11th Lecture: Methods of Structural Reliability Analysis

The aim of the present lecture is to introduce the most common techniques of structural reliability analysis, namely, First Order Reliability Methods (FORM) and Monte-Carlo simulation. First the concept of limit state equations and basic random variables is introduced. Thereafter the problem of error propagation is considered and it is shown that FORM provides a generalization of the classical solution to this problem. Different cases of limit state functions and probabilistic characteristics of basic random variables are then introduced with increasing generality. Furthermore, FORM results are related to partial safety factors used in common design codes. Subsequently, crude Monte-Carlo and Importance sampling is introduced as an alternative to FORM methods. The introduced methods of structural reliability theory provide strong tools for the calculation of failure probabilities for individual failure modes or components. On the basis of the present lecture, it is expected that the students should acquire knowledge and skills in regard to:

- What is a basic random variable and what is a limit state function?
- What is the graphical interpretation of the reliability index?
- What is the principle for the linearization of non-linear limit state functions?
- How to transform non-normal distributed random variables into normal distributed variables?
- How to consider dependent random variables?
- How are FORM results related to partial safety factors?
- What is the principle of Monte-Carlo simulation methods?
- Why is importance sampling effective and what does it require in terms of information additional to crude Monte-Carlo methods?

11.1 Introduction

The first developments of First Order Reliability Methods, also known as FORM methods took place almost 30 years ago. Since then the methods have been refined and extended significantly and by now they form one of the most important methods for reliability evaluations in structural reliability theory. Several commercial computer codes have been developed for FORM analysis and the methods are widely used in practical engineering problems and for code calibration purposes.

In the present chapter first the basic idea behind FORM methods are highlighted and thereafter the individual steps of the methods are explained in detail.

Thereafter the relationship between the results of FORM analysis and partial safety factors for design codes will be explained. Finally the basic concepts of Monte Carlo methods, in structural reliability will be outlined.

11.2 Failure Events and Basic Random Variables

In reliability analysis of technical systems and components the main problem is to evaluate the probability of failure corresponding to a specified reference period. However, also other non-failure states of the considered component or system may be of interest, such as excessive damage, unavailability, etc.

In general any state, which may be associated with consequences in terms of costs, loss of lives and impact to the environment are of interest. In the following we will not differentiate between these different types of states but for simplicity refer to all these as being failure events, however, bearing in mind that also non-failure states may be considered in the same manner.

It is convenient to describe failure events in terms of functional relations, which if they are fulfilled define that the considered event will occur. A failure event may be described by a functional relation, the limit state function $g(\mathbf{x})$ in the following way:

$$\mathbf{F} = \{g(\mathbf{x}) \leq 0\} \quad (11.1)$$

where the components of the vector $\mathbf{x}$ are realisations of the so-called basic random variables $\mathbf{X}$ representing all the relevant uncertainties influencing the probability of failure. In Equation (11.1) the failure event $\mathbf{F}$ is simply defined as the set of realisation of the function $g(\mathbf{x})$, which is zero or negative.

As already mentioned other events than failure may be of interest in reliability analysis and e.g. in reliability updating problems also events of the following form are highly relevant:

$$\mathbf{I} = \{h(\mathbf{x}) = 0\} \quad (11.2)$$

Having defined the failure event the probability of failure may be determined by the following integral:

$$P_f = \int_{g(\mathbf{x})\leq 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{11.3}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of the random variables $\mathbf{X}$. This integral is, however, non-trivial to solve and numerical approximations are expedient. Various methods for the solution of the integral in Equation (11.3) have been proposed including numerical integration techniques, Monte Carlo simulation and asymptotic Laplace expansions. Numerical integration techniques very rapidly become inefficient for increasing dimension of the vector $\mathbf{X}$ and are in general irrelevant. In the following we will direct the focus on the widely applied and quite efficient FORM methods, which furthermore can be shown to be consistent with the solutions obtained by asymptotic Laplace integral expansions.

11.3 Linear Limit State Functions and Normal Distributed Variables

For illustrative purposes we will first consider the case where the limit state function $g(\mathbf{x})$ is a linear function of the basic random variables X. Then we may write the limit state function as:

$$g(x) = a_0 + \sum_{i=1}^{n} a_i x_i \tag{11.4}$$

If the basic random variables are normally distributed we furthermore have that the linear safety margin M defined through:

$$M = a_0 + \sum_{i=1}^{n} a_i X_i \tag{11.5}$$

is also normally distributed with mean value and variance

$$\mu_M = a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}$$

$$\sigma_M^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \rho_{ij} a_i a_j \sigma_i \sigma_j \tag{11.6}$$

where ρ_{ij} are the correlation coefficients between the variables X_i and X_j.

Defining the failure event by Equation (11.1) we can write the probability of failure as:

$$P_F = P(g(\mathbf{X}) \leq 0) = P(M \leq 0) \tag{11.7}$$

which in this simple case reduces to the evaluation of the standard normal distribution function:

$$P_F = \Phi(-\beta) \tag{11.8}$$

where β the so-called reliability index (due to Cornell (1969) and Basler (1961)) is given as:

$$\beta = \frac{\mu_M}{\sigma_M} \tag{11.9}$$

The reliability index β as defined in Equation (11.9) has a geometrical interpretation as illustrated in Figure 11.1 where a two dimensional case is considered:

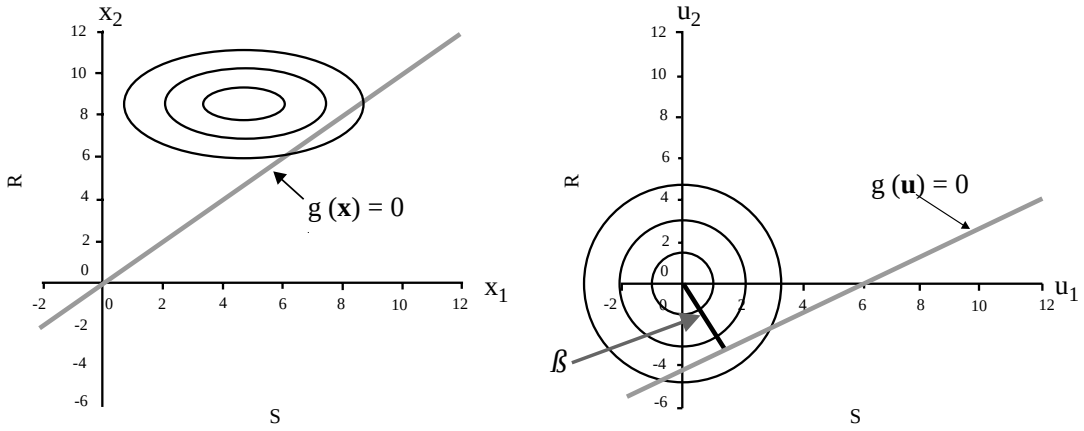

Figure 11.1 Illustration of the two-dimensional case of a linear limit state function and standardised normally distributed variables U .

