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SFEM for Spatial Variability Problems

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ETH ***Zurich***

Introduction

- Basic structural parameters were assumed discrete
- Representation by single-valued parameters
- Valid assumption for quantities concentrated at discrete points in space
  - Loads of joints and supports
  - Stiffnesses of joints and supports

Introduction

- Most parameters are distributed in space
  - Loads
  - Material properties
  - Geometric properties

- Variation over length and/or area

- Representation as a collection of random variables- random processes or fields

Content

- Random processes/fields
- Discretization of random processes/fields
- Sensitivity analysis

Random processes / fields

$X(t)$- *material property*

The ensemble of all realizations $X^i(t)$ is called a random process

x_i - value of $X(t)$ at location t_i

↓

Representation as a random variable

If variation is described as a random process → *Variation at a particular cross section is a random variable*

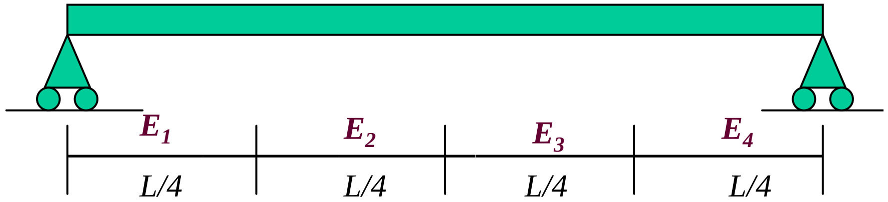
Random processes / fields

- Two types of variability:
  - across samples
  - over space

- Variation in 1 or more dimensions:
  - One dimensional random process
  - Multidimensional random processes (random fields)

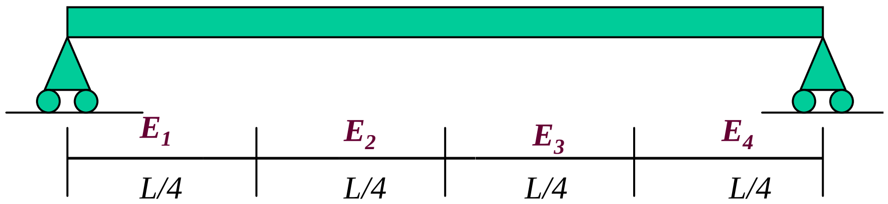
Random processes and reliability analysis

How to include the variability in the reliability analysis?



- Discretization into segments
- Random field represented by 1 random variable in each segment
- Accuracy depends on:
  - size of segment
  - spatial variability of property
- Principles of previous subjects can be applied
- Should account for statistical correlations between the random variables
  (since they originate fro the same field)

Random processes and reliability analysis



Cumulative distribution function

$$F(x,t_1) = P\left\{X(t_1) \le x\right\}$$

Joint cumulative distribution function

$$F(x_1,t_1;x_2,t_2) = P\left\{X(t_1) \le x_1; X(t_2) \le x_2\right\}$$

Ensemble average

$$E\left\{g(X(t_1), X(t_2), ...X(t_n)\right\} =$$

$$\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} g(x_1, x_2, ..., x_n) f_n(x_1, t_1, x_2, t_2, ..., x_n, t_n) dx_1 dx_2 ... dx_n$$

Mean and Variance

$$\mu_{X(t)} = E\left[X(t)\right] \qquad \mathrm{Var}\left[X(t)\right] = E\left\{\left[X(t_1) - \mu_{X(t_1)}\right]\left[X(t_2) - \mu_{X(t_2)}\right]\right\}$$

Random processes and reliability analysis

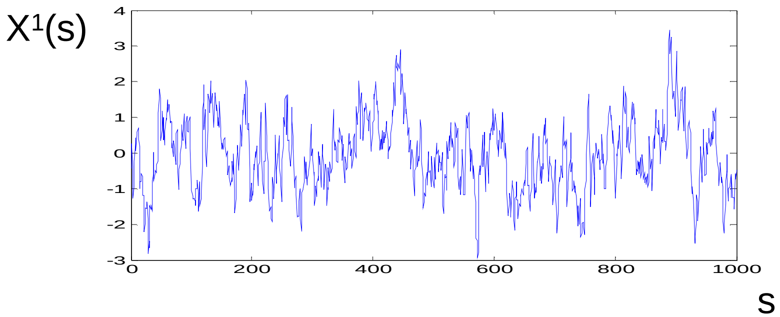
Properties:

