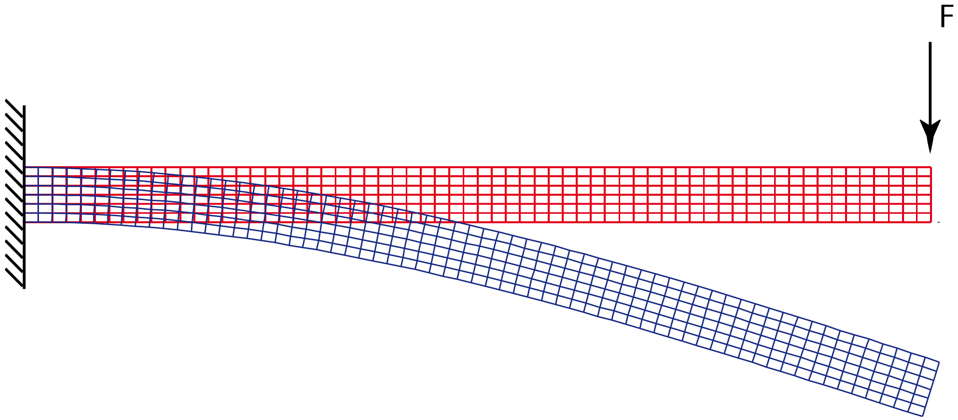
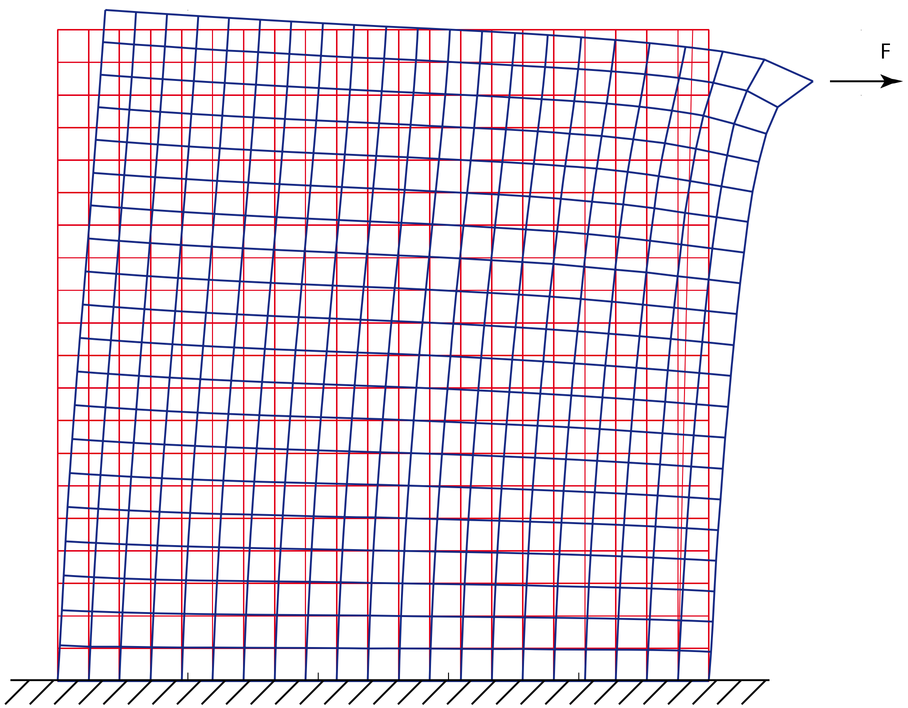
# The Finite Element Method and the Analysis of Systems with Uncertain Properties




Example “4 Node Isoparametric Element”
PhD Seminar
Matthias Schubert

Group Risk and Safety
ETH – Zürich
10.01.07

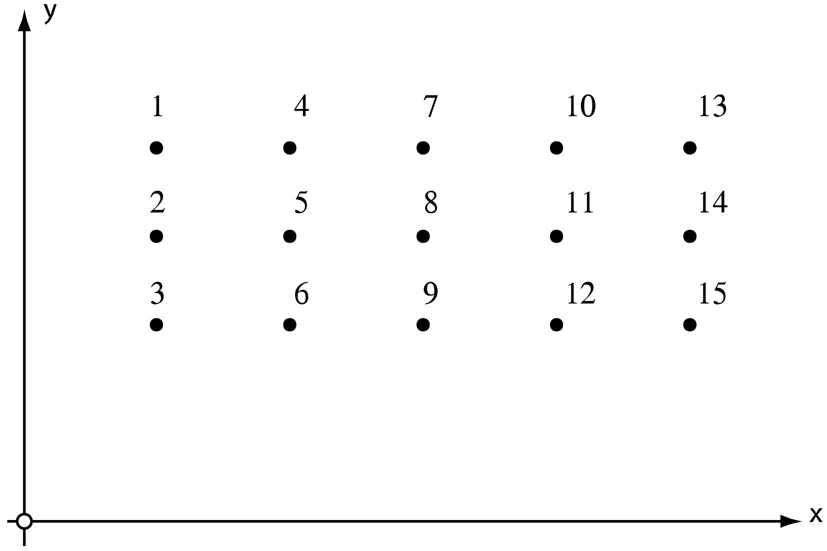
# Overview




- Defining nodes / Geometry
- Assembling Elements
- Isoparametric 4 node elements
- The local stiffness matrix
- Assembling global stiffness matrix
- Convergence and results

# Defining nodes / Geometry and Elements

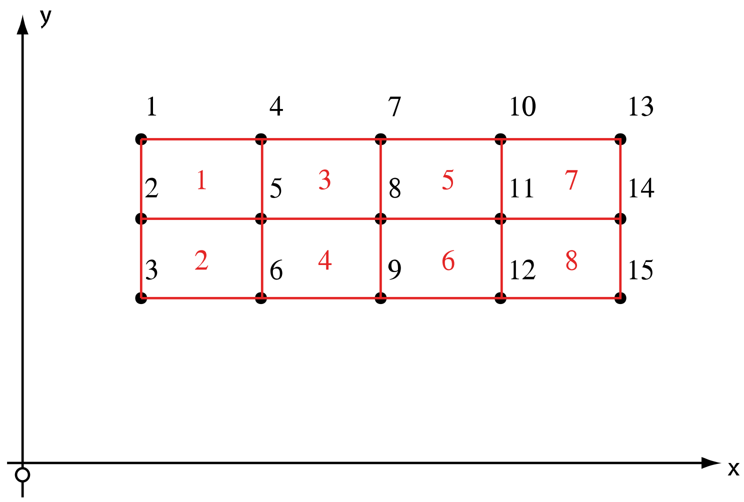
- An efficient node numbering reduces the bandwidth of the stiffness matrix