In Figure 1 the limit state function $g(\mathbf{x})$ has been transformed into the limit state function $g(\mathbf{u})$ by normalisation of the random variables in to standardized normally distributed random variables as:

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \tag{11.10}$$

such that the random variables U_i have zero means and unit standard deviations.

Then the reliability index β has the simple geometrical interpretation as the smallest distance from the line (or generally the hyper-plane) forming the boundary between the safe domain and the failure domain, i.e. the domain defined by the failure event. It should be noted that this definition of the reliability index (due to Hasofer and Lind (1974)) does not depend on the limit state function but rather the boundary between the safe domain and the failure domain. The point on the failure surface with the smallest distance to origin is commonly denoted the design point or most likely failure point.

It is seen that the evaluation of the probability of failure in this simple case reduces to some simple evaluations in terms of mean values and standard deviations of the basic random variables, i.e. the first and second order information.

11.4 The Error Accumulation Law

The results given in Equation (11.6) have been applied to study the statistical characteristics of errors ε accumulating in accordance with some differentiable function $f(\mathbf{x})$, i.e.:

$$\varepsilon = f(\mathbf{x}) \tag{11.11}$$

where $\mathbf{x} = (x_1, x_2, .., x_n)^T$ is a vector of realizations of the random variables $\mathbf{X}$ representing measurement uncertainties with mean values $\boldsymbol{\mu}_\mathbf{X} = (\mu_{X_1}, \mu_{X_2}, .., \mu_{X_n})^T$ and covariances $Cov[X_i, X_j] = \rho_{ij}\sigma_{X_i}\sigma_{X_j}$ where σ_{X_i} are the standard deviations and ρ_{ij} the correlation coefficients. The idea is to approximate the function $f(\mathbf{x})$ by its Taylor expansion including only the linear terms, i.e.:

$$\varepsilon \cong f(\mathbf{x}_0) + \sum_{i=1}^{n} (x_i - x_{i,0}) \left. \frac{\partial f(\mathbf{x})}{\partial x_i} \right|_{\mathbf{x}=\mathbf{x}_0} \tag{11.12}$$

where $\mathbf{x}_0 = (x_{1,0}, x_{2,0}, .., x_{n,0})^T$ is the point in which the linearization is performed normally chosen as the mean value point and $\left. \frac{\partial f(\mathbf{x})}{\partial x_i} \right|_{\mathbf{x}=\mathbf{x}_0}, i = 1,2,..n$ are the first order partial derivatives of $f(\mathbf{x})$ taken in $\mathbf{x} = \mathbf{x}_0$.

From Equation (11.12) and Equation (11.6) it is seen that the expected value of the error $E[\varepsilon]$ can be assessed by:

$$E[\varepsilon] = f(\boldsymbol{\mu}_\mathbf{X}) \tag{11.13}$$

and its variance $Var[\varepsilon]$ can be determined by:

$$Var[\varepsilon] = \sum_{i=1}^{n} \left(\left. \frac{\partial f(\mathbf{x})}{\partial x_i} \right|_{\mathbf{x}=\mathbf{x}_0} \right)^2 \sigma_{X_i}^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left(\left. \frac{\partial f(\mathbf{x})}{\partial x_i} \right|_{\mathbf{x}=\mathbf{x}_0} \right) \left(\left. \frac{\partial f(\mathbf{x})}{\partial x_j} \right|_{\mathbf{x}=\mathbf{x}_0} \right) \rho_{ij}\sigma_{X_i}\sigma_{X_j} \tag{11.14}$$

Provided that the distribution functions for the random variables are known, e.g. normal distributed the probability distribution function of the error is easily assessed. It is, however, important to notice that the variance of the error as given by Equation (11.14) depends on the linearization point, i.e. $\mathbf{x}_0 = (x_{1,0}, x_{2,0}, .., x_{n,0})^T$.

The results given in Equation (11.6) have been applied to study the statistical characteristics of errors ε accumulating in accordance with some differentiable function $f(\mathbf{x})$, i.e.:

$$\varepsilon = f(\mathbf{x}) \tag{11.15}$$

where $\mathbf{x} = (x_1, x_2, .., x_n)^T$ is a vector of realizations of the random variables $\mathbf{X}$ representing measurement uncertainties with mean values $\boldsymbol{\mu}_\mathbf{X} = (\mu_{X_1}, \mu_{X_2}, .., \mu_{X_n})^T$ and covariances $Cov[X_i, X_j] = \rho_{ij}\sigma_{X_i}\sigma_{X_j}$ where σ_{X_i} are the standard deviations and ρ_{ij} the correlation coefficients. The idea is to approximate the function $f(\mathbf{x})$ by its Taylor expansion including only the linear terms, i.e.:

$$\varepsilon \cong f(\mathbf{x}_0) + \sum_{i=1}^{n} (x_i - x_{i,0}) \left. \frac{\partial f(\mathbf{x})}{\partial x_i} \right|_{\mathbf{x}=\mathbf{x}_0} \tag{11.16}$$

where $\mathbf{x}_0 = (x_{1,0}, x_{2,0}, .., x_{n,0})^T$ is the point in which the linearization is performed normally chosen as the mean value point and $\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_0}, i = 1,2,..n$ are the first order partial derivatives of $f(\mathbf{x})$ taken in $\mathbf{x} = \mathbf{x}_0$.