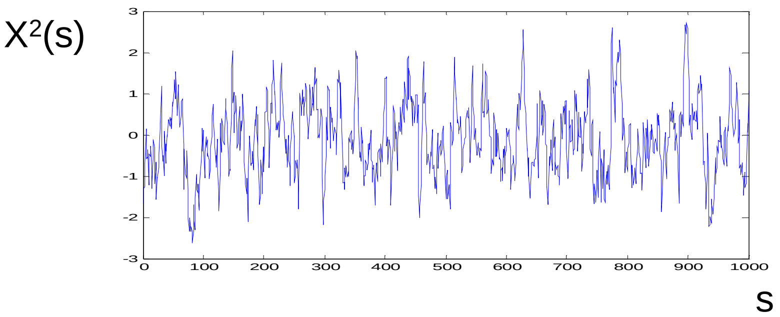
<u>Stationarity:</u>

Statistical properties invariant to a shift of the origin

$$\mu_{X(s)} = E\left[X(s)\right]$$

$$R(s_1 - s_2) = E\left[X(s + s_1 - s_2)X(s)\right]$$





Random processes and reliability analysis

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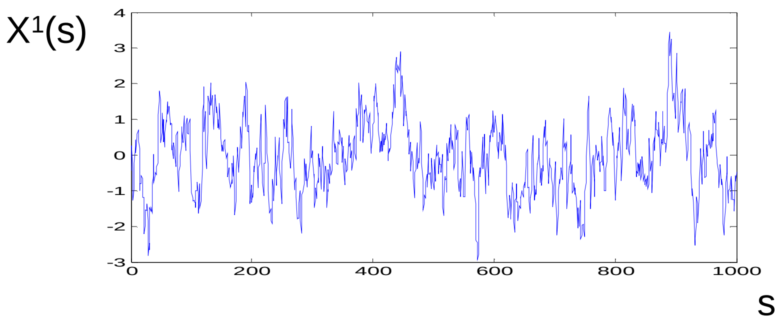
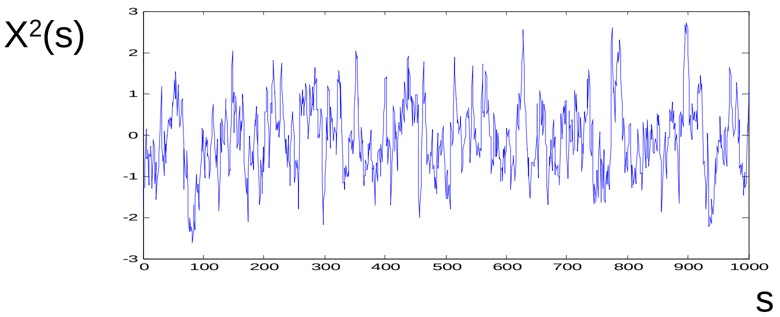
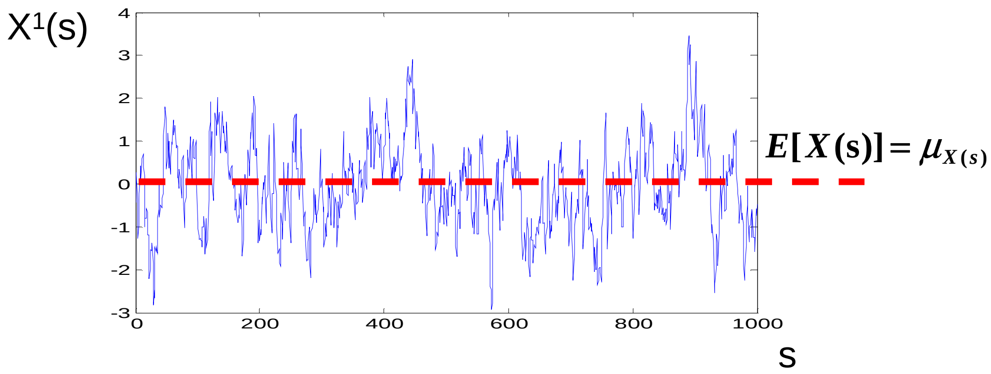




Ergodicity:

The expected value and covariance function can be estimated from one realization



Random field discretization

Finite Element discretization

- Representation of the deterministic behaviour of the structure

Random Field discretization

- Representation of the stochastic spatial variation of any parameter of the structure

Number, size and shape not necessary the same

Random field discretization

Random Field discretization

- Midpoint method

  - value of random field over an element is represented by its value at the midpoint of the element

  $$X_i = X(t_i)$$

  t_i- mid of element

  - Distribution of random variables remains the same with the distribution of the random field

