Solution of static finite element problems: the LDL^T-solution

FEM seminar talk

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Outline

- Gauss elimination
- Physical interpretation of Gauss elimination in the context of finite element problems
- The LDL^T-solution:
  - Introduction to the procedure
  - Algorithm used in computational implementations
- Properties of **K**
- Error considerations
- Related methods

Matrices

- Revision:

Positive-definiteness: $\mathbf{v}^T\mathbf{A}\mathbf{v} > 0$ for all vectors $\mathbf{v}$ (semi-positive definite: $\mathbf{v}^T\mathbf{A}\mathbf{v} \geq 0$)
(analogous: negative-definite)

Bandwidth of a matrix **A**:
$p_1 + p_2 + 1$, where $a_{ij} = 0$ for $j > i + p_2$ or $i > j + p_1$

p_1=2, p_2=1 $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 & 0 & 0 \\ 6 & 7 & 8 & 9 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 & 7 & 8 & 0 \\ 0 & 0 & 0 & 9 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 & 5 & 6 \end{bmatrix}$

Skyline of a matrix:
for j, $j = 1,\ldots,n$: $m_j = i'$ with $a_{ij} = 0$ for $i < i'$

$\mathbf{m}^T = [1\ 1\ 2\ 3\ 4\ 5\ 6]$

Column heights of a matrix: $h_i = i - m_i$ for $i = 1,\ldots,n$

$\mathbf{h}^T = [0\ 1\ 1\ 1\ 1\ 1\ 1]$

(maximum column height = half-bandwidth m_K)

Gauss elimination

→ Carl F. Gauss, ca. 1850,
in the framework of the solution of linear systems of equations

<u>In general:</u>

Solve **Ax=b** for **x**

where **A** is a matrix of coefficients,

x is the vector of unknowns,

b is the right-hand side vector

<u>In the context of finite element problems:</u>

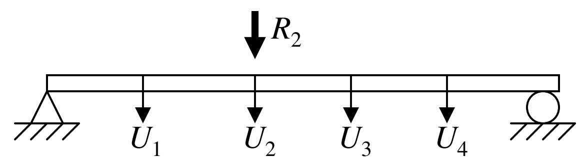
Solve **KU=R** for **U**

where **K** is the stiffness matrix,

U is the displacement vector,

R is the load vector

e.g. simply supported beam with 4 transl. dofs



$$\underbrace{\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{\mathbf{U}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{R}}$$

Gauss elimination

→ In a Gauss elimination, we reduce the matrix of coefficients to an upper triangular form, by a successive addition of multiples of the i^{th} row ($i = 1,\ldots,n-1$) to the remaining $n-i$ rows j ($j = i+1,\ldots,n$).

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

r2 = r2 + 4/5 r1;
r3 = r3 + (-1/5) r1;
r4 = r4;

⇩

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

r3 = r3 + 16/14 r2;
r4 = r4 + (-5/14) r2;

⇩

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ -\frac{5}{14} \end{bmatrix}$$

r4 = r4 + 20/15 r3;

Gauss elimination

→ The result is an upper-triangular matrix which we can solve for the unknowns U_i in the order $U_n, U_{n-1}, \ldots, U_1$.

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & 14/5 & -16/5 & 1 \\ 0 & 0 & 15/7 & -20/7 \\ 0 & 0 & 0 & 5/6 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 8/7 \\ 7/6 \end{bmatrix}$$

$$U_4 = \frac{\frac{7}{6}}{\frac{5}{6}} = \frac{7}{5}; \qquad U_3 = \frac{\frac{8}{7} - (-\frac{20}{7})U_4}{\frac{15}{7}} = \frac{12}{5}$$

$$U_2 = \frac{1 - (-\frac{16}{5})U_3 - (1)U_4}{\frac{14}{5}} = \frac{13}{5}$$

$$U_1 = \frac{0 - (-4)\frac{13}{5} - (1)\frac{12}{5} - (0)\frac{7}{5}}{5} = \frac{8}{5}$$

<u>Note:</u>

- After step i (*i.e.* after the full addition procedure involving multiples of row i), the lower right $(n-i)$ x $(n-i)$ submatrix is symmetric → storage implications
- Solution based on non-vanishing i^{th} diagonal element of coefficient matrix in step i.
- The operations on the coefficient matrix are independent of the right-hand side vector.
- Any desirable order of eliminations may be chosen.