From Equation (11.12) and Equation (11.6) it is seen that the expected value of the error $E[\varepsilon]$ can be assessed by:

$$E[\varepsilon] = f(\boldsymbol{\mu}_{\mathbf{X}}) \tag{11.17}$$

and its variance $Var[\varepsilon]$ can be determined by:

$$Var[\varepsilon] = \sum_{i=1}^{n}\left(\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_0}\right)^2 \sigma_{X_i}^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\left(\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_0}\right)\left(\left.\frac{\partial f(\mathbf{x})}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_0}\right)\rho_{ij}\sigma_{X_i}\sigma_{X_j} \tag{11.18}$$

Provided that the distribution functions for the random variables are known, e.g. normal distributed the probability distribution function of the error is easily assessed. It is, however, important to notice that the variance of the error as given by Equation (11.14) depends on the linearization point, i.e. $\mathbf{x}_0 = (x_{1,0}, x_{2,0}, .., x_{n,0})^T$.

Example 1 – Linear Safety Margin

Consider a steel rod under pure tension loading. The rod will fail if the applied stresses on the rod cross-sectional area exceed the steel yield stress. The yield stress R of the rod and the loading stress on the rod S are assumed to be uncertain modelled by uncorrelated normal distributed variables. The mean values and the standard deviations of the yield strength and the loading are given as $\mu_R = 350, \sigma_R = 35$ MPa and $\mu_S = 200, \sigma_S = 40$ MPa respectively.

The limit state function describing the event of failure may be written as:

$$g(\mathbf{x}) = r - s$$

whereby the safety margin M may be written as:

$$M = R - S$$

The mean value and standard deviation of the safety margin M are thus:

$$\mu_M = 350 - 200 = 150$$

$$\sigma_M = \sqrt{35^2 + 40^2} = 53.15$$

whereby we may calculate the reliability index as:

$$\beta = \frac{150}{53.15} = 2.84$$

Finally we have that the failure probability is determined as:

$$P_F = \Phi(-2.84) = 2.4 \cdot 10^{-3}$$

Example 2 – Error Accumulation Law

As an example of the use of the error propagation law consider a right angle triangle *ABC*, where *B* is the right angle. The lengths of the opposite side *b* and adjacent side *a* are measured. Due to measurement uncertainty the length of the sides *a* and *b* are modelled as independent normal distributed random variables with expected values μ_a = 12.2, μ_b = 5.1 and standard deviations σ_a = 0.4 and σ_b = 0.3, respectively. It is assumed that a critical condition will occur if the hypotenuse *c* is larger than 13.5 and the probability that this condition should happen is to be assessed.

Based on the probabilistic model of *a* and *b* the statistical characteristics of the hypotenuse *c* given by:

$$c = \sqrt{a^2 + b^2}$$

may be assessed through the error propagation model given by Equations (11.13)-(11.14), yielding:

$$E[c] = \sqrt{\mu_a^2 + \mu_b^2}$$

$$Var[c] = \sum_{i=1}^{n} \left(\left. \frac{\partial f(\mathbf{x})}{\partial x_i} \right|_{\mathbf{x}=\mathbf{x}_0} \right)^2 \sigma_{X_i}^2 = \frac{a}{\sqrt{a^2 + b^2}} \sigma_a^2 + \frac{b}{\sqrt{a^2 + b^2}} \sigma_b^2$$

which by inserting for *a* and *b* their expected values yield:

$$E[c] = \sqrt{12.2^2 + 5.1^2} = 13.22$$

$$Var[c] = \frac{12.2}{\sqrt{12.2^2 + 5.1^2}} \sigma_a^2 + \frac{5.1}{\sqrt{12.2^2 + 5.1^2}} \sigma_b^2 = 0.182$$

As seen from the above the variance of the hypotenuse *c* depends on the chosen linearization point. If instead of the mean value point a value corresponding to the mean value plus two standard deviations was chosen the variance of *c* would have been:

$$Var[c] = \frac{13}{\sqrt{13^2 + 5.7^2}} 0.4^2 + \frac{5.7}{\sqrt{13^2 + 5.7^2}} 0.3^2 = 0.149$$

which can be shown to imply a 5.55% reduction of the probability that the hypotenuse is larger than 13.5. Even though such a change seems small it could be of importance in a practical importance situation where the consequences of errors can be significant.

11.5 Non-linear Limit State Functions

When the limit state function is non-linear in the basic random variables **X** the situation is not as simple as outlined in the previous. An obvious approach is, however, considering the error propagation law explained in the foregoing to represent the failure domain in terms of a linearization of the boundary between the safe domain and the failure domain, i.e. the failure surface, but the question remain how to do this appropriately.

Hasofer and Lind (1974)suggested performing this linearization in the design point of the failure surface represented in normalised space. The situation is illustrated in the two dimensional space in Figure 11.2.

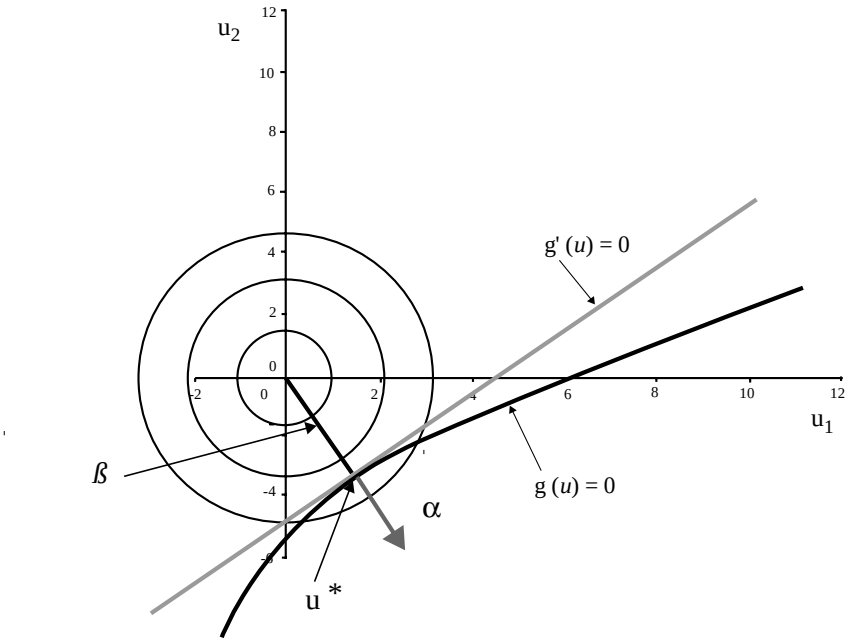

Figure 11.2 Illustration of the linearization proposed by Hasofer and Lind (1974) in standard normal space.