- Spatial averaging method

  - value of random field over an element is represented by the spatial average of the random field over the element

  $$X_i = \frac{1}{T}\int_{-T/2}^{T/2} X(t)dt$$

  ***T*- averaging interval**

  - Only for Gaussian random fields

Random field discretization

Random Field discretization

- Midpoint method

$$X_i = X(t_i)$$ t_i **- mid of element**

$$E_1 = E(\frac{1}{8}L)$$

$$E_2 = E(\frac{3}{8}L)$$

$$E_3 = E(\frac{5}{8}L)$$

$$E_4 = E(\frac{7}{8}L)$$

- Spatial averaging method

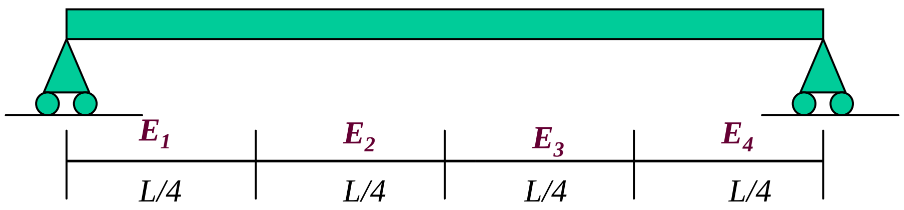
$$X_i = \frac{1}{T}\int_{-T/2}^{T/2} X(t)dt$$ T**- averaging interval**