Physical interpretation of Gauss elimination

→A physical interpretation of the operations performed in a Gauss elimination:

Example:

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ & 6 & -4 & 1 \\ & & 6 & -4 \\ \text{symmetric} & & & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

First equation: $5\ U_1 - 4\ U_2 + U_3 = 0 \iff U_1 = 4/5\ U_2 - 1/5\ U_3$
Elimination of U_1 from equations 2, 3 and 4 yields the lower right 3 x 3 submatrix which we get after the first step of the Gauss elimination of the original matrix:

$$\begin{bmatrix} 14/5 & -16/5 & 1 \\ -16/5 & 29/5 & -4 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Stiffness matrix corresponding to beam after release of dof 1.
(dof 1“statically condensed out”)

….. 5/6 is stiffness matrix of beam after release of dofs 1, 2 and 3 (cf. Gauss elimination: final upper triangular matrix).

Physical interpretation of Gauss elimination

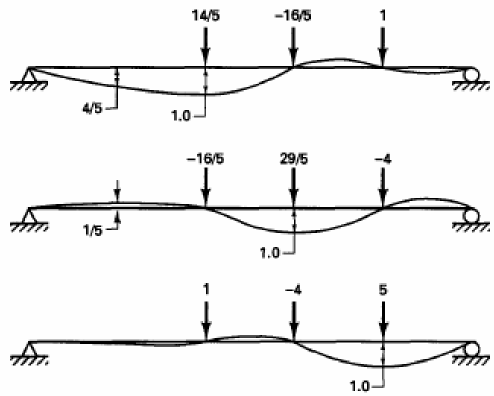


Figure 8.4 Experimental results of forces in clamps due to unit displacements with clamp 1 not present.

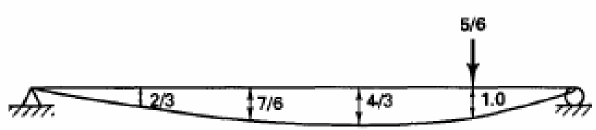


Figure 8.6 Experimental results of forces in clamps due to unit displacement with clamps 1, 2, and 3 not present.

Physical interpretation of Gauss elimination

• We get a total of n stiffness matrices of decreasing order (n, n-1,…, 2, 1), each describing a set of n-i degrees of freedom (i = 0, 1,...,n-1) of the same physical system.

• If $\mathbf{R}\neq\mathbf{0}$, then we also establish the load vectors pertaining to these stiffness matrices.

• The physical picture suggests that the diagonal elements remain positive during the Gauss elimination: Stiffness should be positive; a non-positive diagonal element implies an unstable structure.

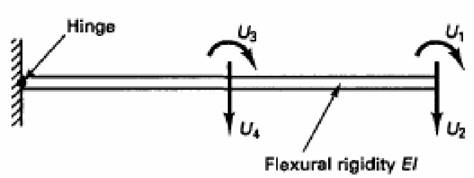


Figure 8.7 Example of an unstable structure

Here, after release of dofs U_1, U_2 and U_3 the last diagonal element (*i.e.* the stiffness at dof U_4) is zero.

The LDL^T-solution

The successive matrix operations during a Gauss elimination can be cast into a general form, which leads, likewise, to the reduction of **K** to an upper triangular form, **S**,

$$\mathbf{L}_{n-1}^{-1}\ldots\ldots\mathbf{L}_2^{-1}\mathbf{L}_1^{-1}\mathbf{K} = \mathbf{S}$$

where

$$\mathbf{L}_i^{-1} = \begin{bmatrix} 1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & 1 & & & \\ & & & & -l_{i+1,i} & \cdot & & \\ & & & & -l_{i+2,i} & & \cdot & \\ & & & & \cdots & & & \cdot \\ & & & & -l_{n,i} & & & & 1 \end{bmatrix}$$

with

$$\underbrace{l_{i+j,i} = \frac{k_{i+j,i}^{(i)}}{k_{ii}^{(i)}}}$$

Gauss factors for the matrix

$$\mathbf{L}_{i-1}^{-1}\ldots\ldots\mathbf{L}_2^{-1}\mathbf{L}_1^{-1}\mathbf{K}$$