In Figure 11.2 a principal sketch is given illustrating that the failure surface is linearized in the design point $\mathbf{u}^*$ by the line $g'(\mathbf{u}) = 0$. The α-vector is the out ward directed normal vector to the failure surface in the design point $\mathbf{u}^*$ i.e. the point on the linearized failure surface with the shortest distance - β - to the origin.

As the limit state function is in general non-linear one does not know the design point in advance and this has to be found iteratively e.g. by solving the following optimisation problem:

$$\beta = \min_{\mathrm{u}\in\{g(\mathrm{u})=0\}} \sqrt{\sum_{i=1}^{n} u_i^2} \tag{11.19}$$

This problem may be solved in a number of different ways. Provided that the limit state function is differentiable the following simple iteration scheme may be followed

$$\alpha_i = \frac{-\frac{\partial g}{\partial u_i}(\beta \cdot \alpha)}{\left[\sum_{i=1}^{n}\left(\frac{\partial g}{\partial u_i}(\beta \cdot \alpha)\right)^2\right]^{1/2}}, \quad i = 1,2,..n \tag{11.20}$$

$$g(\beta \cdot \alpha_1, \beta \cdot \alpha_2, ... \beta \cdot \alpha_n) = 0 \tag{11.21}$$

First a design point is guessed $\mathbf{u}^* = \beta\ \boldsymbol{\alpha}$ and inserted into Equation (11.20) whereby a new normal vector $\boldsymbol{\alpha}$ to the failure surface is achieved. Then this $\boldsymbol{\alpha}$-vector is inserted into Equation (11.8) from which a new β-value is calculated.

The iteration scheme will converge in a few, say normally 6-10 iterations and provides the design point $\mathbf{u}^*$ as well as the reliability index β and the outward normal to the failure surface in the design point $\boldsymbol{\alpha}$. As already mentioned the reliability index β may be related directly to the probability of failure. The components of the $\boldsymbol{\alpha}$-vector may be interpreted as sensitivity factors giving the relative importance of the individual random variables for the reliability index β.

Second Order Reliability Methods (SORM) follow the same principles as FORM, however, as a logical extension of FORM the failure surface is expanded to the second order in the design point. The result of a SORM analysis may be given as the FORM β multiplied with a correction factor evaluated on the basis of the second order partial derivatives of the failure surface in the design point. Obviously the SORM analysis becomes exact for failure surfaces, which may be given as second order polynomial of the basic random variables. However, in general the result of a SORM analysis can be shown to be asymptotically exact for any shape of the failure surface as β approaches infinity. The interested reader is referred to the literature for the details of SORM analyses; see e.g. Madsen et al. (1986).

Example 3 – Non-linear Safety Margin

Consider again the steel rod from the previous example. However, now it is assumed that the cross sectional areas of the steel rod A is also uncertain.

The steel yield stress R is normal distributed with mean values and standard deviation $\mu_R = 350, \sigma_R = 35$ MPa and the loading S is normal distributed with mean value and standard deviation $\mu_S = 1500, \sigma_S = 300$ N. Finally the cross sectional area A is assumed normally distributed with mean value and standard deviation $\mu_A = 10, \sigma_A = 1\ mm^2$.

The limit state function may be written as:

$$g(\mathbf{x}) = r \cdot a - s$$

Now the first step is to transform the normally distributed random variables R, A and S into standardized normally distributed random variables, i.e.:

$$U_R = \frac{R - \mu_R}{\sigma_R}$$

$$U_A = \frac{A - \mu_A}{\sigma_A}$$

$$U_S = \frac{S - \mu_S}{\sigma_S}$$

The limit state function may now be written in the space of the standardized normally distributed random variables as:

$$g(u) = (u_R\sigma_R + \mu_R)(u_A\sigma_A + \mu_A) - (u_S\sigma_S + \mu_S)$$

$$= (35u_R + 350)(u_A + 10) - (300u_S + 1500)$$
$$= 350\mathrm{u}_\mathrm{R} + 350u_A - 300u_S + 35u_Ru_A + 2000$$

The reliability index and the design point may be determined in accordance with Equation (11.11) as:

$$\beta = \frac{-2000}{350\alpha_R + 350\alpha_A - 300\alpha_S + 35\beta\alpha_R\alpha_A}$$

$$\alpha_R = -\frac{1}{k}(350 + 35\beta\alpha_A)$$

$$\alpha_A = -\frac{1}{k}(350 + 35\beta\alpha_R)$$

$$\alpha_S = \frac{300}{k}$$

with

$$k = \sqrt{(350 + 35\beta\alpha_A)^2 + (350 + 35\beta\alpha_R)^2 + (300)^2}$$

which by calculation gives the iteration history shown in Table 11.1.

Iteration	Start	1	2	3	4	5
β	3.0000	3.6719	3.7399	3.7444	3.7448	3.7448
α_R	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_A	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_S	0.5800	0.5916	0.6084	0.6086	0.6087	0.6087

Table 11.1. Iteration history for the non-linear limit state example.

From Table 11.1 it is seen that the basic random variable *S* modelling the load on the steel rod is slightly dominating with an α-value equal to 0.6087. Furthermore it is seen that both the variables *R* and *A* are acting as resistance variables as their α-values are negative. The failure probability for the steel rod is determined as $P_F = \Phi(-3.7448) = 9.02 \cdot 10^{-5}$.

11.6 Correlated and Dependent Random Variables

The situation where basic random variables **X** are stochastically dependent is often encountered in practical problems. For normally distributed random variables we remember that the joint probability distribution function may be described in terms of the first two moments, i.e. the mean value vector and the covariance matrix. This is, however, only the case for normally or log-normally distributed random variables.