- The elements can now be defined using the node numbering

Element 1 = [1 4 5 2]

- Each node has two degrees of freedom



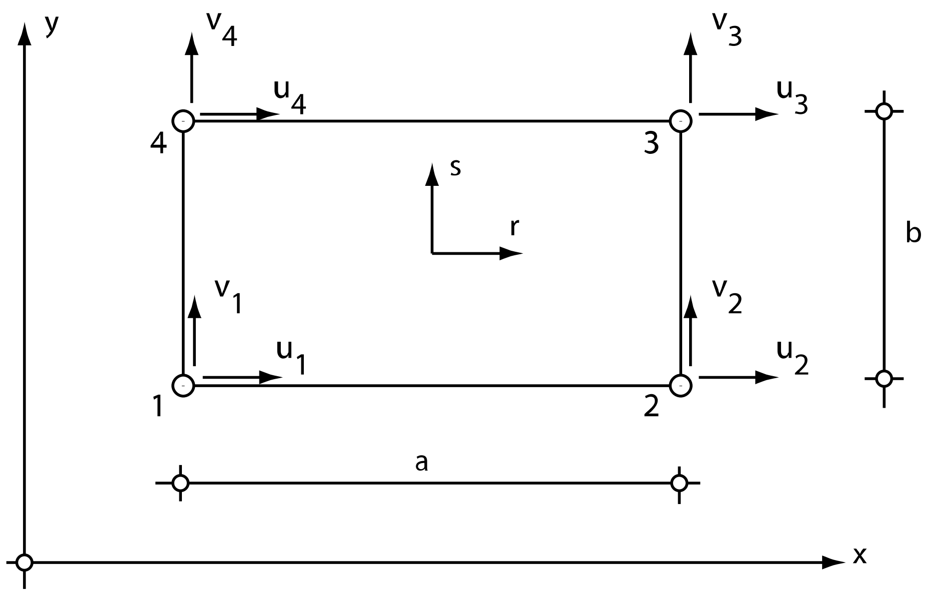
# 4 Node Isoparametric Elements

- Here: rectangular bilinear element

→ rectangular

→ 4 Nodes

→ 8 Degrees of freedom



# 4 Node Isoparametric Elements

→ Bilinear shape function

$$\Omega_1 = \frac{1}{4}(1-r)(1-s)$$

$$\Omega_1 = \frac{1}{4}(1+r)(1-s)$$

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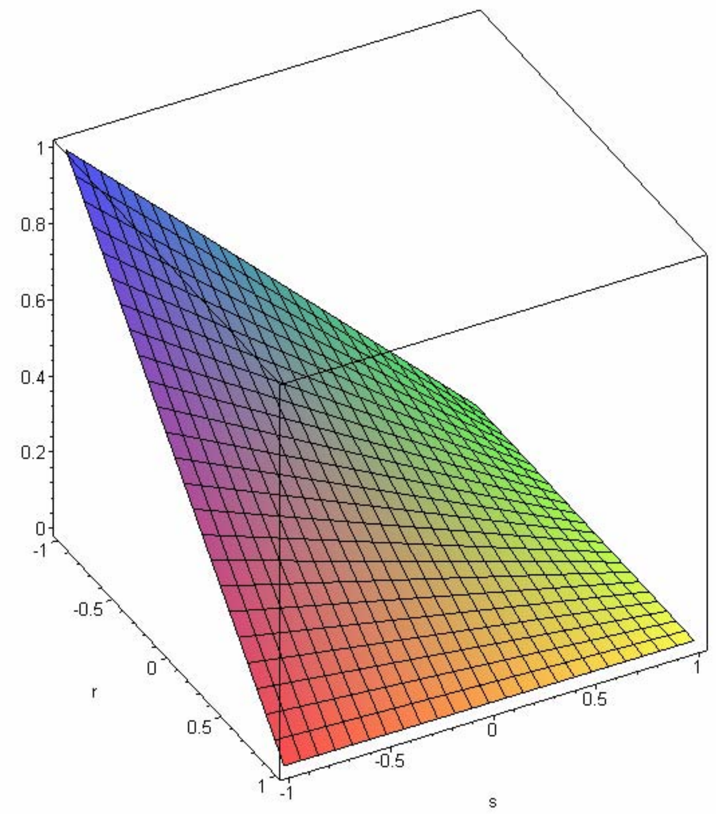


## 4 Node Isoparametric Elements

### Strain-displacement matrix B

$$\mathbf{B} = \mathbf{D}_k \boldsymbol{\Omega} = \begin{bmatrix} dx & 0 \\ 0 & dy \\ dy & dx \end{bmatrix} \begin{bmatrix} \Omega_1 & 0 & \Omega_2 & 0 & \Omega_3 & 0 & \Omega_4 & 0 \\ 0 & \Omega_1 & 0 & \Omega_2 & 0 & \Omega_3 & 0 & \Omega_4 \end{bmatrix}$$

$$\mathbf{B} = \frac{1}{4ab} \begin{bmatrix} -b(1-s) & 0 & b(1-s) & 0 & b(1+s) & 0 & -b(1+s) & 0 \\ 0 & -a(1-r) & 0 & -a(1+r) & 0 & a(1+r) & 0 & a(1-r) \\ -a(1-r) & -b(1-s) & -a(1+r) & b(1-s) & a(1+r) & b(1+s) & a(1-r) & -b(1+s) \end{bmatrix}$$

### Stress-strain matrix E – usually constant in elastic problems

$$\mathbf{E} = \frac{Et}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & 0.5(1-\upsilon) \end{bmatrix}$$

## The local stiffness matrix

$$\mathbf{k} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{E}\,\mathbf{B}\,\frac{ab}{4}\,dr\;ds$$

In this case the stiffness matrix can be expressed in closed form

-numerical integration

- analytically

## The local stiffness matrix

How can the local stiffness matrix be controlled?