The LDL^T-solution

$$\mathbf{L}_{n-1}^{-1}\ldots\ldots\mathbf{L}_2^{-1}\mathbf{L}_1^{-1}\mathbf{K} = \mathbf{S}$$

Solve for **K**:

$$\mathbf{K} = \mathbf{L}_1\mathbf{L}_2\ldots\ldots\mathbf{L}_{n-1}\mathbf{S}$$

where

$$\mathbf{L}_i = \begin{bmatrix} 1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & 1 & & & \\ & & & & l_{i+1,i} & \cdot & & \\ & & & & l_{i+2,i} & & \cdot & \\ & & & & \ldots & & & \cdot & \\ & & & & l_{n,i} & & & & 1 \end{bmatrix}$$

Hence,

$$\mathbf{K} = \mathbf{LS}$$

with

$$\mathbf{L} = \mathbf{L}_1\mathbf{L}_2\ldots\ldots\mathbf{L}_{n-1} = \begin{bmatrix} 1 & & & & & & & & \\ l_{21} & \cdot & & & & & & & \\ l_{31} & l_{32} & \cdot & & & & & & \\ l_{41} & l_{42} & & \cdot & & & & & \\ \cdot & \cdot & & & 1 & & & & \\ \cdot & \cdot & & & & \cdot & & & \\ \cdot & \cdot & & & & & \cdot & & \\ \cdot & \cdot & & & & & & \cdot & \\ l_{n1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & l_{n,n\text{-}1} & 1 \end{bmatrix}$$

The LDL^T-solution

$$\mathbf{K} = \mathbf{LS}$$

Now, write $\mathbf{S} = \mathbf{D}\tilde{\mathbf{S}}$ where $d_{ij}=\delta_{ij}s_{ij}$, hence $\mathbf{K} = \mathbf{LD}\tilde{\mathbf{S}}$ and since $k_{ij}=k_{ji}$, $\tilde{\mathbf{S}} = \mathbf{L}^T$, so

$$\boxed{\mathbf{K} = \mathbf{LDL}^T}$$

In practice:

$$\mathbf{V} = \mathbf{L}^{-1}\mathbf{R} \qquad \Leftarrow \mathbf{LV} = \mathbf{R}$$

$$\mathbf{U} = \left(\mathbf{L}^T\right)^{-1}\mathbf{D}^{-1}\mathbf{V} \qquad \Leftarrow \mathbf{DL}^T\mathbf{U} = \mathbf{V}$$

Example: Compute $\mathbf{L}_i^{-1}$, i = 1, 2, 3 , **L**, **S**, **D** and **V** from

$$\mathbf{K} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The LDL^T-solution

Recall the Gauss multiplication factors – they enter into $\mathbf{L}_i^{-1}$, i = 1, 2, 3:

Step 1:
r2 = r2 + 4/5 r1;
r3 = r3 + (-1/5) r1;
r4 = r4;

Step 2:
r3 = r3 + 8/7 r2;
r4 = r4 + (-5/14) r2;

Step 3:
r4 = r4 + 4/3 r3;

$$\mathbf{L}_1^{-1} = \begin{bmatrix} 1 & & & \\ 4/5 & 1 & & \\ -1/5 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{L}_2^{-1} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 8/7 & 1 & \\ 0 & -5/14 & 0 & 1 \end{bmatrix} \quad \mathbf{L}_3^{-1} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 4/3 & 1 \end{bmatrix}$$

(*i.e.* the i^{th} column of $\mathbf{L}_i^{-1}$ contains the multipliers of the i^{th} step)

$$\mathbf{L} = \mathbf{L}_1\mathbf{L}_2\mathbf{L}_3 = \begin{bmatrix} 1 & & & \\ -4/5 & 1 & & \\ 1/5 & -8/7 & 1 & \\ 0 & 5/14 & -4/3 & 1 \end{bmatrix}$$