Considering in the following the case of normally distributed random variables these situations may be treated completely along the same lines as described in the foregoing. However, provided that we in addition to the transformation by which we go from a limit state

function expressed in $\mathbf{X}$ variables to a limit state function expressed in $\mathbf{U}$ variables, introduce a transformation in between where we obtain that the considered random variables first are standardized before they are made uncorrelated. I.e. the row of transformations yields:

$$\mathbf{X} \rightarrow \mathbf{Y} \rightarrow \mathbf{U}$$

In the following we will see how this transformation may be implemented in the iterative procedure outlined previously.

Let us assume that the basic random variables $\mathbf{X}$ are correlated with covariance matrix given as:

$$\mathbf{C}_{\mathbf{X}} = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] \ldots & Cov[X_1, X_n] \\ \vdots & \vdots & \vdots \\ Cov[X_n, X_1] & \cdots & Var[X_n] \end{bmatrix} \tag{11.22}$$

and correlation coefficient matrix $\boldsymbol{\rho}_{\mathbf{X}}$:

$$\boldsymbol{\rho}_{\mathbf{X}} = \begin{bmatrix} 1 & \cdots & \rho_{1n} \\ \vdots & 1 & \vdots \\ \rho_{n1} & \cdots & 1 \end{bmatrix} \tag{11.23}$$

If only the diagonal elements of these matrixes are non-zero clearly the basic random variables are uncorrelated.

As before the first step is to transform the *n*-vector of basic random variables $\mathbf{X}$ into a vector of standardised random variables $\mathbf{Y}$ with zero mean values and unit variances. This operation may be performed by

$$Y_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}, i = 1,2,..n \tag{11.24}$$

whereby the covariance matrix of $\mathbf{Y}$, i.e. $\mathbf{C}_{\mathbf{Y}}$ is equal to the correlation coefficient matrix of $\mathbf{X}$, i.e. $\boldsymbol{\rho}_{\mathbf{X}}$.

The second step is to transform the vector of standardized basic random variables $\mathbf{Y}$, into a vector of uncorrelated basic random variables U. This last transformation may be performed in several ways. The approach described in the following utilises the Choleski factorisation from matrix algebra and is efficient for both hand calculations and for implementation in computer programs.

The desired transformation may be written as

$$Y = TU \tag{11.25}$$

where $\boldsymbol{T}$ is a lower triangular matrix such that $T_{ij} = 0$ for $j > i$. It is then seen that the covariance matrix $\mathbf{C}_{\mathbf{Y}}$ can be written as:

$$\mathbf{C}_{\mathbf{Y}} = E\left[Y \cdot Y^T\right] = E\left[T \cdot U \cdot U^T \cdot T^T\right] = T \cdot E\left[U \cdot U^T\right] \cdot T^T = T \cdot T^T = \boldsymbol{\rho}_X \tag{11.26}$$

from which it is seen that the components of T may be determined as:

$$
\begin{aligned}
T_{11} &= 1 \\
T_{21} &= \rho_{12} \\
T_{31} &= \rho_{13} \\
T_{22} &= \sqrt{1 - T_{21}^2} \\
T_{32} &= \frac{\rho_{23} - T_{31} \cdot T_{21}}{T_{22}} \\
T_{33} &= \sqrt{1 - T_{31}^2 - T_{32}^2} \\
&\vdots
\end{aligned}
\tag{11.27}
$$

Considering the example from before but now with the additional information that the random variables *A* and *R* are correlated with correlation coefficient matrix:

$$
\rho_X = \begin{bmatrix} 1 & \rho_{AR} & \rho_{AS} \\ \rho_{RA} & 1 & \rho_{RS} \\ \rho_{SA} & \rho_{SR} & 1 \end{bmatrix} \tag{11.28}
$$

with $\rho_{AR} = \rho_{RA} = 0.1$ and all other correlation coefficients equal to zero we can calculate the transformation matrix *T* as:

$$
T = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.995 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11.29}
$$

The components of the vector ***Y*** may then be calculated as:

$$
\begin{aligned}
Y_A &= U_A \\
Y_R &= 0.1 \cdot U_A + 0.995 \cdot U_R \\
Y_S &= U_S
\end{aligned}
\tag{11.30}
$$

and finally the components of the vector ***X*** are determined as:

$$
\begin{aligned}
X_A &= U_A \cdot \sigma_A + \mu_A \\
X_R &= (0.1 \cdot U_A + 0.995 \cdot U_R) \cdot \sigma_R + \mu_R \\
X_S &= U_S \cdot \sigma_S + \mu_S
\end{aligned}
\tag{11.31}
$$

whereby we can write the limit state function in terms of the uncorrelated and normalised random variables ***U*** as follows:

$$
g(u) = ((0.1 \cdot u_A + 0.995 \cdot u_R) \cdot \sigma_R + \mu_R)(u_A \sigma_A + \mu_A) - (u_S \sigma_S + \mu_S) \tag{11.32}
$$

from which the reliability index can be calculated as in the previous example.

In case the stochastically dependent basic random variables are not normally or log-normally distributed the dependency can no longer be described completely in terms of correlation coefficients and the above-described transformation is thus not appropriate. In such cases other transformations must be applied as described in the next section.

11.7 Non-Normal and Dependent Random Variables

As stated in the previous the joint probability distribution function of a random vector $\boldsymbol{X}$ can only be completely described in terms of the marginal probability distribution functions for the individual components of the vector $\boldsymbol{X}$ and the correlation coefficient matrix when all the components of $\boldsymbol{X}$ are either normally or log-normally distributed.

I the following we shall first consider the simple case where the components of $\boldsymbol{X}$ are independent but non-normally distributed. Thereafter we shall see how in some cases the situation of jointly dependent and non-normally distributed random variables may be treated.

The Normal-tail Approximation

One approach to consider the problem of non-normally distributed random variables within the context of the iterative scheme given in Equations (11.20)-(11.21) for the calculation of the reliability index β is to approximate the real probability distribution by a normal probability distribution in the design point.

As the design point is usually located in the tails of the distribution functions of the basic random variables the scheme is often referred to as the "normal tail approximation".