-Symmetry

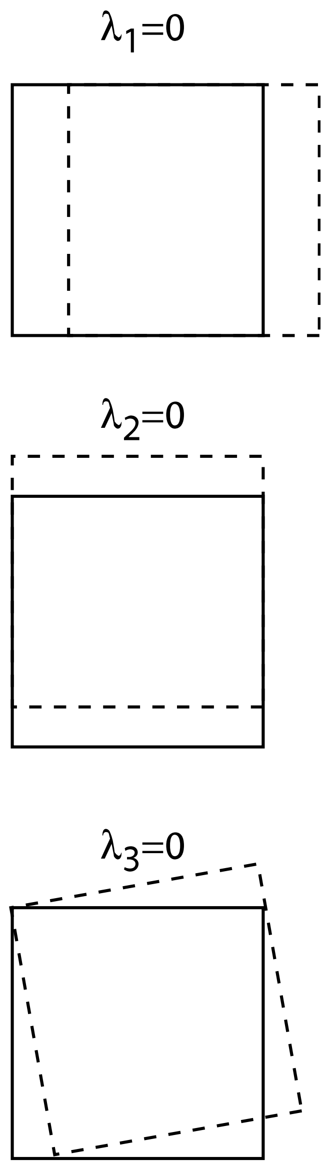
-Diagonal elements are positive

-Controll the rigid-body modes

We have 3 possible ridig body modes

$$\det\left(\mathbf{k}-\lambda\mathbf{E}\right)=0$$

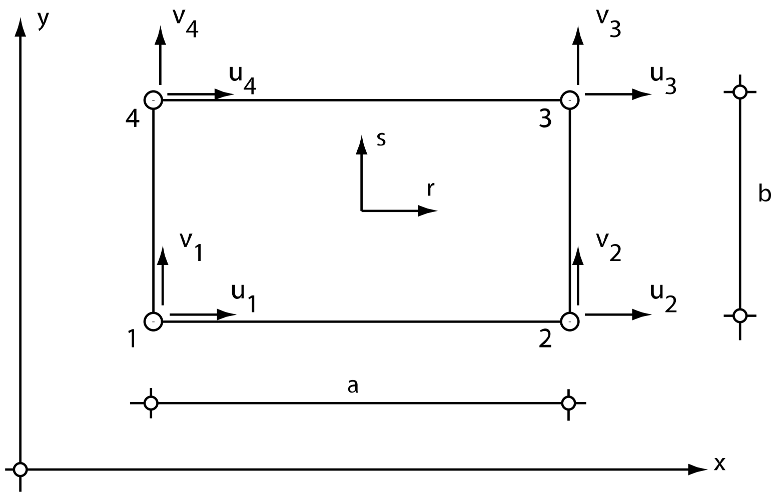
$\mathbf{E}$ denotes the unit matrix



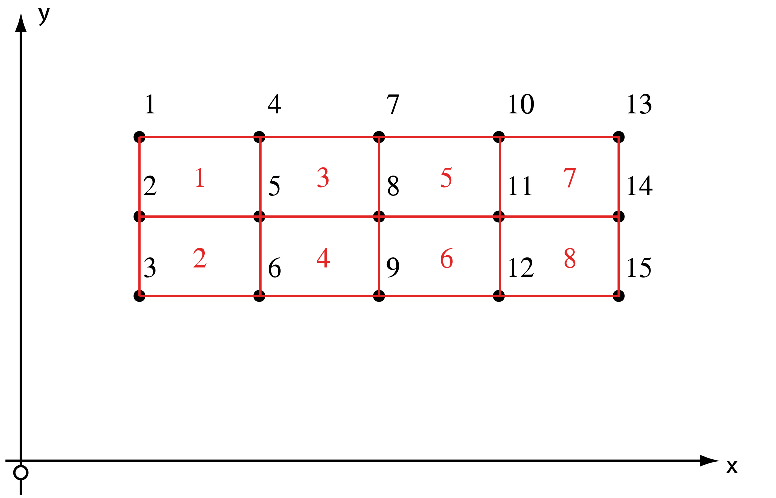
# Assembling the gobal stiffness matrix

Local stiffness – local coordinates

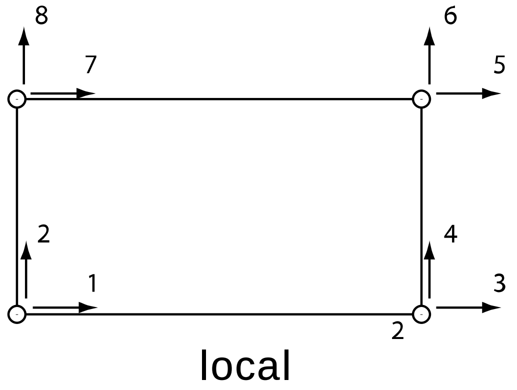
Global coordinates – global stiffness

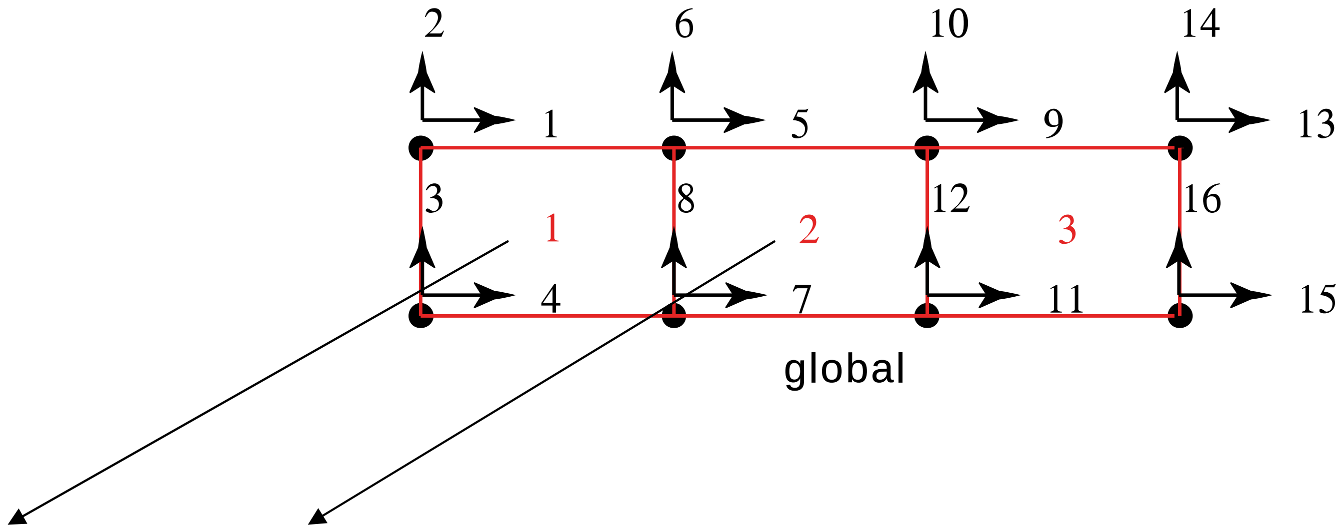



Transformation local - global



# Assembling the gobal stiffness matrix

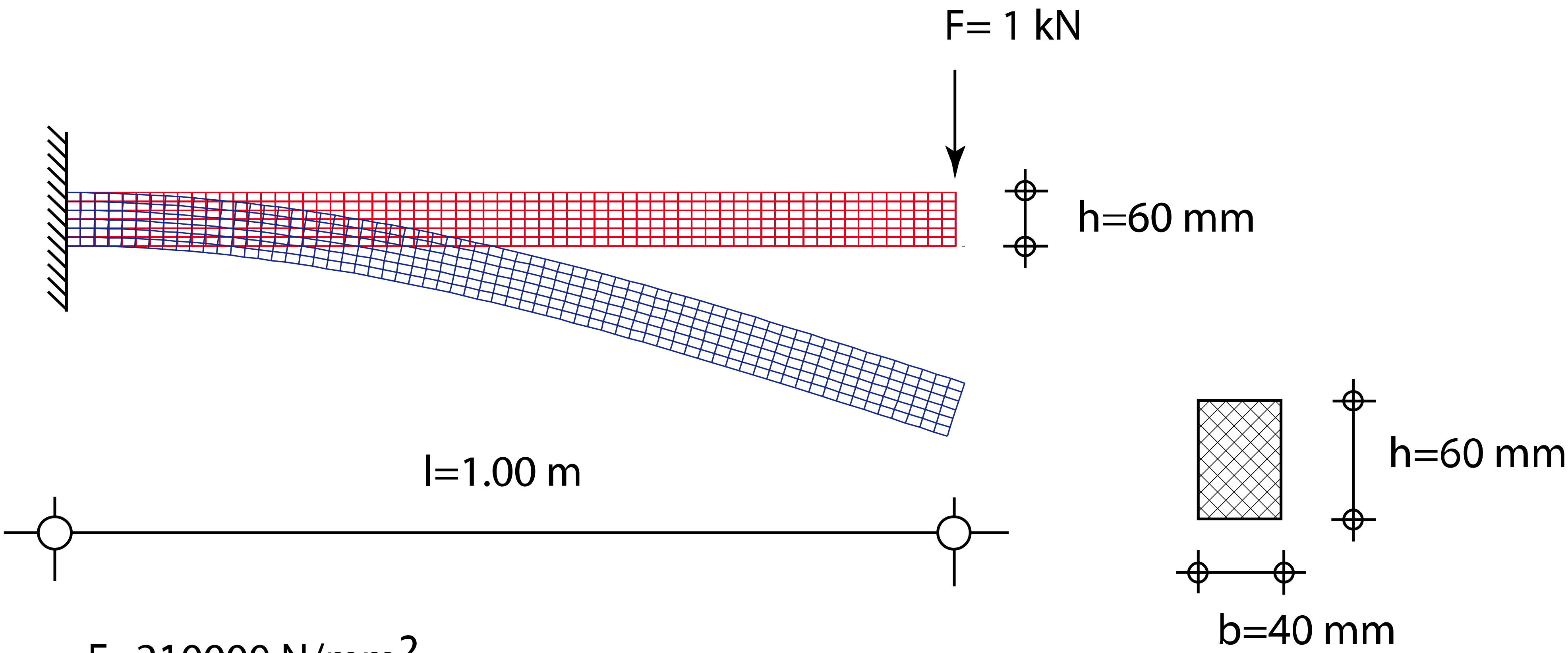



$$k_{global}(5,5) = k_{local}(5,5) + k_{local}(7,7)$$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6.54E+06 | -1.50E+06 | -5.71E+06 | -1.15E+05 | 2.44E+06 | 1.15E+05 | -3.27E+06 | 1.50E+06 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | -1.50E+06 | 1.73E+07 | 1.15E+05 | -1.70E+07 | -1.15E+05 | 8.35E+06 | 1.50E+06 | -8.64E+06 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | -5.71E+06 | 1.15E+05 | 6.54E+06 | 1.50E+06 | -3.27E+06 | -1.50E+06 | 2.44E+06 | -1.15E+05 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | -1.15E+05 | -1.70E+07 | 1.50E+06 | 1.73E+07 | -1.50E+06 | -8.64E+06 | 1.15E+05 | 8.35E+06 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 2.44E+06 | -1.15E+05 | -3.27E+06 | -1.50E+06 | | 0 | -1.14E+07 | 0 | 2.44E+06 | 1.15E+05 | -3.27E+06 | 1.50E+06 | 0 | 0 | 0 | 0 |