The LDL^T-solution

Recall the pivots in the Gauss elimination – they enter into S (*i.e.* reduced **K**)***:***

$$\mathbf{S}=\begin{bmatrix} 5 & -4 & 1 & 0 \\ & 14/5 & -16/5 & 1 \\ & & 15/7 & -20/7 \\ & & & 5/6 \end{bmatrix} \begin{array}{l} \longrightarrow \text{First row of } \mathbf{K} \\ \longrightarrow \text{Second row of } \mathbf{K} \text{ after step 1} \\ \longrightarrow \text{Third row of } \mathbf{K} \text{ after step 2} \\ \longrightarrow \text{Fourth row of } \mathbf{K} \text{ after step 3} \end{array}$$

For the matrix **D:** $d_{ij}=\delta_{ij}s_{ij}$

$$\mathbf{D}=\begin{bmatrix} 5 & & & \\ & 14/5 & & \\ & & 15/7 & \\ & & & 5/6 \end{bmatrix}$$

V is the right-hand side after the reduction of **K** to upper triangular form, *i.e.*

$$\mathbf{V}=\begin{bmatrix} 0 & 1 & 8/7 & 7/6 \end{bmatrix}^T$$

The LDL^T-solution

Practical issues (1)

- Simultaneous computation of **V** and $\mathbf{L}_i^{-1}$.
- **L** and **V** not computed from scratch, but by modifications of **K** and **R.**
- **K** is symmetric, banded and positive-definite; the two former properties permit the compact storage in a 1-D Array, complemented by a 1-D address array:

$$\mathbf{K} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

$\mathbf{a} = [5 \quad 6 \quad -4 \quad 6 \quad -4 \quad 1 \quad 5 \quad -4 \quad 1]$ non-zero elements of upper half and diagonal

$\mathbf{b} = [1 \quad 2 \quad 4 \quad 7]$

array indices of the diagonal elements in **a**
(here: Fortran-type indexing!)

The LDL^T-solution

Algorithm:

- Columnwise calculation of l_{ij} and d_{jj} for $j = 2,\ldots,n$, starting with $d_{11} = k_{11}$:

$$g_{m_j,j} = k_{m_j,j}$$

$$g_{ij} = k_{ij} - \sum_{r=m_m}^{i-1} l_{ri} g_{rj} \qquad i = m_j + 1,\ldots, j-1$$

where **m** = skyline of **K**, $m_m = \max\{m_i, m_j\}$

Here: l_{ij} denotes an element of $\mathbf{L}^T$

now: decomposition of **K** to the factors **D** and **L** (or: $\mathbf{L}^T$)

$$l_{ij} = \frac{g_{ij}}{d_{ii}} \qquad i = m_j + 1,\ldots, j-1$$

$$d_{jj} = k_{jj} - \sum_{r=m_j}^{j-1} l_{rj} g_{rj}$$

The LDL^T-solution

Algorithm:

- compute **U** *via* **V**: $$\mathbf{U} = \left(\mathbf{L}^T\right)^{-1}\mathbf{D}^{-1}\mathbf{V}$$

$$V_i = R_i - \sum_{r=m_i}^{i-1} l_{ri} V_r$$ Starting with $V_1 = R_1$, compute V_i for $i = 2,\dots,n$

now backsubstitution: first, compute $\overline{\mathbf{V}} = \mathbf{D}^{-1}\mathbf{V}$

Get $U_n = \overline{V}_n^{(n)}$ and

then get (successively) for U_{i-1} ($i = n,\ldots,2$)

$$\overline{V}_r^{(i-1)} = \overline{V}_r^{(i)} - l_{ri} U_i \qquad r = m_i,\ldots,i-1$$

$$U_{i-1} = \overline{V}_{i-1}^{(i-1)}$$ in the $(n - i + 1)^{\text{th}}$ evaluation, *i.e.* in the evaluation of U_{i-1}

The LDL^T-solution

Practical issues (2)

- The sums over the products $l_{rj}g_{rj}$ and $l_{ri}g_{rj}$ do not involve terms outside the skyline

  ("skyline reduction method", "column reduction method", "active column solution")

- Storage: **K** as compact 1-D array
  - l_{ij} replaces g_{ij}
  - d_{jj} replaces k_{jj}

$$\mathbf{a} = [5 \quad 6 \quad -4 \quad 6 \quad -4 \quad 1 \quad 5 \quad -4 \quad 1] \quad \mathbf{b} = [1 \quad 2 \quad 4 \quad 7]$$

$$\Longrightarrow \quad \mathbf{a} = [d_{11} \quad d_{22} \quad l_{12} \quad d_{33} \quad l_{23} \quad l_{13} \quad d_{44} \quad l_{34} \quad l_{24}]$$