Denoting by $\mathbf{x}^*$ the design point the approximation is introduced by:

$$F_{X_{ii}}(x_i^*) = \Phi(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}) \tag{11.33}$$

$$f_{X_{ii}}(x_i^*) = \frac{1}{\sigma_{X_i}}\varphi(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}}) \tag{11.34}$$

where μ'_{X_i} and σ_{X_i} are the unknown mean value and standard deviation of the approximating normal distribution.

Solving Equation (11.33) and (11.34) with respect to μ'_{X_i} and σ_{X_i} we obtain:

$$\sigma'_{X_i} = \frac{\varphi(\Phi^{-1}(F_{X_i}(x_i^*)))}{f_{X_i}(x_i^*)}$$

$$\mu'_{X_i} = x_i^* - \Phi^{-1}(F_{X_i}(x_i^*))\sigma'_{X_i} \tag{11.35}$$

This transformation may easily be introduced in the iterative evaluation of the reliability index β as a final step before the basic random variables are normalised.

The Rosenblatt Transformation

If the joint probability distribution function of the random vector $\boldsymbol{X}$ can be obtained in terms of a sequence of conditional probability distribution functions e.g.:

$$F_X(x) = F_{X_n}(x_n|x_1, x_2, \ldots x_{n-1}) \cdot F_{X_{n-1}}(x_{n-1}|x_1, x_2, \ldots x_{n-2}) \ldots F_{X_1}(x_1) \tag{11.36}$$

the transformation from the ***X***-space to the ***U***-space may be performed using the so-called Rosenblatt transformation:

$$\begin{aligned} &\Phi(u_1) = F_{X_1}(x_1) \\ &\Phi(u_2) = F_{X_2}(x_2|x_1) \\ &\vdots \\ &\Phi(u_n) = F_{X_n}(x_n|x_1, x_2, \ldots x_{n-1}) \end{aligned} \tag{11.37}$$

where n is the number of random variables, $F_{Xi}(x_i|x_1, x_2, \ldots, x_{i-1})$ is the conditional probability distribution function for the i'th random variable given the realisations of $x_1, x_2, \ldots x_{i-1}$ and $\Phi(\cdot)$ is the standard normal probability distribution function. From the transformation given by Equation (11.37) the basic random variables ***X*** may be expressed in terms of standardised normal distributed random variables ***U*** by

$$\begin{aligned} &x_1 = F_{X_1}^{-1}(\Phi(u_1)) \\ &x_2 = F_{X_2}^{-1}(\Phi(u_2|x_1) \\ &\vdots \\ &x_n = F_{X_n}^{-1}(\Phi(u_n|x_1, x_2, \ldots x_{n-1}) \end{aligned} \tag{11.38}$$

In some cases the Rosenblatt transformation cannot be applied because the required conditional probability distribution functions cannot be provided. In such cases other transformations may be useful such as e.g. the Nataf transformation se e.g. Madsen et al. (1986). Standard commercial software for FORM analysis usually include a selection of possibilities for the representation of dependent non-normally distributed random variables.

11.7 Software for Reliability Analysis

Several software packages are available for FORM analysis following the principles out lined in the forgoing sections. Most of the programs are more or less self-explanatory provided that the basic principles of FORM analysis are known.

The reader is referred to software packages such as STRUREL and VaP for which more information is available on the web.

7 Assessment of Partial Safety Factors by FORM Analysis

In code based design formats such as the Eurocodes and the Swisscodes, design equations are prescribed for the verification of the capacity of different types of structural components in regard to different modes of failure. The typical format for the verification of a structural component is given as design equations such as:

$$zR_c / \gamma_m - \left(\gamma_G G_C + \gamma_Q Q_C\right) = 0 \tag{11.39}$$

where:

R_C	is the characteristic value for the resistance
z	is a design variable (e.g. the cross sectional area of the steel rod considered previously)
G_C	is a characteristic value for the permanent load
Q_C	is a characteristic value for the variable load
γ_m	is the partial safety factor for the resistance
γ_G	is the partial safety factor for the permanent load
γ_Q	is the partial safety factor for the variable load

In the codes different partial safety factors are specified for different materials and for different types of loads. Furthermore when more than one variable load is acting load combination factors are multiplied on one ore more of the variable load components to take into account the fact that it is unlikely that all variable loads are acting with extreme values at the same time.

The partial safety factors together with the characteristic values are introduced in order to ensure a certain minimum reliability level for the structural components designed according to the code. As different materials have different uncertainties associated with their material parameters the partial safety factors are in general different for the different materials. The principle is illustrated in Figure 11.3 for the simple *r-s* case.

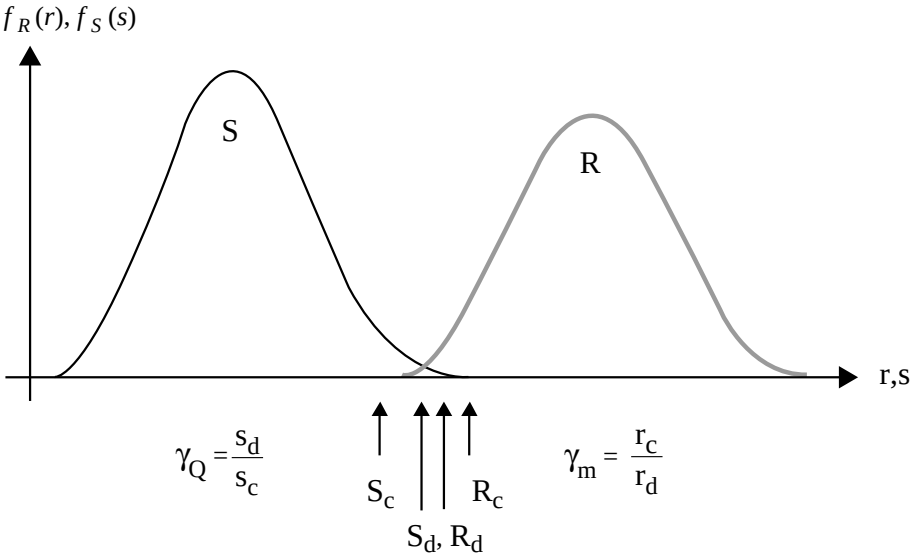

Figure 11.3 Illustration of the relation between design values, characteristic values and partial safety factors.