$$E_1 = \frac{1}{L/4}\int_0^{L/4} E(t)dt$$

$$E_2 = \frac{1}{L/4}\int_{L/4}^{L/2} E(t)dt$$

$$E_3 = \frac{1}{L/4}\int_{L/2}^{3L/4} E(t)dt$$

$$E_4 = \frac{1}{L/4}\int_{3L/4}^{L} E(t)dt$$



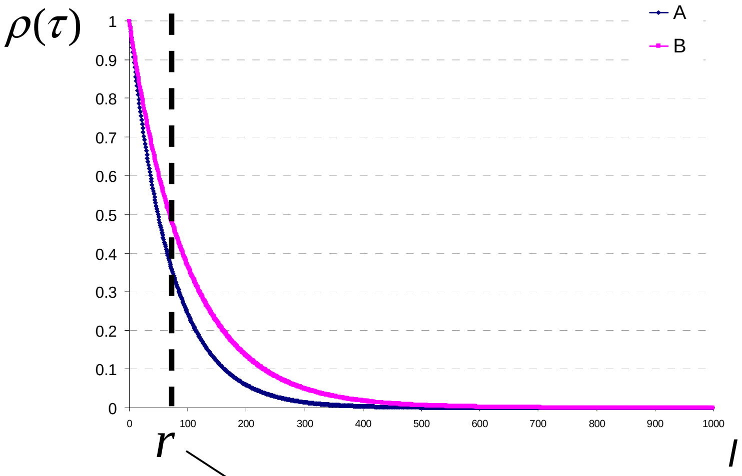
Random field discretization

Coefficients of correlation

$$\rho_{ij} = \frac{COV(X_i, X_j)}{\sigma_i \sigma_j}$$

A: $\rho(\tau) = \exp(-\frac{\tau\sqrt{2}}{r})$

B: $\rho(\tau) = \exp(-\frac{\tau}{r})$

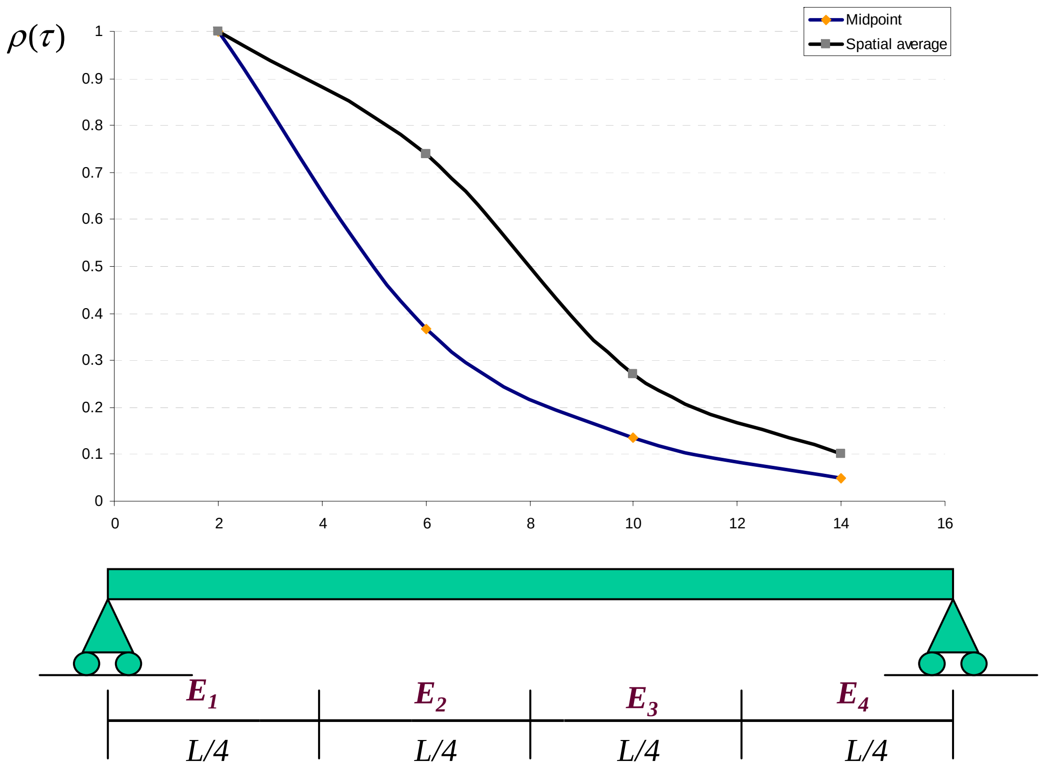


Correlation length: Approximate length over which strong correlation persists

