Implizit Performance Functions (PF)
&
Introduction to Stochastic FEM
(SFEM/PFEM)

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Seminar on
The Finite Element Method and the Analysis of Systems with Uncertain Properties
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Outline

- Introduction

- Performance Functions (PF)

- Computational approaches for implicit PF
  - Monte Carlo Simulation
  - Response Surface Approach
  - Sensitivity-based Approch

- Calculation of response sensitivities
  - Finite Difference Method
  - Classical / iterative perturbation

- Introduction to SFEM

What is the problem?

- Estimate the reliability or safety index or probability of failure
- But…for many correlated or uncorrelated random variables involved, things become complicated
- Plus …implicit, non-continuous PFs can prohibit closed form solutions

→ No analytical solution possible

Why should we care?

- Especially for more complicated structures safety factor design decreases material input enormous

- Failure probabilities are of important interest, since hazards have large social impact

→ Because it is important for design

What can we do about it?

- Computer simulation of entire systems
- Nummeric procedures to estimate properties

→ **Nummeric solutions or PFs are possible**

Performance Functions (PF)s

$g(\mathbf{X})$ ← Random load / resitance related input variables X_i

explicit

e.g. $f(x) = y = \pm\sqrt{\frac{8-x^4}{2}}$

- Closed form function of the input variables
- Derivatives of g(X) with respect to X_i is easy
- FORM

implicit

e.g. $x^4 + 2y^2 = 8$

- For complicated stuctures (most practical cases)
- No closed form expression of g(X) in terms of X_i
- Derivatives of g(X) are not readily available
- FORM2 is usable
- Each evaluation of g(X) is time consuming
- Response must be calculated through a numerical procedure

Computational approaches

Classification based on philosophies …..

1. **Monte Carlo Simulation**
2. **Response surface approach**
3. **Sensitivity-based analysis**

If combined with FEA ► SFEM

Monte Carlo Simulation

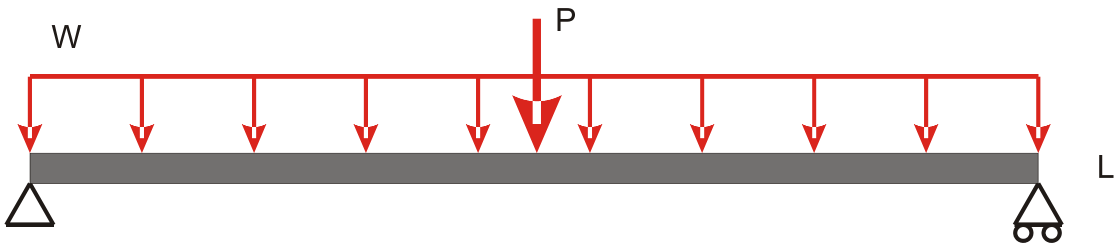
Basic idea: Make many realizations of the problem with random input and make a statistic evaluation of the system answers

1. Defining the problem in terms of all random variables
2. Quantifying the probabilistic characteristics
3. Generating random variables
4. Evaluating the deterministic problem for each realization
5. Extracting probabilistic information from *N* realizations
6. Determining the accuracy and efficiency of the MC simulation

Monte Carlo Simulation

1. Defining the problem in terms of all random variables

Example:



$$M = \frac{WL^2}{8} + \frac{PL}{4}$$

Design bending moment at midspan

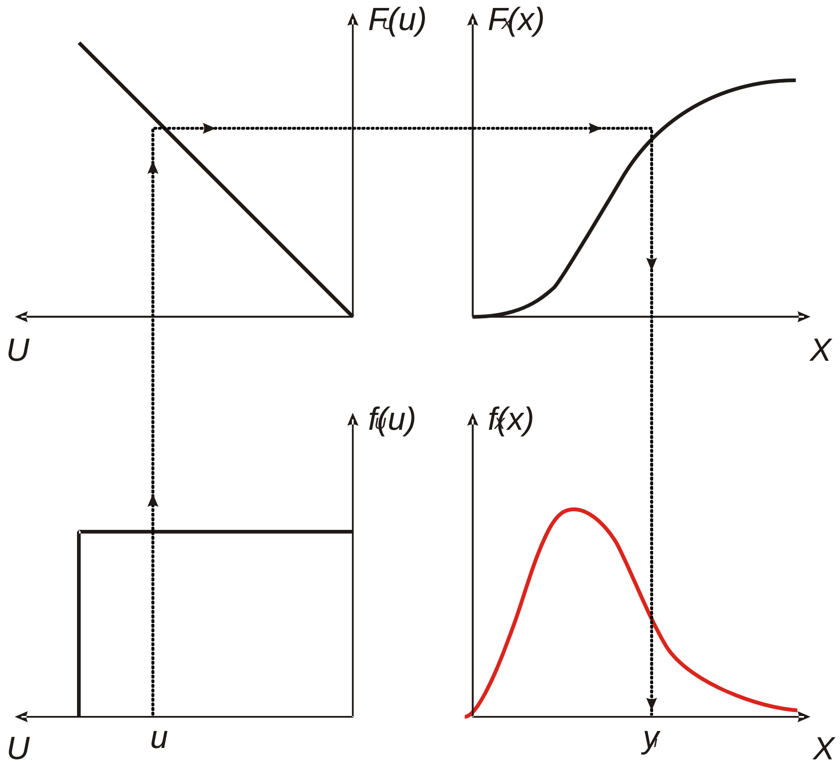
2. Quantifying the probabilistic characteristics of random variables

PDF or CDF with corresponding parameters

Monte Carlo Simulation

3. Generating values for independent random variables

Continuous Random Variables: to get from uniform $a<u_i<b$ to other distributions
→ inverse transformation technique (inverse CDF method)



Monte Carlo Simulation

4. Evaluating the problem for each realization

   Can be made by any approach (analytic, black box, nummeric, FEM…)

5. Extracting probabilistic information from *N* realizations

6. Determining the accuracy and efficiency

Monte Carlo Simulation

MC Simulation is…

a robust, powerful tool

works for uncorrelated & correlated random variables

requires only basic knowledge of probability

provides an estimate for any problem

can be used to verify more sophisticated analytical methods

but….

accurate results only for $N \rightarrow \infty$

Time consuming for highly reliable problems

→ efficient sampling methods and variance reduction techniques can help

Estimates for accuracy needed

Response Surface Approach (RSA)

Basic idea: Construct polynominal closed-form approximation for the PF through a few deterministic analyses and a regression anaysis of results.

1. Select sets of values for *n* random variables x_n to evaluate the PF.

   Eg. Full factorial design; 2 or 3 values/variable; calculate PF for all possible combinations (2^n or 3^n) on higher and lower $\mu \pm k\sigma$ or μ, $\mu \pm k\sigma$

2. Evaluate PF $g(\mathbf{X})$ for all variable sets

3. Put results in 1st or 2nd order regression model for $g(\mathbf{X})$

4. Use FORM/SORM or MC on the closed form expression of $g(\mathbf{X})$ to estimate design point and failure probability.

Response Surface Approach (RSA)

Example: PF

$$g(\mathbf{X}) = a - u(\mathbf{X}), \qquad \mathbf{X} = (X_1, X_2, X_3 ... X_n)$$

First order closed form approximation

$$g(\mathbf{X}) = b_0 + b_1 X_1 + b_2 X_2 + b_3 X_3 + ... + b_n X_n$$

b_i from linear regression scheme

FORM/SORM based on approximated PF

Response Surface Approach (RSA)

RSA is…

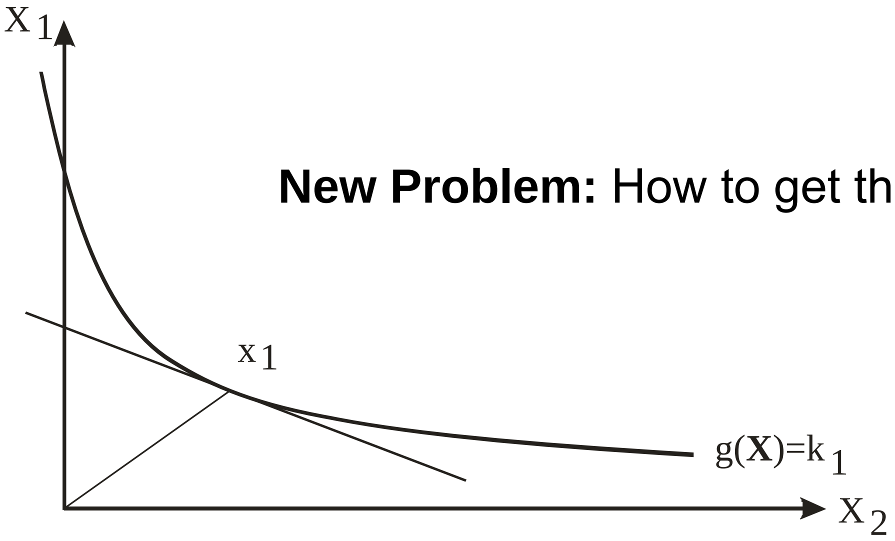
- A simple approximation based on selected values
- Adequate for linear functions

but….