In accordance with a given design equation such as e.g. Equation (11.39) a reliability analysis may be made with a limit state function of the same form as the design equation but where the characteristic values for the resistance and load variables are now replaced by basic random variables, i.e.:

$$zR-(G+Q)=0 \tag{11.40}$$

For given probabilistic models for the basis random variables *R*, *G* and *Q* and with a given requirement to the maximum allowable failure probability it is now possible to determine the value of the design variable *z* which corresponds to this failure probability. Such a design could be interpreted as being an optimal design because it exactly fulfils the given requirements to structural reliability.

Having determined the optimal design *z* we may also calculate the corresponding design point in the original space, i.e. $\mathbf{x}_d$ for the basic random variables. This point may be interpreted as the most likely failure point, i.e. the most likely combination of the outcomes of the basic random variables leading to failure. Now partial safety factors may be derived from the design point for the various resistance variables as:

$$\gamma_m = \frac{x_c}{x_d} \tag{11.41}$$

and for load variables:

$$\gamma_Q = \frac{x_d}{x_c} \tag{11.42}$$

where x_d is the design point for the considered design variable and x_c the corresponding characteristic value.

Example 4 – Calculation of Partial Safety Factors

Consider again the case of the steel rod. Assume that the reliability index of β=3.7448 is considered optimal, implicitly implying that the optimal design variable *z* is equal to 1, the task is to establish a partial safety factor based design format for this problem.

The first task is to establish the design equation, which is simply the limit state equation where the basic random variables are exchanged with characteristic values and multiplied or divided by partial safety factors, i.e.:

$$z \cdot \frac{r_C}{\gamma_R} \cdot \frac{a_C}{\gamma_A} - s_C \cdot \gamma_S = 0$$

The next step is to establish the characteristic values and the partial safety factors and to this end the results of the FORM analysis performed previously may be utilised. The design point for the resistance variable *R* is obtained by:

$$r_d = u_R^* \cdot \sigma_R + \mu_R = -0.561 \cdot 3.7448 \cdot 35 + 350.0 = 276.56$$

defining the characteristic value of the resistance as a lower 5% fractile value, which is a typical definition according to most design codes, this is determined as:

$$r_c = -1.64 \cdot \sigma_R + \mu_R = -1.64 \cdot 35 + 350 = 292.60$$

whereby we get the partial safety factor for the resistance as:

$$\gamma_R = \frac{292.60}{276.56} = 1.06$$

Similarly in accordance with common code practice by defining a_c as the mean value of A and s_c by the upper 98% fractile value of the distribution function for S there is:

$$\gamma_A = \frac{10.0}{7.90} = 1.27\,,\quad \gamma_S = \frac{2242.0}{2115.0} = 1.06$$

Finally we may use the derived partial safety factor design format for the design of the steel rod whereby we get the following equation for the determination of the design *z*:

$$z \cdot \frac{292.6}{1.06} \cdot \frac{10}{1.27} - \cdot 1.06 \cdot 2115 = 0 \Rightarrow z \approx 1$$

8 Simulation Methods

The probability integral considered in Equation (11.3) for the estimation of which we have seen that FORM methods may successfully be applied:

$$P_f = \int_{g(\mathbf{x})\le 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{11.43}$$

may also be estimated by so-called simulation techniques. In the literature a large variety of simulation techniques may be found and a treatment of these will not be given in the present text. Here it is just noted that simulation techniques have proven their value especially for problems where the representation of the limit state function is associated with difficulties. Such cases are e.g. when the limit state function is not differentiable or when several design points contribute to the failure probability.

However, as all simulation techniques have origin in the so-called Monte Carlo method the principles of this – very crude simulation technique - will be shortly outlined in the following. Thereafter one of the most commonly applied techniques for utilisation of FORM analysis in conjunction with simulation techniques, namely the importance sampling method, will be explained.

The basis for simulation techniques is well illustrated by rewriting the probability integral in Equation (11.43) by means of an indicator function as shown in Equation (11.44)

$$P_f = \int_{g(\mathbf{x})\le 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} = \int I\left[g(\mathbf{x}) \le 0\right] f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{11.44}$$

where the integration domain is changed from the part of the sample space of the vector $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ for which $g(\mathbf{x}) \le 0$ to the entire sample space of X and where $I\left[g(\mathbf{x}) \le 0\right]$ is an indicator function equal to 1 if $g(\mathbf{x}) \le 0$ and otherwise equal to zero. Equation (11.44) is in

this way seen to yield the expected value of the indicator function $I[g(\mathbf{x}) \le 0]$. Therefore if now N realisations of the vector X, i.e. $\hat{\mathbf{x}}_j, j=1,2,..,N$ are sampled it follows from sample statistics that:

$$P_f = \frac{1}{N}\sum_{j=1}^{N} I[g(\mathbf{x}) \le 0] \tag{11.45}$$

is an unbiased estimator of the failure probability P_f.

Crude Monte-Carlo Simulation

The crude Monte Carlo simulation technique rests directly on the application of Equation (11.45) A large number of realisations of the basic random variables X, i.e. $\hat{\mathrm{x}}_j, j = 1,2\ldots N$ are generated (or simulated) and for each of the outcomes $\hat{\mathrm{x}}_j$ it is checked whether or not the limit state function taken in $\hat{\mathrm{x}}_j$ is positive. All the simulations for which this is not the case are counted (n_f) and after *N* simulations the failure probability p_f may be estimated through:

$$p_f = \frac{n_f}{N} \tag{11.46}$$

which then may be considered a sample expected value of the probability of failure. In fact for $N \to \infty$ the estimate of the failure probability becomes exact. However, simulations are often costly in computation time and the uncertainty of the estimate is thus of interest. It is easily realised that the coefficient of variation of the estimate is proportional to $1/\sqrt{n_f}$ meaning that if Monte Carlo simulation is pursued to estimate a probability in the order of 10^{-6} it must be expected that approximately 10^8 simulations are necessary to achieve an estimate with a coefficient of variance in the order of 10%. A large number of simulations are thus required using Monte Carlo simulation and all refinements of this crude technique have the purpose of reducing the variance of the estimate. Such methods are for this reason often referred to as variance reduction methods.