| 6 | 1.15E+05 | 8.35E+06 | -1.50E+06 | -8.64E+06 | 0 | 3.46E+07 | 0 | -3.40E+07 | -1.15E+05 | 8.35E+06 | 1.50E+06 | -8.64E+06 | 0 | 0 | 0 | 0 |
| 7 | -3.27E+06 | 1.50E+06 | 2.44E+06 | 1.15E+05 | -1.14E+07 | 0 | 1.31E+07 | 0 | -3.27E+06 | -1.50E+06 | 2.44E+06 | -1.15E+05 | 0 | 0 | 0 | 0 |
| 8 | 1.50E+06 | -8.64E+06 | -1.15E+05 | 8.35E+06 | 0 | -3.40E+07 | 0 | 3.46E+07 | -1.50E+06 | -8.64E+06 | 1.15E+05 | 8.35E+06 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 2.44E+06 | -1.15E+05 | -3.27E+06 | -1.50E+06 | 1.31E+07 | 0 | -1.14E+07 | 0 | 2.44E+06 | 1.15E+05 | -3.27E+06 | 1.50E+06 |
| 10 | 0 | 0 | 0 | 0 | 1.15E+05 | 8.35E+06 | -1.50E+06 | -8.64E+06 | 0 | 3.46E+07 | 0 | -3.40E+07 | -1.15E+05 | 8.35E+06 | 1.50E+06 | -8.64E+06 |
| 11 | 0 | 0 | 0 | 0 | -3.27E+06 | 1.50E+06 | 2.44E+06 | 1.15E+05 | -1.14E+07 | 0 | 1.31E+07 | 0 | -3.27E+06 | -1.50E+06 | 2.44E+06 | -1.15E+05 |
| 12 | 0 | 0 | 0 | 0 | 1.50E+06 | -8.64E+06 | -1.15E+05 | 8.35E+06 | 0 | -3.40E+07 | 0 | 3.46E+07 | -1.50E+06 | -8.64E+06 | 1.15E+05 | 8.35E+06 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2.44E+06 | -1.15E+05 | -3.27E+06 | -1.50E+06 | 6.54E+06 | 1.50E+06 | -5.71E+06 | 1.15E+05 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.15E+05 | 8.35E+06 | -1.50E+06 | -8.64E+06 | 1.50E+06 | 1.73E+07 | -1.15E+05 | -1.70E+07 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -3.27E+06 | 1.50E+06 | 2.44E+06 | 1.15E+05 | -5.71E+06 | -1.15E+05 | 6.54E+06 | -1.50E+06 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.50E+06 | -8.64E+06 | -1.15E+05 | 8.35E+06 | 1.15E+05 | -1.70E+07 | -1.50E+06 | 1.73E+07 |

