Risk and Safety

in

Civil, Surveying and Environmental Engineering

Prof. Dr. Michael Havbro Faber
Swiss Federal Institute of Technology
ETH Zurich, Switzerland

Contents of Today's Lecture

- Probability theory
- Uncertainties in engineering decision making
- Probabilistic modelling
- Engineering model building
- Methods of structural reliability theory
  - Linear normal distributed safety margins
  - Non-linear normal distributed safety margins
  - General case
  - SORM improvements
  - Monte-Carlo simulation

Conditional Probability and Bayes‘s Rule

as there is $P(A \cap E_i) = P(A|E_i)P(E_i) = P(E_i|A)P(A)$

we have

Likelihood

Prior

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{i=1}^{n} P(A|E_i)P(E_i)}$$

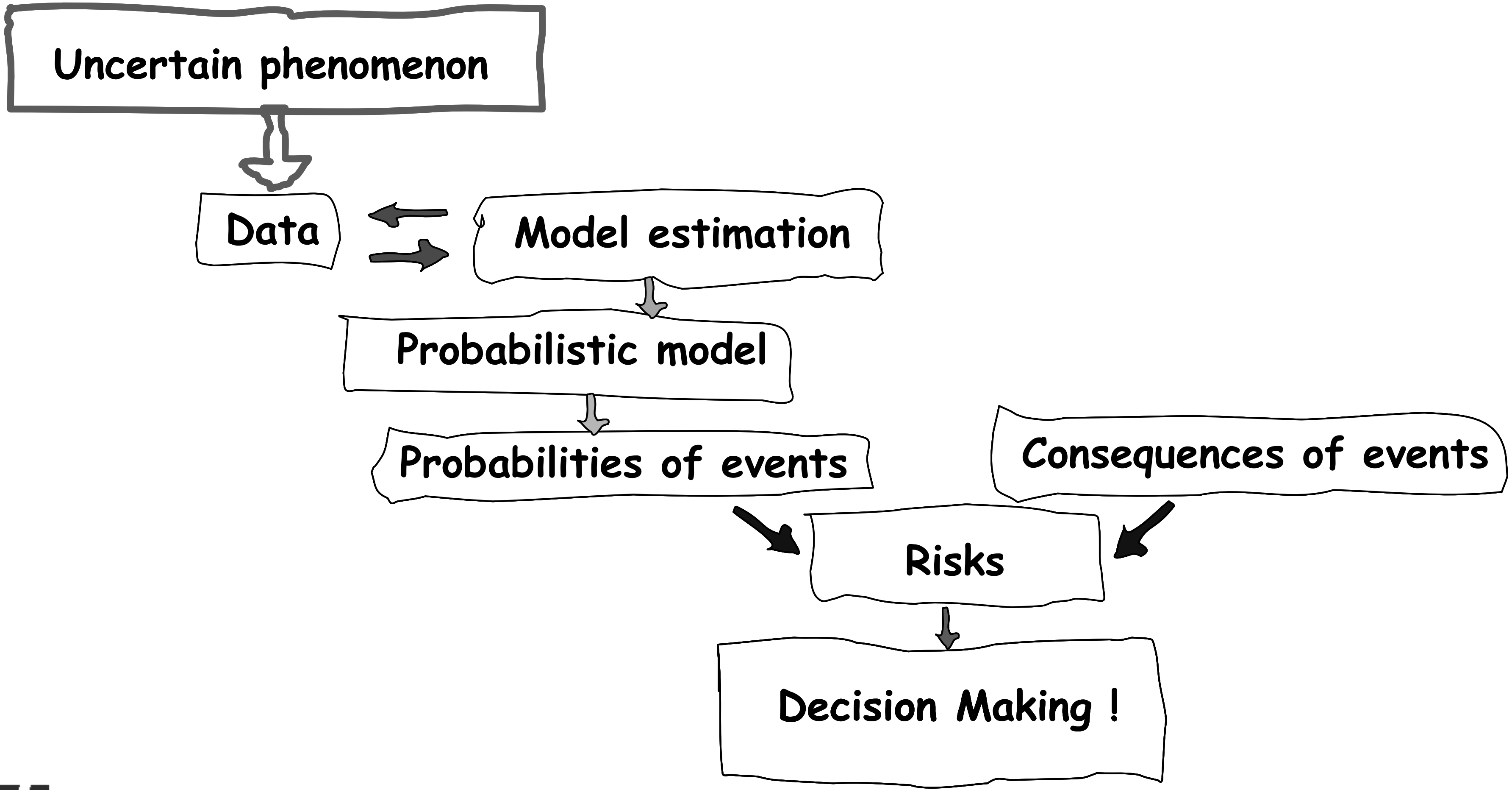
Posterior

Bayes Rule

Reverend Thomas Bayes
(1702-1764)

Overview of Uncertainty Modelling

- Why uncertainty modelling


Uncertain phenomenon

Data

Model estimation

Probabilistic model

Probabilities of events

Consequences of events

Risks

Decision Making !