Random field discretization

Correlation aspects

- Midpoint method
  - Over representation of variability
  - Lower value of reliability index
- Spatial averaging method
  - Under representation of variability
  - Higher value of reliability index



Random field discretization

Stochastic mesh refinement

To be (a fine mesh) OR not to be???

To be:

- Improvement of accuracy

Not to be:

- Large number of random variables
- Increased computational cost
- High correlations between variables in cases of a fine mesh

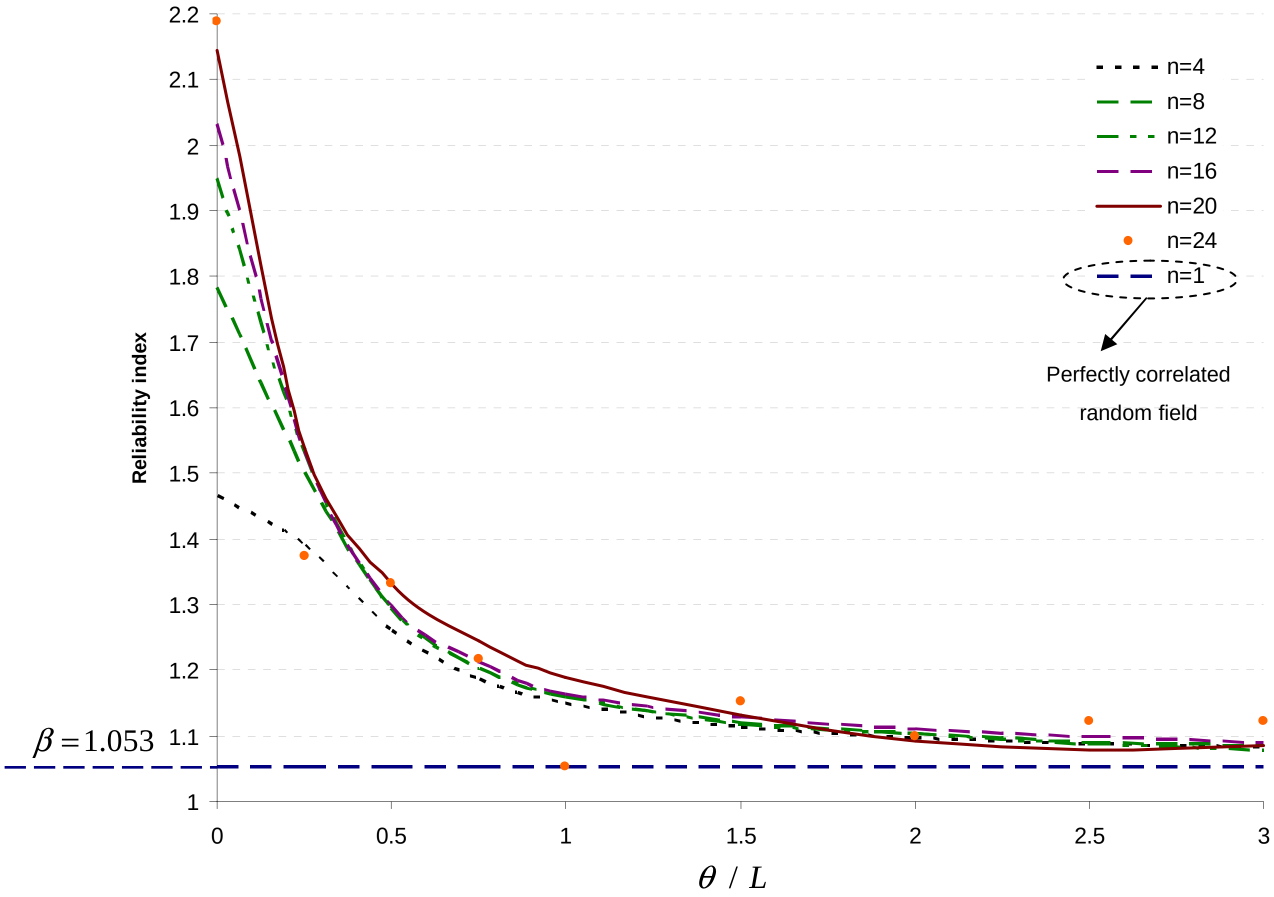
How to be!