  - V_i replaces R_i
  - $\overline{V}_k^{(j)}$ replaces V_k

$$\mathbf{c} = [R_1 \quad R_2 \quad R_3 \quad R_4]$$

$$\Longrightarrow \quad \mathbf{c} = [\overline{V}_1^{(j)} \quad \overline{V}_2^{(j)} \quad \overline{V}_3^{(j)} \quad \overline{V}_4^{(j)}]$$

- Effective, because of skyline reduction, but the sums over the products $l_{rj}g_{rj}$ and $l_{ri}g_{rj}$ can still involve zero terms. "Sparse solvers" skip the vanishing terms.
- Active column solutions: re-order the equations to reduce column heights
- Sparse solvers: re-order the equations to eliminate operations on elements equal to zero

The LDL^T-solution

Example:

$$\mathbf{K} = \begin{bmatrix} 2 & -2 & & & -1 \\ & 3 & -2 & & 0 \\ & & 5 & -3 & 0 \\ \textit{symm.} & & & 10 & 4 \\ & & & & 10 \end{bmatrix} \quad \rightarrow \mathbf{m} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

First, get **D** and **L**^{T}:

$d_{11} = 2$; now loop over all j, $j = 2,\ldots,5$

$j = 2$: $g_{12} = k_{12} = -2$ $\qquad g_{m_j,j} = k_{m_j,j}$

$l_{12} = g_{12}/d_{11} = -2/2 = -1$ $\qquad l_{ij} = \dfrac{g_{ij}}{d_{ii}}$ $\qquad i = m_j + 1, \ldots, \; j - 1$

$d_{22} = k_{22} - l_{12}\, g_{12} = 3 - (-1)(-2) = 1$ $\qquad d_{jj} = k_{jj} - \sum_{r=m_j}^{j-1} l_{rj} g_{rj}$

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & & & -1 \\ & 1 & -2 & & 0 \\ & & 5 & -3 & 0 \\ & & & 10 & 4 \\ & & & & 10 \end{bmatrix}$$

The LDL^T-solution

Example (cont.):

$j = 3$: $g_{23} = k_{23} = -2$

$$g_{m_j,j} = k_{m_j,j}$$

$l_{23} = g_{23}/d_{22} = -2/1 = -2$

$$l_{ij} = \frac{g_{ij}}{d_{ii}} \qquad i = m_j + 1, \dots, \; j-1$$

$d_{33} = k_{33} - l_{23}\, g_{23} = 5 - (-2)(-2) = 1$

$$d_{jj} = k_{jj} - \sum_{r=m_j}^{j-1} l_{rj} g_{rj}$$

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & & & -1 \\ & 1 & -2 & & 0 \\ & & 1 & -3 & 0 \\ & & & 10 & 4 \\ & & & & 10 \end{bmatrix}$$

$j = 4$: $g_{34} = k_{34} = -3$

$$g_{m_j,j} = k_{m_j,j}$$

$l_{34} = g_{34}/d_{33} = -3/1 = -3$

$$l_{ij} = \frac{g_{ij}}{d_{ii}} \qquad i = m_j + 1, \dots, \; j-1$$

$d_{44} = k_{44} - l_{34}\, g_{34} = 10 - (-3)(-3) = 1$

$$d_{jj} = k_{jj} - \sum_{r=m_j}^{j-1} l_{rj} g_{rj}$$

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & & & -1 \\ & 1 & -2 & & 0 \\ & & 1 & -3 & 0 \\ & & & 1 & 4 \\ & & & & 10 \end{bmatrix}$$

The LDL^T-solution

Example (cont.):