Uncertainties in Engineering Problems

Different types of uncertainties influence decision making

- **Inherent natural variability – aleatory uncertainty**
  - **result of throwing dices**
  - **variations in material properties**
  - **variations of wind loads**
  - **variations in rain fall**

- **Model uncertainty – epistemic uncertainty**
  - **lack of knowledge (future developments)**
  - **inadequate/imprecise models (simplistic physical modelling)**

- **Statistical uncertainties – epistemic uncertainty**
  - **sparse information/small number of data**

Uncertainties in Engineering Problems

- Consider as an example a dike structure

  - the design (height) of the dike will be determining the frequency of floods

  - if exact models are available for the prediction of future water levels and our knowledge about the input parameters is perfect then we can calculate the frequency of floods (per year) - a deterministic world !

  - even if the world would be deterministic – we would not have perfect information about it – so we might as well consider the world as random

Uncertainties in Engineering Problems

In principle the so-called

inherent physical uncertainty (aleatory – Type I)

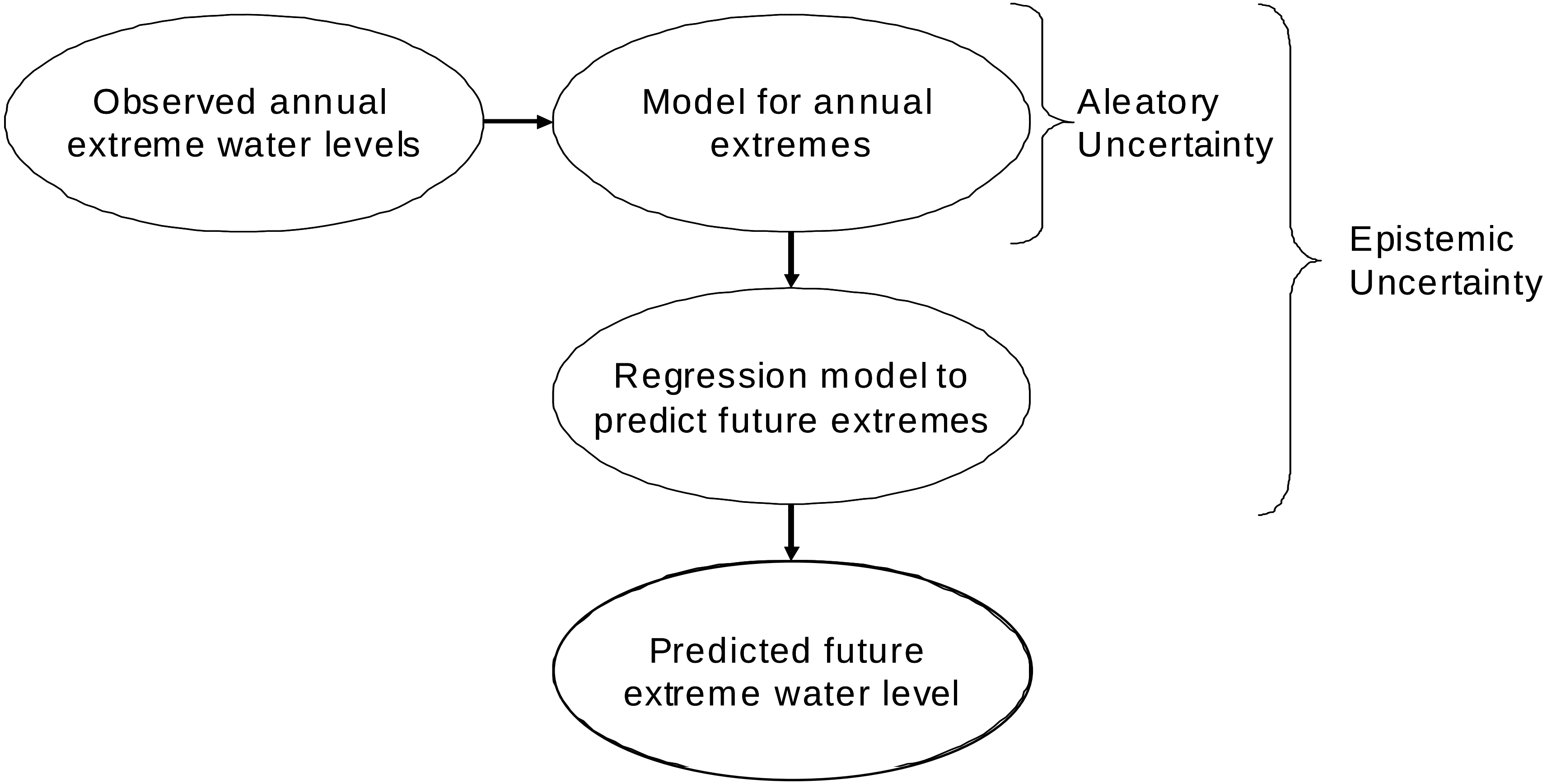
is the uncertainty caused by the fact that the world is random, however, another pragmatic viewpoint is to define this type of uncertainty as