- Use of correlation length (scale of fluctuation)
  Indication of the distance over which strong correlation persists

Random field discretization

Stochastic mesh refinement

$\theta\uparrow \quad \beta\downarrow$

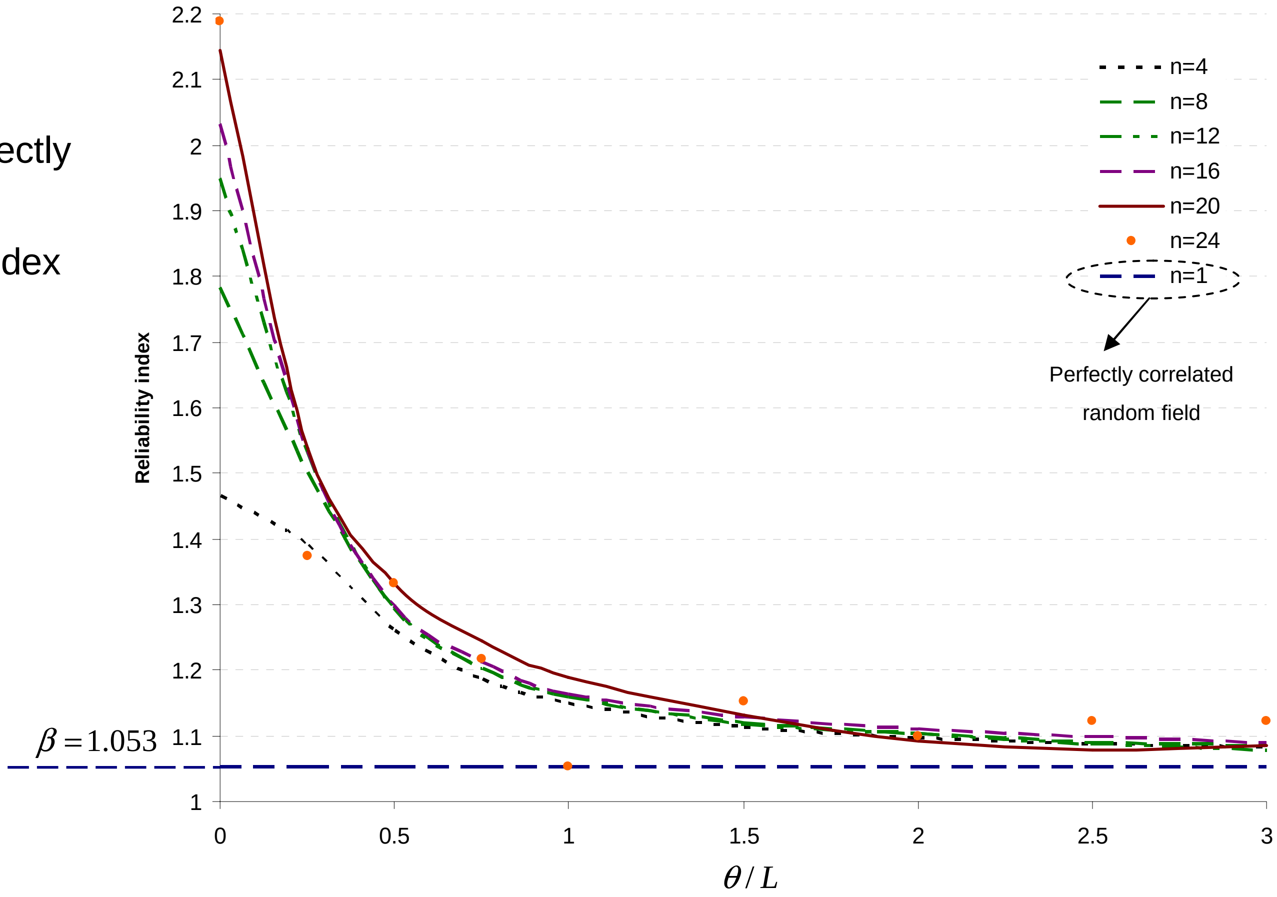


Random field discretization

Stochastic mesh refinement

$\theta \uparrow \quad \beta \downarrow$

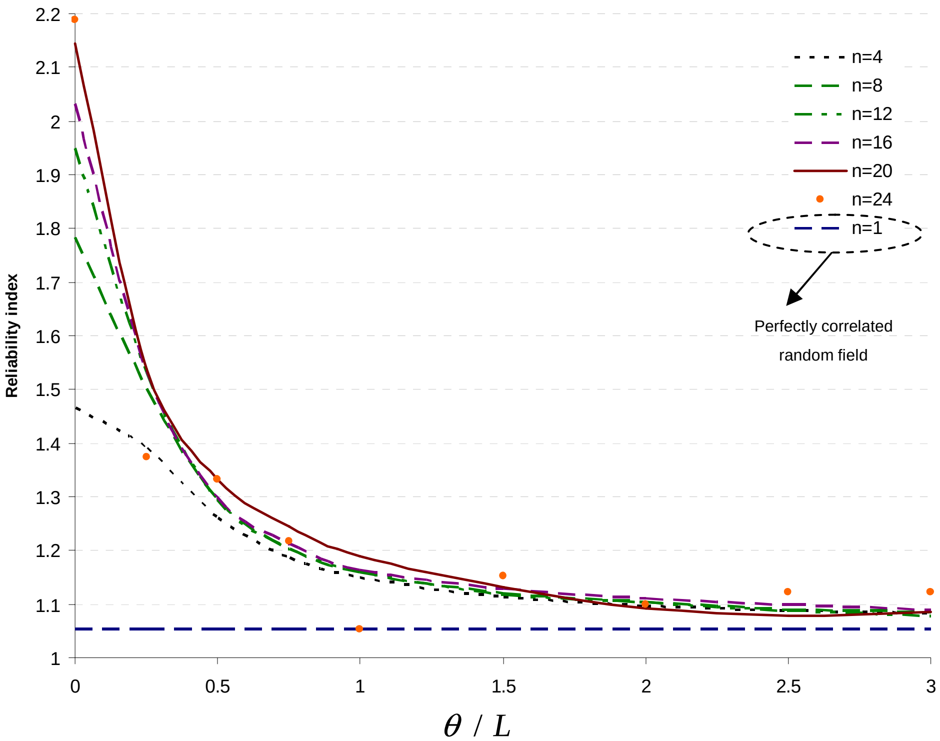
- As $\theta \uparrow$ approaching the perfectly correlated random field
- Finer mesh: higher reliability index
- As mesh becomes even finer: convergence

