The Finite Element Method and the Analysis of Systems with Uncertain Properties

Exersice "SFEM for linear static problems"

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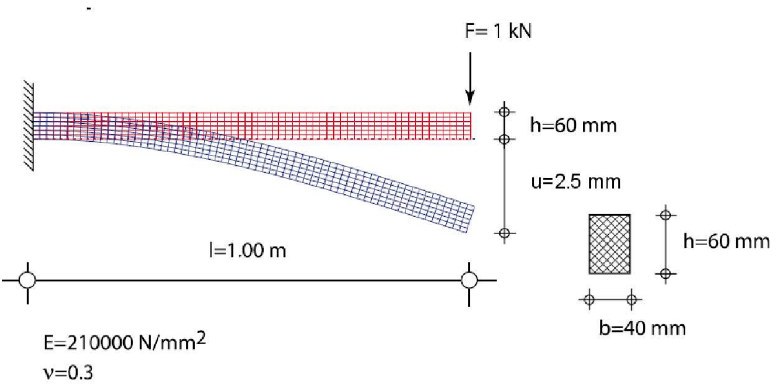
Outline

- Problem definition
- The Newton-Raphson method
- FORM method 2
- Derivation of the gradient
- Results

Problem definition



What is the probabilty that the maximum desplacement of the cantilever is greater than 25 mm?

Introduction of uncertainty

Some quantities in the simulation will be modeled by random variables:

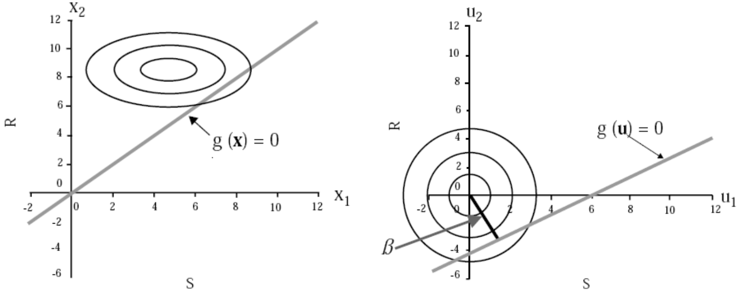
- Critical displacement ($u0$)
- Stiffness of the material (E)
- Force induced by the load (P)

Moreover, we define that the system fails if a limit state function $\mathrm{g} : \mathbb{R}^n \rightarrow \mathbb{R}$ is not fulfilled, i.e. $\mathrm{g} < 0$. For our problem, the limit state function is

$$\mathrm{g}(u0, E, F) = \mathbf{u}_m - u0,$$

where $\mathbf{u} = \mathbf{K}(E)^{-1}\mathbf{f}(P)$ are the displacements of the nodes, $\mathbf{u}_m$ the displacement of the lower-right-most node in the y–direction and $u0$ the displacement limit.

The idea of the First Order Reliability Method 2 (FORM2)



The area below the line $\mathrm{g}(\mathbf{X})$ is the failure region. Failure probability is equal to the density function integrated over this region. FORM2 computes the point (*design point*) nearest to the origin in the transformed space and its distance β to the origin. Note β is also called *reliability index* and the probability of failure is $\Phi(-\beta)$ if $\mathbf{X} \sim \mathcal{N}$.

How to find the design point?

For the design point $\mathbf{x}^*$, the following holds:

1. $\mathrm{g}(\mathbf{x}^*) = 0$,
2. the distance $\beta = \sqrt{(\mathbf{x}^*)^T \mathbf{x}^*}$ to the origin is minimal, and
3. $\nabla \mathrm{g}(\mathrm{T}(\mathbf{x}^*)) \parallel \mathrm{T}(\mathbf{x}^*)$

To find this point, the Newton–Raphson method can be used.

Newton–Raphson method

Taylor series: $\mathrm{g}(\mathbf{y}_{k+1}) = \mathrm{g}(\mathbf{y}_k) + \nabla \mathrm{g}(\mathbf{y}_k)(\mathbf{y}_{k+1} - \mathbf{y}_k) + \mathbf{R}_1$
Setting $\mathrm{g}(\mathbf{y}_{k+1}) := 0$ and separating $\mathbf{y}_{k+1}$ results in

$$\mathbf{y}_{k+1} \approx \frac{1}{(\nabla \mathrm{g}(\mathbf{y}_k))^T \nabla \mathrm{g}(\mathbf{y}_k)}((\nabla \mathrm{g}(\mathbf{y}_k))^T \mathbf{y}_k - \mathrm{g}(\mathbf{y}_k))\nabla \mathrm{g}(\mathbf{y}_k).$$

This equation is applied recursively until $\|\mathbf{y}_{k+1} - \mathbf{y}_k\| \leq \delta$ and $|\mathrm{g}(\mathbf{y}_k| \leq \epsilon$.