## Example

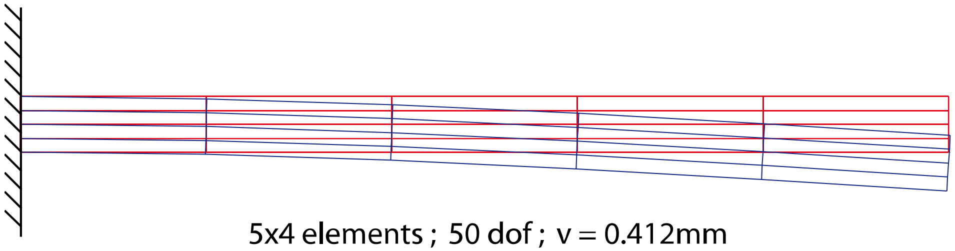



E=210000 N/mm$^2$

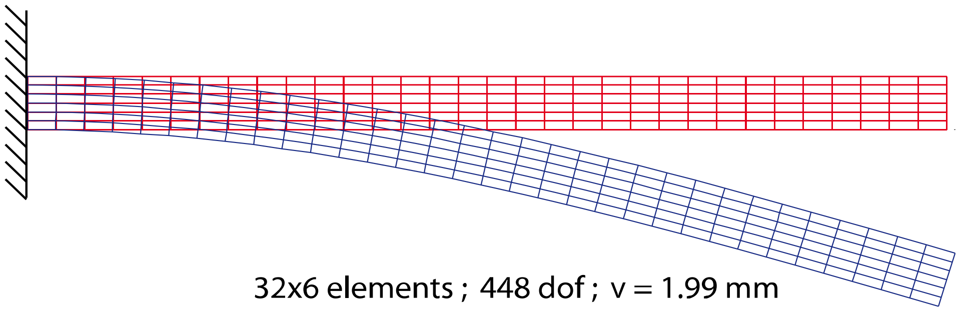
ν=0.3

The Matlab-Code will be provided on our Web site

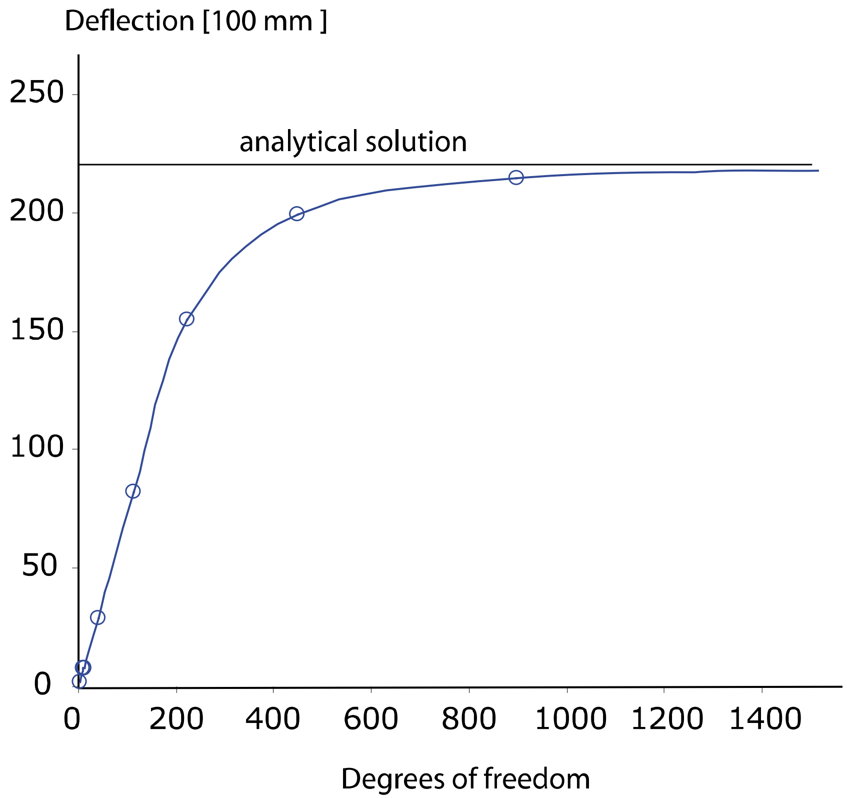
# Results



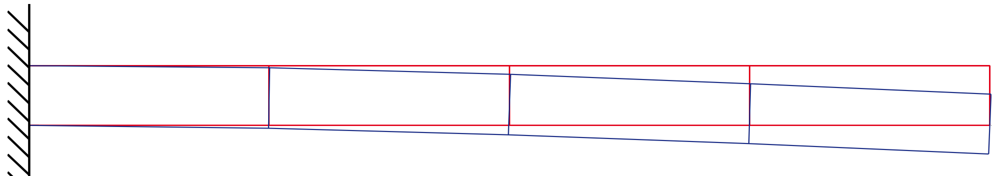
5x4 elements ; 50 dof ; v = 0.412mm



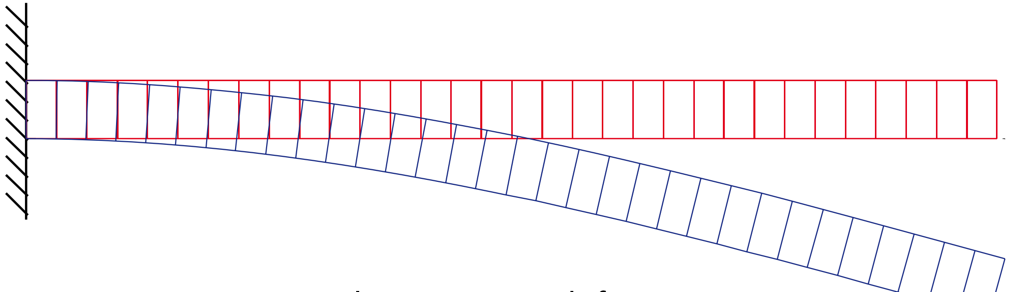
32x6 elements ; 448 dof ; v = 1.99 mm