$j = 5$: $g_{15} = k_{15} = -1$

$g_{25} = k_{25} - l_{12}\, g_{15} = 0 - (-1)(-1) = -1$

$g_{35} = k_{35} - l_{23}\, g_{25} = 0 - (-2)(-1) = -2$

$g_{45} = k_{45} - l_{34}\, g_{35} = 4 - (-3)(-2) = -2$

$$g_{m_j,j} = k_{m_j,j}$$

$l_{15} = g_{15}/d_{11} = -1/2$

$l_{25} = g_{25}/d_{22} = -1/1 = -1$

$l_{35} = g_{35}/d_{33} = -2/1 = -2$

$l_{45} = g_{45}/d_{44} = -2/1 = -2$

$$l_{ij} = \frac{g_{ij}}{d_{ii}} \qquad i = m_j + 1, \ldots, j - 1$$

$$d_{jj} = k_{jj} - \sum_{r=m_j}^{j-1} l_{rj}\, g_{rj}$$

$d_{55} = k_{55} - l_{15}\, g_{15} - l_{25}\, g_{25} - l_{35}\, g_{35} - l_{45}\, g_{45} = 10 - (-1/2)(-1) - (-1)(-1) - (-2)(-2) - (-2)(-2) = 1/2$

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & & & -1/2 \\ & 1 & -2 & & -1 \\ & & 1 & -3 & -2 \\ & & & 1 & -2 \\ & & & & 1/2 \end{bmatrix}$$

The LDL^T-solution

Example (cont.):

Now, get the solution **U** of **KU**=**R** with **R**=[0 1 0 0 0]^T

$V_1 = R_1 = 0$
$V_2 = R_2 - l_{12} V_1 = 1 - 0 = 1$
$V_3 = R_3 - l_{23} V_2 = 0 - (-2)(1) = 2$
$V_4 = R_4 - l_{34} V_3 = 0 - (-3)(2) = 6$
$V_5 = R_5 - l_{15} V_1 - l_{25} V_2 - l_{35} V_3 - l_{45} V_4 = 0 - 0 - (-1)(1) - (-2)(2) - (-2)(6) = 17$

$$V_i = R_i - \sum_{r=m_i}^{i-1} l_{ri} V_r$$

forward reduction

Hence: **V**=[0 1 2 6 17]^T and $\bar{\mathbf{V}} = \mathbf{D}^{-1}\mathbf{V} = [0 \quad 1 \quad 2 \quad 6 \quad 34]^T$

backsubstitution

$\bar{\mathbf{V}}^{(5)} = \bar{\mathbf{V}} \qquad \rightarrow U_5 = \bar{V}_5 = 34$

$$\bar{V}_r^{(i-1)} = \bar{V}_r^{(i)} - l_{ri} U_i$$

$$r = m_i, \dots, i-1$$

$i = 5$

$\bar{V}_1^{(4)} = \bar{V}_1^{(5)} - l_{15} U_5 = 0 - (-1/2)(34) = 17$
$\bar{V}_2^{(4)} = \bar{V}_2^{(5)} - l_{25} U_5 = 1 - (-1)(34) = 35$
$\bar{V}_3^{(4)} = \bar{V}_3^{(5)} - l_{35} U_5 = 2 - (-2)(34) = 70$
$\bar{V}_4^{(4)} = \bar{V}_4^{(5)} - l_{45} U_5 = 6 - (-2)(34) = 74$

$U_4 = \bar{V}_4^{(4)} = 74$

$\mathbf{R} = [17 \quad 35 \quad 292 \quad 74 \quad 34]^T$

The LDL^T-solution

Example (cont.): $\overline{V}_r^{(i-1)} = \overline{V}_r^{(i)} - l_{ri} U_i$ $\quad r = m_i, \ldots, i-1$

$i = 4$

$$\overline{V}_3^{(3)} = \overline{V}_3^{(4)} - l_{34} U_4 = 70 - (-3)(74) = 292$$

$$U_3 = \overline{V}_3^{(3)} = 292$$

$$\mathbf{R} = [17 \quad 35 \quad 292 \quad 74 \quad 34]^T$$

$i = 3$

$$\overline{V}_2^{(2)} = \overline{V}_2^{(3)} - l_{23} U_3 = 35 - (-2)(292) = 619$$

$$U_2 = \overline{V}_2^{(2)} = 619$$

$$\mathbf{R} = [17 \quad 619 \quad 292 \quad 74 \quad 34]^T$$

$i = 2$

$$\overline{V}_1^{(1)} = \overline{V}_1^{(2)} - l_{12} U_2 = 17 - (-1)(619) = 636$$

$$U_1 = \overline{V}_1^{(1)} = 636$$

$$\mathbf{R} = \mathbf{U} = [636 \quad 619 \quad 292 \quad 74 \quad 34]^T$$

Properties of K

- for $m_K = i - m_i$, for all $i > m_K$: $\mathbf{LDL}^T$ decomposition requires ½ $n m_K^2$ operations, reduction and backsubstitution requires $2 n m_K$ operations; in general: ½ $\Sigma(i - m_i)^2 + 2\,\Sigma(i - m_i)$ operations

