \times q)$ can be added if and only if

$m = n$ and $p = q$ and

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$c_{ij} = a_{ij} + b_{ij}$$

Basic mathematical tools

- **Matrix multiplication with a scalar**

A matrix **A** is multiplied by a scalar *c* by multiplying all elements of **A** with *c*

$$\mathbf{B} = c\mathbf{A}$$

$$b_{ij} = c a_{ij}$$

Basic mathematical tools

- **Multiplication of matrixes**

Two matrixes $\mathbf{A}(p \times m)$ and $\mathbf{B}(n \times` q)$ can be multiplied only if $m = n$

$$\mathbf{C} = \mathbf{BA}$$

$$c_{ij} = \sum_{r=1}^{m} a_{ir} b_{rj}, \ \mathbf{C}(p \times q)$$

Basic mathematical tools

- **Multiplication of matrixes**

The commutative law does not hold, i.e.

$\mathbf{AB} \neq \mathbf{BA},$ unless $\mathbf{A}$ and $\mathbf{B}$ commute

The distributive law holds, i.e.

$$\mathbf{E} = (\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

The associative law holds, i.e.

$$\mathbf{G} = (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \mathbf{ABC}$$

$\mathbf{AB} = \mathbf{CB},$ does not imply that $\mathbf{A} = \mathbf{C}$

however does hold for special cases (e.g. for $\mathbf{B} = \mathbf{I}$)

Special rule for the transpose of matrix products

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

Basic mathematical tools

- **The inverse of a matrix**

The inverse of a matrix $\mathbf{A}$ is denoted $\mathbf{A}^{-1}$

If the inverse matrix exist then there is:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

the matrix $\mathbf{A}$ is said to be non-singular

The inverse of a matrix product:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Basic mathematical tools

- **Sub matrixes**

A matrix **A** may be sub divided as:

$$\mathbf{A} = \left[\begin{array}{c:cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hdashline a_{31} & a_{32} & a_{33} \end{array}\right]$$

$$\mathbf{A} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix}$$

Basic mathematical tools

- **Trace of a matrix**

The trace of a matrix $\mathbf{A}(n \times n)$ is defined through:

$$tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

Basic mathematical tools

- **The determinant of a matrix**

The determinant of a matrix is defined through the recurrence formula

$$\det(\mathbf{A}) = \sum_{i=1}^{n} (-1)^{1+j} a_{1j} \det \mathbf{A}_{1j}$$

where $\mathbf{A}_{1j}$ is the $(n\text{-}1)\times(n\text{-}1)$ matrix obtained by eliminating the 1^{st} row and the j^{th} column from the matrix $\mathbf{A}$ and where there is

if $\mathbf{A} = \left[a_{11}\right], \det \mathbf{A} = a_{11}$

Basic mathematical tools

- **The determinant of a matrix**

It is convenient to decompose a matrix **A** by the socalled Cholesky decomposition

$$\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

where **L** is a lower triangular matrix with all diagonal elements equal to 1 and **D** is a diagonal matrix with components d_{ii} then the determinant of the matrix **A** can be writtes as

$$\det \mathbf{A} = \prod_{i=1}^{n} d_{ii}$$