The simulation of the *N* outcomes of the joint density function in Equation (11.46) is in principle simple and may be seen as consisting of two steps. Here we will illustrate the steps assuming that the *n* components of the random vector ***X*** are independent.

In the first step a "pseudo random" number between 0 and 1 is generated for each of the components in $\hat{\mathbf{x}}_j$ i.e. $\hat{x}_{ji}$ *i=1,..,n*.The generation of such numbers may be facilitated by build-in functions of basically all programming languages and spreadsheet software.

In the second step the outcomes of the "pseudo random" numbers z_{ji} are transformed to outcomes of $\hat{x}_{ji}$ by:

$$x_{ji} = F_{X_i}^{-1}(z_{ji}) \tag{11.47}$$

where $F_{X_i}(\,)$ is the probability distribution function for the random variable X_i.

The principle is also illustrated in Figure 11.4.

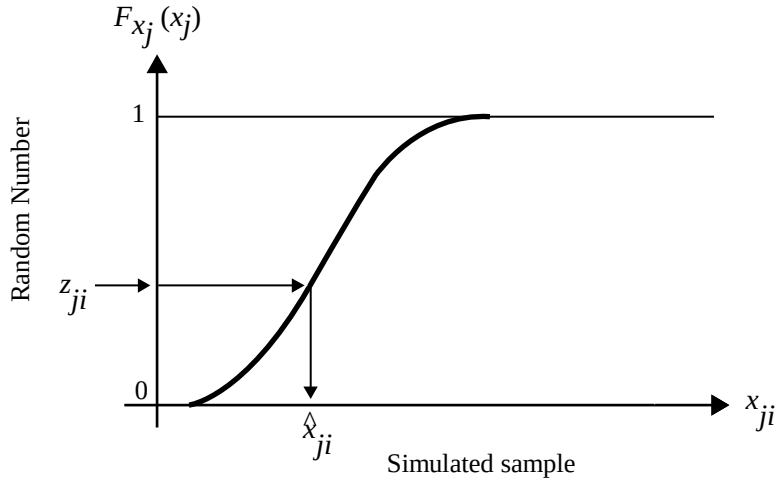

Figure 11.4 Principle for simulation of a random variable.

This process is the continued until all components of the vector $\hat{\mathbf{x}}_j$ have been generated.

Importance Sampling Simulation Method

As already mentioned the problem in using Equation (11.45) is that the sampling function $f_{\mathbf{x}}(\mathbf{x})$ typically is located in a region far away from the region where the indicator function $I[g(\mathbf{x}) \le 0]$ attains contributions. The success rate in the performed simulations is thus low. In practical reliability assessment problems where typical failure probabilities are in the order of $10^{-3} - 10^{-6}$ this in turn leads to the effect that the variance of the estimate of failure probability will be rather large unless a substantial amount of simulations are performed.

To overcome this problem different variance reduction techniques have been proposed aiming at, with the same number of simulations to reduce the variance of the probability estimate. In the following we shall briefly consider one of the most commonly applied techniques for variance reduction in structural reliability applications, namely the importance sampling method.

The importance sampling method takes basis in the utilisation of prior information about the domain contribution to the probability integral, i.e. the region that contributes to the indicator function. Let us first assume that we know which point in the sample space $\mathbf{x}^*$ contributes the most to the failure probability. Then by centring the simulations on this point, the important point, we would obtain a higher success rate in the simulations and the variance of the estimated failure probability would be reduced. Sampling centred on an important point may be accomplished by rewriting Equation (11.42) in the following way:

$$P_f = \int I[g(\mathbf{x}) \le 0] f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = \int I[g(\mathbf{x}) \le 0] \frac{f_{\mathbf{X}}(\mathbf{x})}{f_s(\mathbf{x})} f_s(\mathbf{x}) d\mathbf{x} \tag{11.48}$$

in which $f_s(\mathbf{x})$ is denoted the importance sampling density function. It is seen that the integral in Equation (11.48) represents the expected value of the term $I[g(\mathbf{x}) \le 0]\frac{f_x(\mathbf{x})}{f_s(\mathbf{x})}$ where the components of s are distributed according to $f_{\mathbf{V}}(\mathbf{x})$. The question in regard to the choice of an appropriate importance sampling function $f_S(s)$, however, remains open.

One approach to the selection of an importance sampling density function $f_S(s)$ is to select a n-dimensional joint Normal probability density function with uncorrelated components, mean values equal to the design point as obtained from FORM analysis, i.e. $\boldsymbol{\mu}_{\mathbf{S}} = \mathbf{x}^*$ and standard deviations e.g. corresponding to the standard deviations of the components of X, i.e. $\boldsymbol{\sigma}_{\mathbf{s}} = \boldsymbol{\sigma}_{\mathbf{X}}$. In this case Equation (11.48) may be written as:

$$P_f = \int I[g(\mathbf{x}) \le 0]\frac{f_{\mathrm{X}}(\mathbf{x})}{f_s(\mathbf{x})} f_s(\mathbf{x})d\mathbf{x} = \int I[g(s) \le 0]\frac{f_{\mathrm{X}}(s)}{\boldsymbol{\varphi}(s)}\boldsymbol{\varphi}(s)ds \tag{11.49}$$

in equivalence to Equation (11.44) leading to:

$$P_f = \frac{1}{N}\sum_{j=1}^{N} I[g(s) \le 0]\frac{f_{\mathrm{X}}(s)}{\boldsymbol{\varphi}(s)} \tag{11.50}$$

which may be assessed by sampling over realisations of s as described in the above.

Application of Equation (11.50) instead of Equation (11.44) greatly enhances efficiency of the simulations. If the limit state function is not too non-linear around the design point x^* the success rate of the simulations will be close to 50%. If the design point is known in advance in a reliability problem where the probability of failure is in the order of 10-6 the number of simulations required to achieve a coefficient of variance in the order of 10% is thus around 200. This number stands in strong contrast to the 108 required using the crude Monte Method discussed before, but of course also requires knowledge about the design point.

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