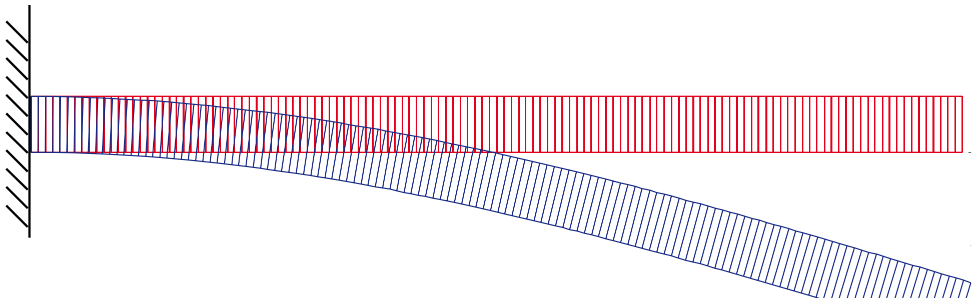
# Results



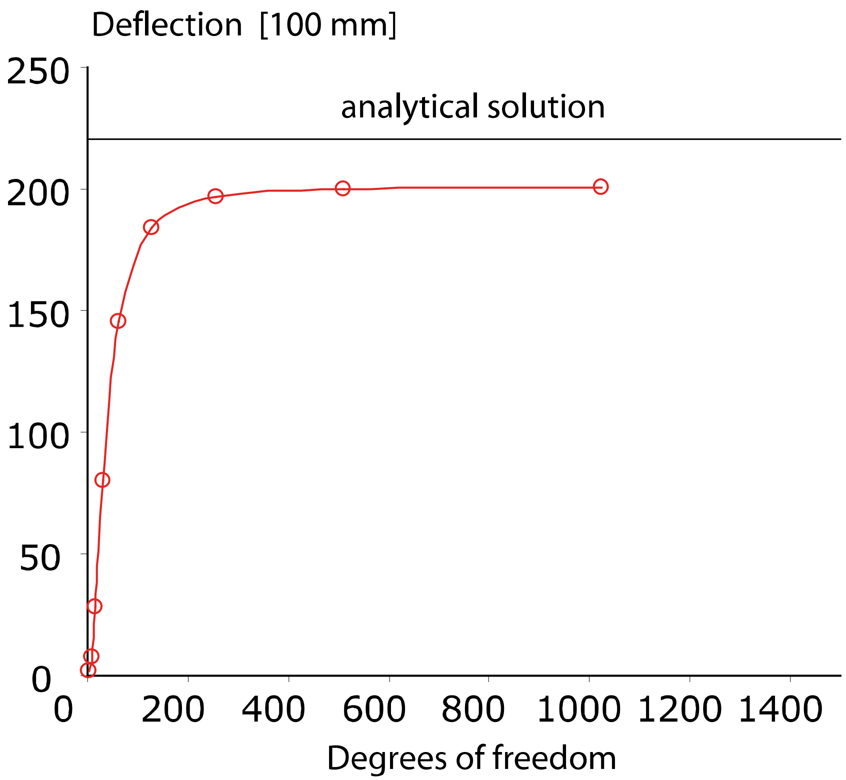
4x1 elements ; 16 dof ; v = 0.28 mm



32x1 elements ; 128 dof ; v = 1.83 mm



128x1 elements ; 512 dof ; v = 1.99 mm



## Stress

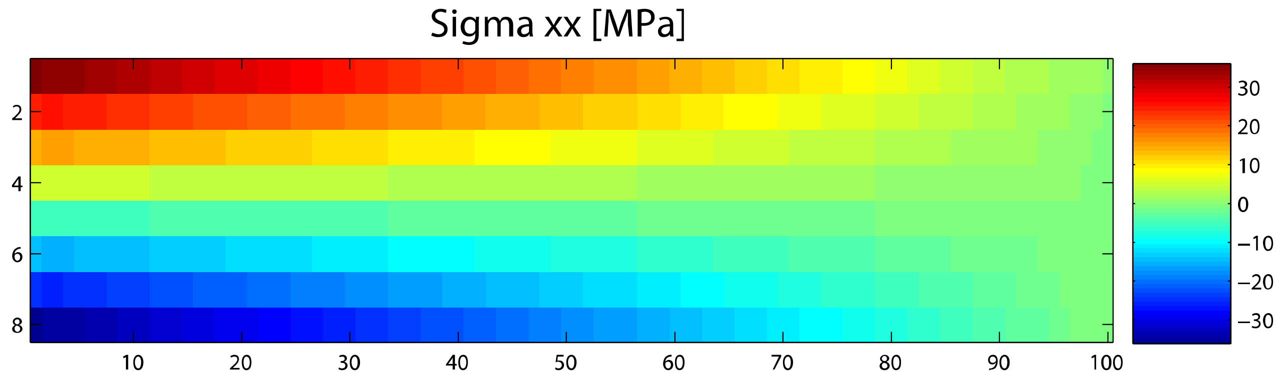
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & 0.5(1-\upsilon) \end{bmatrix} \mathbf{B}\ \mathbf{V}$$