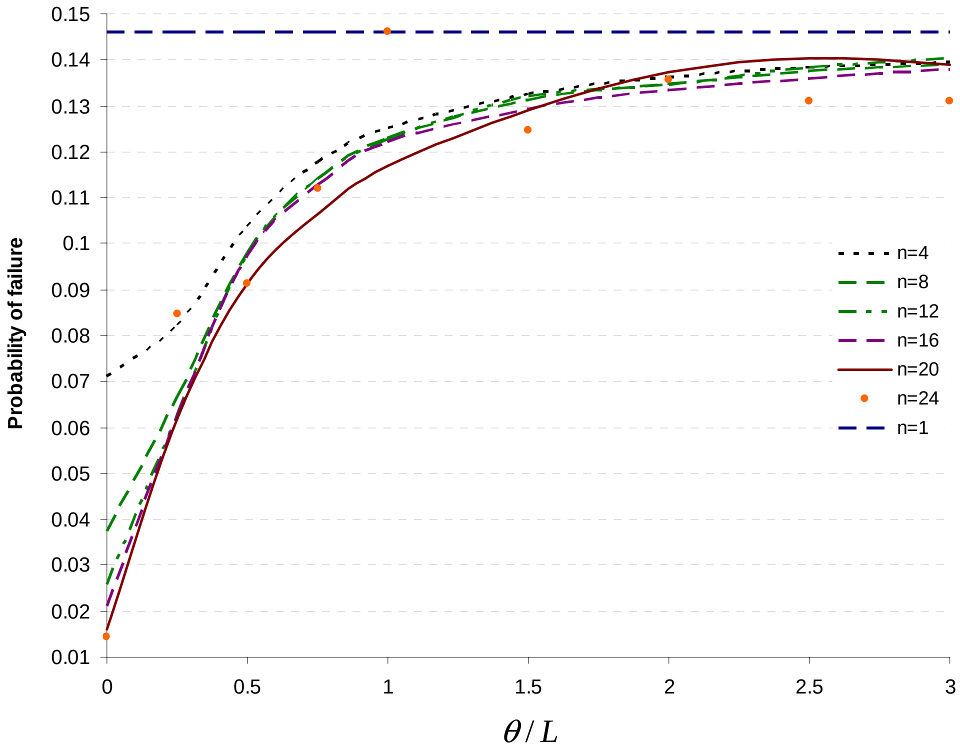


Random field discretization

Stochastic mesh refinement

θ ↑ β ↓ P_f ↑





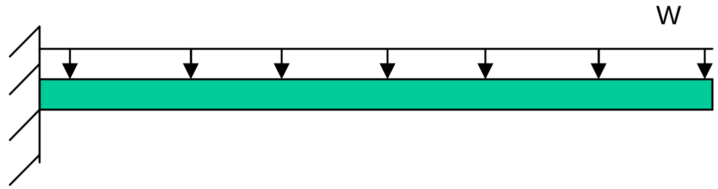
Selective consideration of random fields

Should all distributed parameters be considered as random fields?

Sensitivity indices

- measure the relative influence of random variables on the reliability
- if sensitivity index is high, better to model the parameter as random field
- BUT if a parameter has a large correlation length,
  - ⇨ approximation to the perfectly correlated random field
  - ⇨ model as random variable EVEN if it has a high sensitivity index