- Could be inadequate for highly nonlinear PF
- Valid only within the range of the considered values,
  - → extrapolaton is not accurate
- Good physical knowledge on the system is required to choose the right approximation order
- Time consuming for problems with many variables (scales with power *n*)

Sensititvity-based approach

Basic idea: Use actual value and actual gradient of PF directly at each iteration of the search for the design point and an optimization scheme to converge



New Problem: How to get the actual gradient?

Approximate methods:

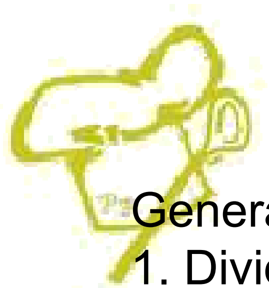
1. Finite Difference method (FD)
2. Classical perturbation methods
3. Iterative perturbation analysis

Finite Difference Approach

<u>Basic idea:</u>Replace **differential equations** by **finite difference equations**

algebraic in form / solutions are related to grid points.

$$\frac{dZ}{dX} = \lim_{\Delta X \to 0} \frac{\Delta Z}{\Delta X}$$



General steps of FD Solution:

1. Dividing the solution into grids of nodes.
2. Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.
3. Solving the difference equations subject to the prescribed boundary conditions and/or initial conditions.

Finite Difference Approach

Input/basic variables

usually $Z = g(X_1, X_2, ..., X_n)$

Output/response variable

Forward difference approach at the point $(X_1^0, X_2^0, ..., X_n^0)$:

1. Compute $Z_0 = g(X_1^0, X_2^0, ..., X_n^0)$
2. Change X_1^0 to $X_1^0 + \Delta X_1$ and fix all other variables. Compute the new value

   $$Z_1 = g(X_1^0 + \Delta X_1, X_2^0, ..., X_n^0)$$

3. Compute the change of Z and the derivative $\Delta Z / \Delta X_1 = (Z_1 - Z_0)/X_1$
4. Repeat 2-3 for each other variable

backward difference approach / central difference approach

Finite Difference Approach

FDA is…

- A simple approximation based on selected values
- Adequate for linear functions
- Can be done with any type of analysis e.g. commercial software

but….

- Brute force method
- Requires many evalutations of g(X)
- (n+1)m iterations become time consuming

Number of FORM iterations (m)

Number of random variables (n)

Classical Perturbation Methods

Basic idea: Estimate the variation of the response by tracking the variation at every step in terms of the variation of the basic variabes.

→ Application of the **chain rule** of differentiation

$$g(\mathbf{X}) = a - u(\mathbf{X}), \qquad \mathbf{X} = (X_1, X_2, X_3, ..., X_n)$$

Chain rule of differentiation:

$$\frac{\partial g}{\partial X_i} = \frac{\partial g}{\partial X_1} \cdot \frac{\partial X_1}{\partial X_2} \cdot \ldots \cdot \frac{\partial X_n}{\partial X_i}$$

Iterative Perturbation Analysis

Basic idea: Compute the sensitivity of a PF to changes in the random variable values.

2 Methods of iterative perturbation:

Method 1: simple iterative procedure for geometrical nonlinearety
can be used for simpler problems where the analysis computer program can be modified by the user

Method 2: iterative perturbation by residual force calculation
useful with structural analysis