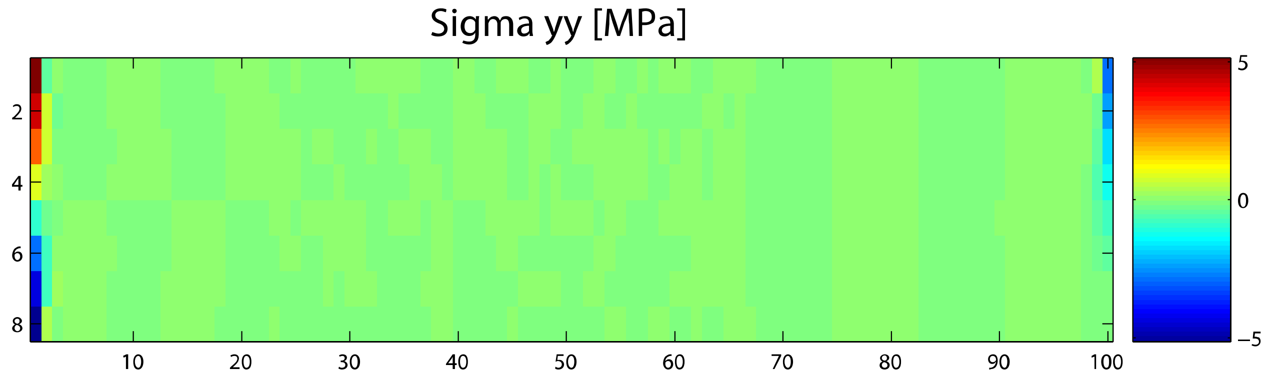
Displacement of the element

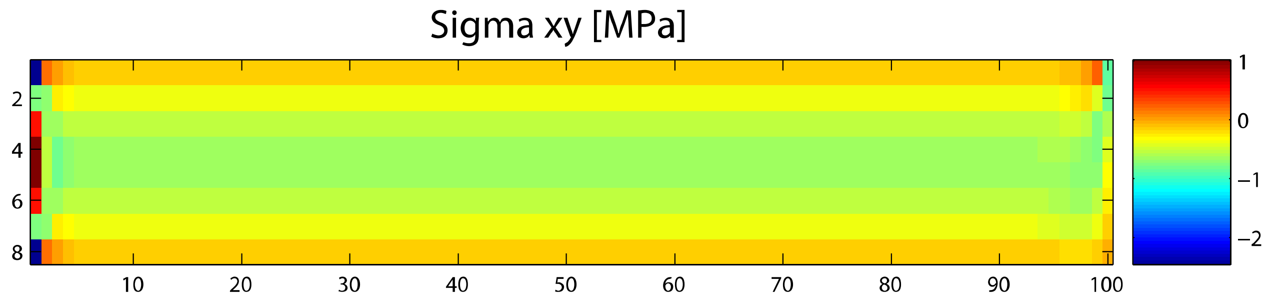
Transformation of the global displacement into local displacements