any uncertainty which cannot be reduced by means of collection of additional information

the uncertainty which can be reduced is then the

model and statistical uncertainties (epistemic – Type II)

Uncertainties in Engineering Problems



Observed annual extreme water levels

Model for annual extremes

Regression model to predict future extremes

Predicted future extreme water level

Aleatory Uncertainty

Epistemic Uncertainty

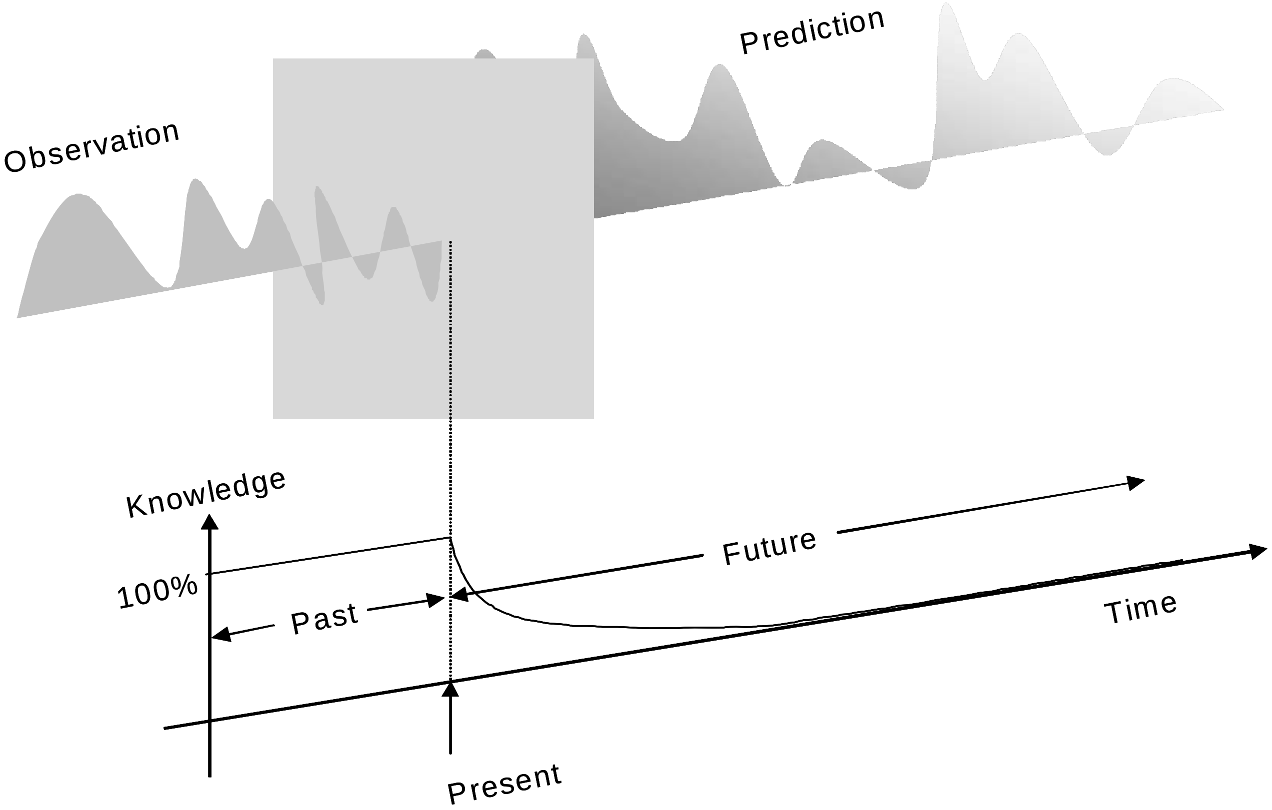
Uncertainties in Engineering Problems

The relative contribution of aleatory and epistemic uncertainty to the prediction of future water levels is thus influenced directly by the applied models

refining a model might reduce the epistemic uncertainty – but in general also changes the contribution of aleatory uncertainty

the uncertainty structure of a problem can thus be said to be scale dependent !

Uncertainties in Engineering Problems