Method 1: Example axially loaded bar

$$K_0 = \frac{AE}{L_0},$$

$$\Delta U = \frac{\Delta F}{K_0}$$

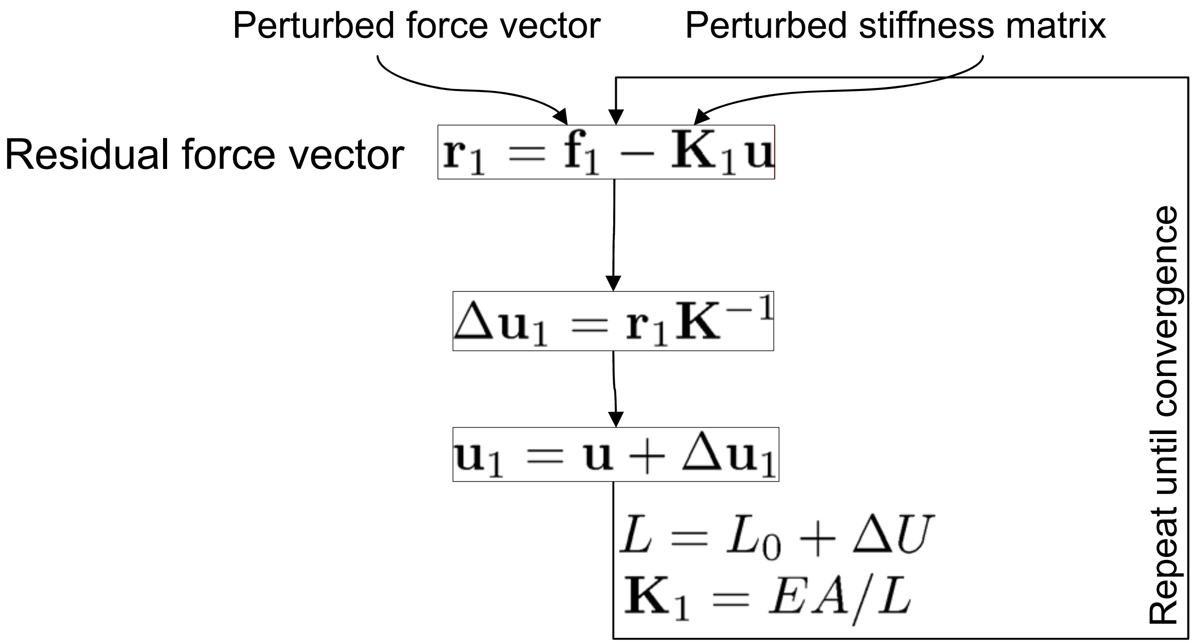
$$L = L_0 + \Delta U$$

Repeat until convergence

$$\frac{\delta U}{\delta F} \approx \frac{\Delta U}{\Delta F}$$

For a small variation in the load, an iterative solution for the response is computed

Method 2: Example axially loaded bar



$$\frac{\delta U}{\delta F} \approx \frac{\Delta U}{\Delta F}$$

Original stiffness is used in every iteration to calculate the perturbed displacements. A new residual vector is defined and predictor-corrector sequence is repeated until convergence of the perturbed solution

Basic Concept of SFEM

Get derivatives (FD, cl. / Iter. Perturbation)

Construct an approximate closed-form PF

FORM/SORM

Use derivatives directly in the reliability analysis

FORM Method 2

Combination of FE analysis with sensitivity analysis
→Probabilistic analysis
→PFEM / SFEM

Finite Difference Approach with FEM

Input/basic variables

usually $Z = g(X_1, X_2, ..., X_n)$

Output/response variable

Forward difference approach at the point $(X_1^0, X_2^0, ..., X_n^0)$:

1. Compute $Z_0 = g(X_1^0, X_2^0, ..., X_n^0)$ FEM
2. Change X_1^0 to $X_1^0 + \Delta X_1$ and fix all other variables. Compute the new value $Z_1 = g(X_1^0 + \Delta X_1, X_2^0, ..., X_n^0)$ FEM
3. Compute the change of Z and the derivative $\Delta Z / \Delta X_1 = (Z_1 - Z_0)/X_1$
4. Repeat 2-3 for each other variable

backward difference approach / central difference approach

Classical Perturbation with FEM

Form global

$$\mathbf{K}, \mathbf{F}$$

Solve FE equation

$$\mathbf{K}\mathbf{U} = \mathbf{F}$$

Compute response vector

$$\mathbf{S} = \mathbf{Q}^{\mathbf{t}}\mathbf{U} + \mathbf{S}_0$$

Construct PF

$$g(\mathbf{X}) = g\{\mathbf{R}(\mathbf{X}), \mathbf{S}(\mathbf{X})\}$$

→ More about this in next weeks lecture

Thank you for your attention