# Stress

100x8 Elements

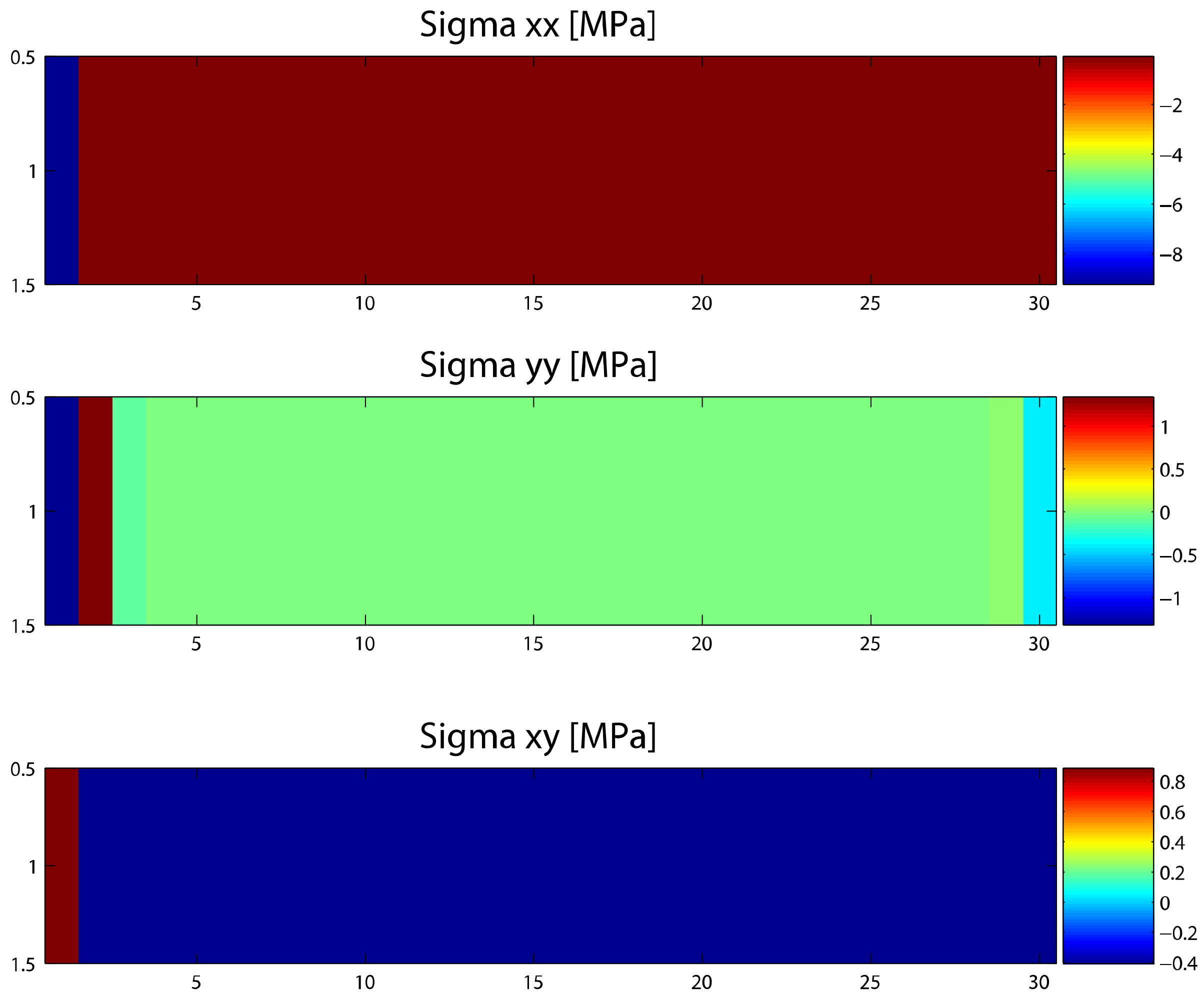


# Stress

20x1 Elements



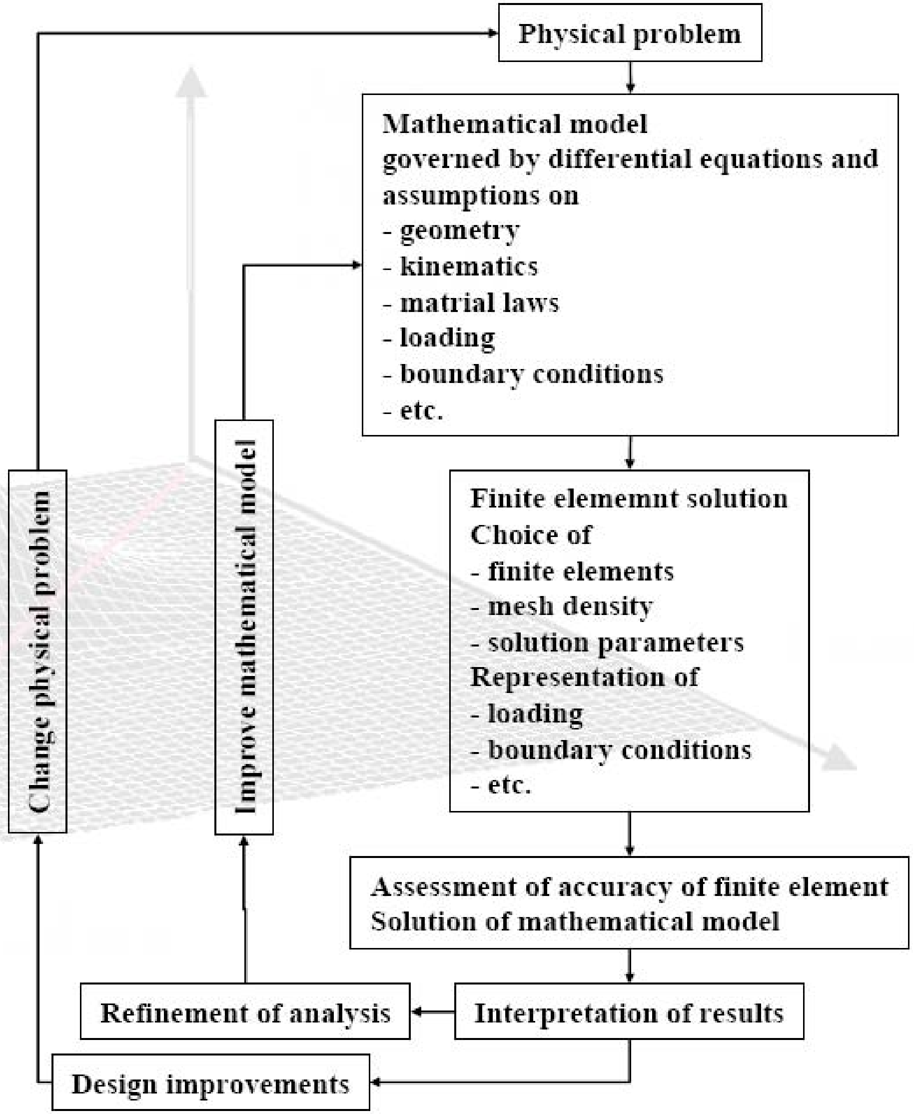
# Introduction to the use of finite element

- Physical problem, mathematical modeling and finite element solutions

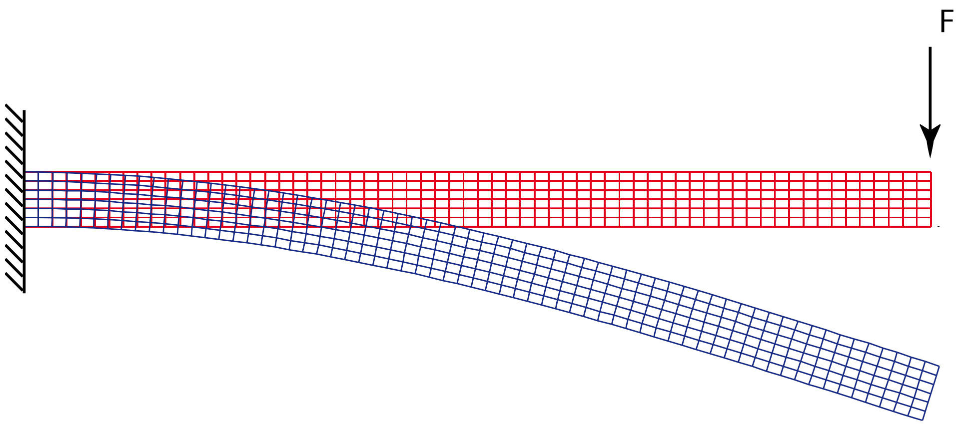
  - we are only working on the basis of mathematic models !

  - choice of mathematical model is crucial !

  - mathematical models must be *reliable* and *effective*



# The Finite Element Method and the Analysis of Systems with Uncertain Properties

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