The uncertainty structure changes also as function of time – is thus time dependent !

Random Variables

- Probability distribution and density functions

A random variable is denoted with capital letters : X

A realization of a random variable is denoted with small letters : x

We distinguish between

- *continuous random variables* :	can take any value in a given range
- *discrete random variables* :	can take only discrete values

Random Variables

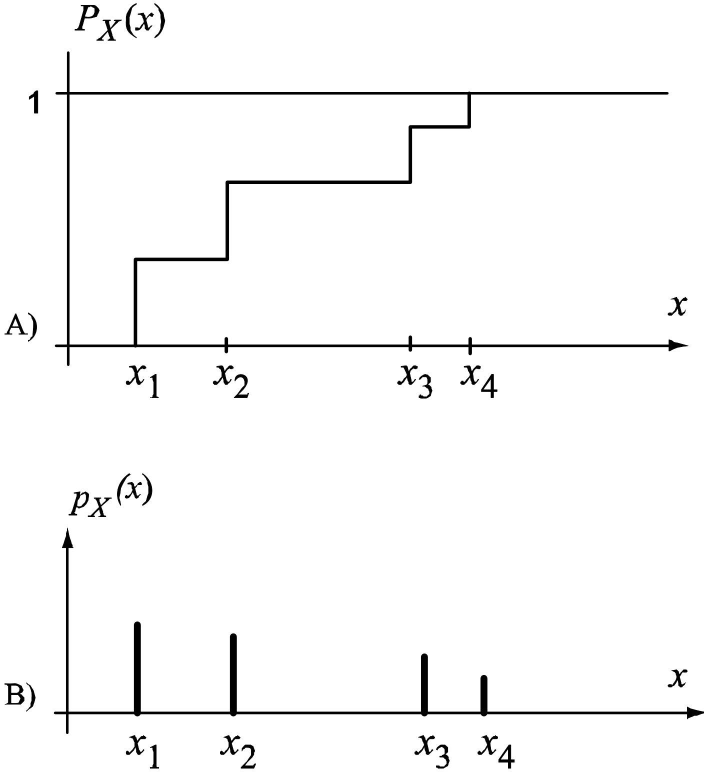
- **Probability distribution and density functions**

The probability that the outcome of a discrete random variable X is smaller than x is denoted the *probability distribution function*

$$P_X(x) = \sum_{x_i < x} p_X(x_i)$$

The *probability density function* for a discrete random variable is defined by

$$p_X(x_i) = P(X = x)$$



Random Variables

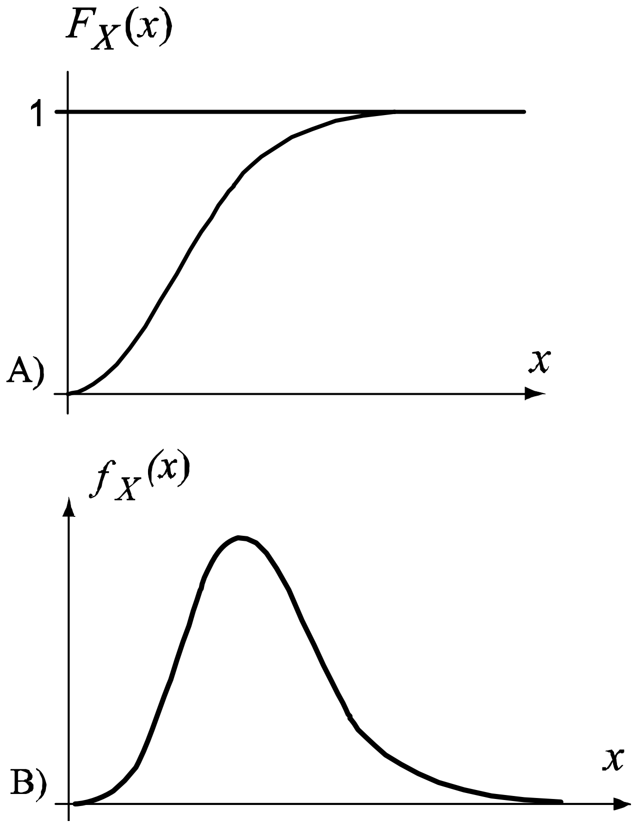
- **Probability distribution and density functions**