Selective consideration of random fields

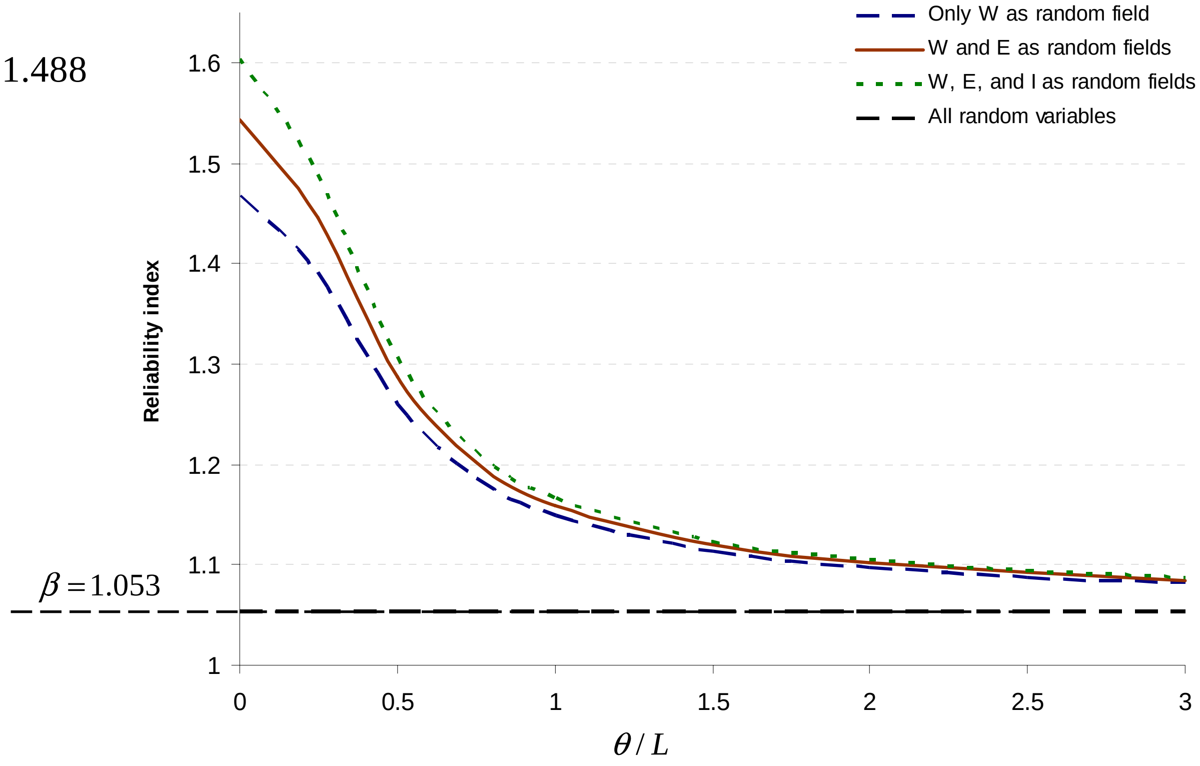


Sensitivity indices

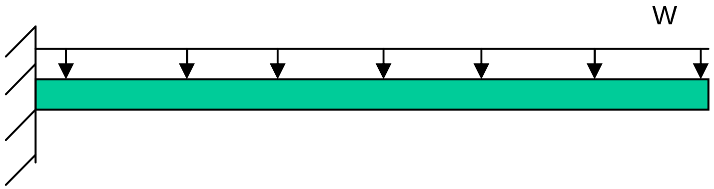
W as random field: $\beta = 1.391$

W and E as random fields: $\beta = 1.446$

All as random fields: $\beta = 1.488$



Selective consideration of random fields



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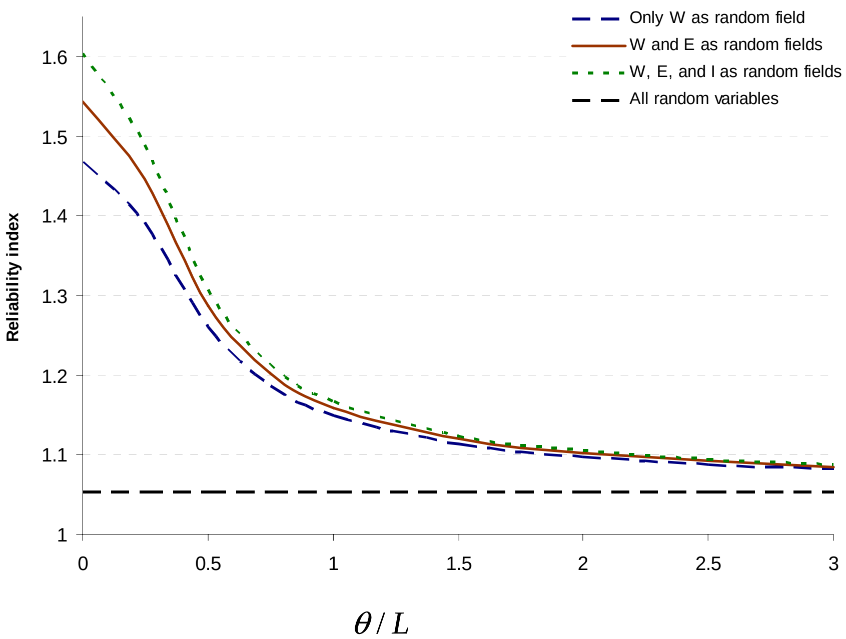

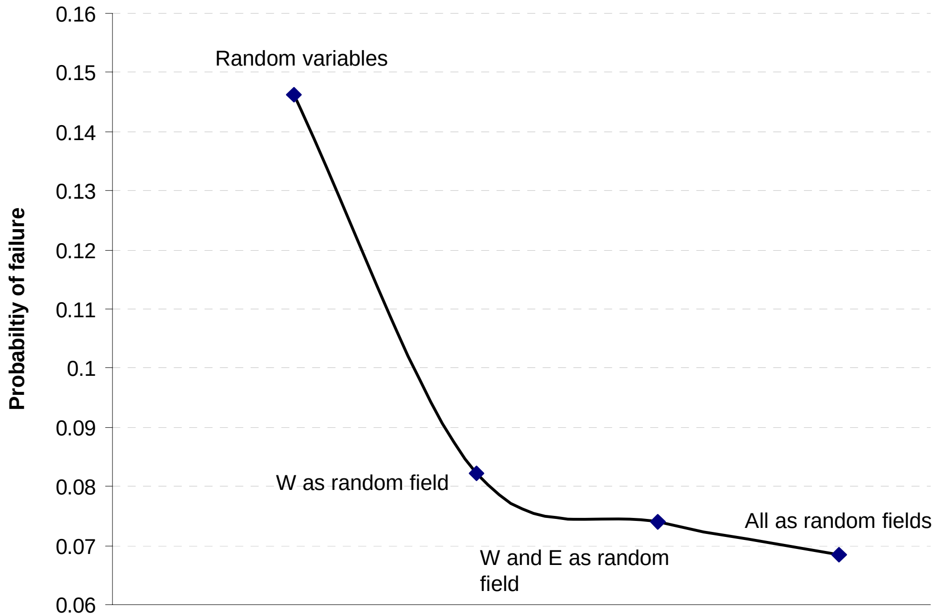

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Thank you for your attention☺

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