FORM2 algorithm

Given the limit state function $\mathrm{g}(\mathbf{x})$, the transformation function $\mathrm{T} : \mathcal{P}(\mu_x, \sigma_x) \rightarrow \mathcal{P}(0, 1)$, a stopping criteria ϵ and a starting point $\mathbf{x}^{(1)}$.

$$\mathbf{y}^{(1)} := \mathrm{T}(\mathbf{x}^{(1)})$$

$$i := 1$$

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DO
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Compute $\nabla \mathrm{g}(\mathbf{y}^{(i)})$, where $\nabla = [\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}]$.

$$\beta := \sqrt{(\mathbf{y}^{(i)})^T \mathbf{y}^{(i)}}]^T$$

$$\mathbf{y}^{(i+1)} := \frac{1}{(\nabla \mathrm{g}(\mathbf{y}_k))^T \nabla \mathrm{g}(\mathbf{y}_k)}((\nabla \mathrm{g}(\mathbf{y}_k))^T \mathbf{y}_k - \mathrm{g}(\mathbf{y}_k)) \nabla \mathrm{g}(\mathbf{y}_k)$$

$$\mathbf{x}^{(i+1)} := \mathrm{t}^{-1}(\mathbf{y}^{(i+1)})$$

$$i := i + 1$$

UNTIL $|\mathrm{g}(\mathbf{x}^{(i+1)})| \leq \epsilon$ and $|\mathbf{x}^{(i+1)} - \mathbf{x}^{(i)}| \leq \epsilon$

$$P_f = \Phi(-\beta)$$

How to compute the gradient?

In our example, the following is given

$$\epsilon_0 = u0, \qquad \epsilon_x = \mathbf{u}_m$$

$$R = \epsilon_0, \qquad S = \epsilon_x$$

$$\mathrm{g}(R, S) = \epsilon_x - \epsilon_0, \qquad T_i = \frac{X_i - \mu_i}{\sigma_i}$$

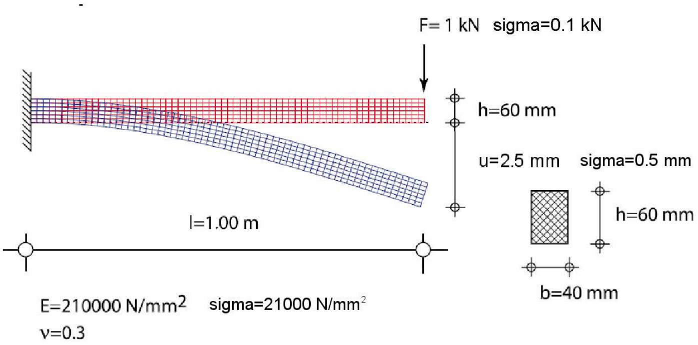
Applying the chain rule to $\partial G / \partial Y_i$

$$\frac{\partial G}{\partial Y_1} = \frac{\partial g}{\partial R}\frac{\partial R}{\partial X_1}\frac{\partial X_1}{\partial Y_1} = (-1) \cdot 1 \cdot \sigma_1$$

$$\frac{\partial G}{\partial Y_2} = \frac{\partial g}{\partial S}\frac{\partial S}{\partial X_2}\frac{\partial X_2}{\partial Y_2} = 1 \cdot \frac{\mathbf{K}(x_2 + dx_2)^{-1}\mathbf{f} - \mathbf{K}(x_2)^{-1}\mathbf{f}}{dx_2} \cdot \sigma_2$$

$$\frac{\partial G}{\partial Y_3} = \frac{\partial g}{\partial S}\frac{\partial S}{\partial X_3}\frac{\partial X_3}{\partial Y_3} = 1 \cdot \frac{\mathbf{K}^{-1}\mathbf{f}(x_3 + dx_3) - \mathbf{K}^{-1}\mathbf{f}(x_3)}{dx} \cdot \sigma_3$$

Result



Reliability index $\beta = 5.1722$

Probability of failure $P_f = 0.1986$

Thank you for your attention!