- stiffness matrix of an element with suppressed rigid-body modes is positive-definite

- if $\mathbf{K}$ is positive-definite, then the matrix $\mathbf{K}^{(r)}$ (matrix of the r^{th} associated constraint problem) is also positive-definite (Sturm sequence property) and all $d_{ii} > 0$

$$p(\lambda) = \det(\mathbf{K} - \lambda \mathbf{I}) \Rightarrow \det \mathbf{K} = \det \mathbf{L} \det \mathbf{D} \det \mathbf{L}^T = \prod_{i=1}^{n} d_{ii} > 0$$

$$\mathbf{K}^{(i)} = \mathbf{L}^{(i)} \mathbf{D}^{(i)} \mathbf{L}^{(i)T} \qquad i = 1, \ldots, n-1$$

$$\lambda_1^{(i)} > 0 \qquad i = 1, \ldots, n-1$$

$$d_{ii} > 0 \qquad i = 1, \ldots, n$$

- if $\mathbf{K}$ is positive-semidefinite, $d_{kk} = 0$ if $\lambda_1^{(n-k)} = 0$

- if $d_{kk} = 0$ row interchanges can establish $d_{kk} \neq 0$ unless $k = n - m_\lambda + 1$

m_λ : multiplicity of λ=0

Error estimate

Due to truncation and roundoff errors, we solve (**K** + δ**K**) (**U**+ δ**U**) **= R** rather than **KU = R**.

$$\delta\mathbf{U} = -\mathbf{K}^{-1}\delta\mathbf{K}\mathbf{U}$$

A large condition number implies high probability of erroneous solutions.

$$\frac{\|\delta\mathbf{U}\|}{\|\mathbf{U}\|} \le \mathrm{cond}(\mathbf{K})\frac{\|\delta\mathbf{K}\|}{\|\mathbf{K}\|} \qquad \text{with} \qquad \mathrm{cond}(\mathbf{K}) = \frac{\lambda_n}{\lambda_1}$$

In practice: Approx., based on upper bound ||**K**|| and lower bound from inverse iteration $\mathrm{cond}(\mathbf{K}) \approx \frac{\|\mathbf{K}\|}{\lambda_1^{inv.it}}$

For t-bit double precision: $\frac{\|\delta\mathbf{K}\|}{\|\mathbf{K}\|} = 10^{-t}$

The number precision in the solution **U**, s, with $\frac{\|\delta\mathbf{U}\|}{\|\mathbf{U}\|} = 10^{-s}$ is thus $s \ge t - \log_{10}(\mathrm{cond}(\mathbf{K}))$

cond(**K**)	s (t=16)
10.00	15
100.00	14
1000.00	13
10000.00	12

Error estimate

- Bathe, page 749: Summary on truncation and roundoff errors in solving **KU = R**

1. Both types of errors can be expected to be large if structures with widely varying stiffness are analyzed. Large stiffness differences may be due to different material moduli, or they may be the result of the finite element modeling used, in which case a more effective model can frequently be chosen. This may be achieved by the use of finite elements that are nearly equal in size and have almost the same lengths in each dimension, the use of master-slave degrees of freedom, i.e., constraint equations (see Section 4.2.2 and Example 8.19), and relative degrees of freedom (see Example 8.20).
2. Since truncation errors are most significant, to improve the solution accuracy it is necessary to evaluate both the stiffness matrix **K** and the solution of **KU** = **R** in double precision. It is not sufficient (a) to evaluate **K** in single precision and then solve the equations in double precision (see Example 8.18), or (b) to evaluate **K** in single precision, solve the equations in single precision using a Gauss elimination procedure, and then iterate for an improvement in the solution employing, for example, the Gauss-Seidel method.

Related methods

- Cholesky factorization
- Static condensation
- Substructure analysis
- Frontal solution

References

Bathe, K.J., Finite Element Procedures, Prentice Hall, 1996:
Chapter 8