The probability that the outcome of a continuous random variable X is smaller than x is denoted the *probability distribution function*

$$F_X\left(x\right) = P\left(X < x\right)$$

The probability density function for a continuous random variable is defined by

$$f_X\left(x\right) = \frac{\partial F_X\left(x\right)}{\partial x}$$



Random Variables

- **Moments of random variables and the expectation operator**

  Probability distribution and density function can be described in terms of their parameters $\mathbf{p}$ or their moments

  Often we write

$$F_X(x,\mathbf{p}) \qquad f_X(x,\mathbf{p})$$

Parameters

The parameters can be related to the moments and visa versa

Random Variables

- **Moments of random variables and the expectation operator**

The i'th moment *m_i* for a continuous random variable *X* is defined through

$$m_i = \int_{-\infty}^{\infty} x^i \cdot f_X(x)dx$$

The *expected value E[X]* of a continuous random variable *X* is defined accordingly as the first moment

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$$

Random Variables

- **Moments of random variables and the expectation operator**

The i'th moment m_i for a discrete random variable X is defined through

$$m_i = \sum_{j=1}^{n} x_j^i \cdot p_X(x_j)$$

The *expected value E[X]* of a discrete random variable *X* is defined accordingly as the first moment

$$\mu_X = E[X] = \sum_{j=1}^{n} x_j \cdot p_X(x_j)$$

Random Variables

- **Moments of random variables and the expectation operator**

The *standard deviation* σ_X of a continuous random variable is defined as the second central moment i.e. for a continuous random variable *X* we have

$$\sigma_X{}^2 = \underset{\substack{\uparrow\\ \textbf{Variance}}}{\mathrm{Var}}[\mathrm{X}] = E\left[(X-\mu_X)^2\right] = \int_{-\infty}^{\infty} \left(x - \underset{\substack{\uparrow\\ \textbf{Mean value}}}{\mu_X}\right)^2 \cdot f_X(x) dx$$

for a discrete random variable we have correspondingly

$$\sigma_X^2 = Var[X] = \sum_{j=1}^{n} (x_j - \mu_X)^2 \cdot p_X(x_j)$$

Random Variables

- **Moments of random variables and the expectation operator**

The ratio between the standard deviation and the expected value of a random variable is called the *Coefficient of Variation CoV* and is defined as

$$CoV\left[X\right]=\frac{\sigma_X}{\mu_X}$$

Dimensionless

a useful characteristic to indicate the variability of the random variable around its expected value

Random Variables

- Typical probability distribution functions in engineering

  Normal : sum of random effects

  Log-Normal: product of random effects

  Exponential: waiting times

  Gamma: Sum of waiting times

  Beta: Flexible modeling function

Distribution type	Parameters	Moments
Rectangular a ≤ x ≤ b $f_X(x) = \frac{1}{b-a}$ $F_X(x) = \frac{x-a}{b-a}$	a b	$\mu = \frac{a+b}{2}$ $\sigma = \frac{b-a}{\sqrt{12}}$
Normal $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$ $F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$	μ $\sigma > 0$	μ σ
Shifted Lognormal x > ε $f_X(x) = \frac{1}{(x-\varepsilon)\zeta\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(x-\varepsilon)-\lambda}{\zeta}\right)^2\right)$ $F_X(x) = \Phi\left(\frac{ln(x-\varepsilon)-\lambda}{\zeta}\right)^2$	λ $\zeta > 0$ ε	$\mu = \varepsilon + \exp\left(\lambda + \frac{\zeta^2}{2}\right)$ $\sigma = \exp\left(\lambda + \frac{\zeta^2}{2}\right)\sqrt{\exp(\zeta^2)-1}$
Shifted Exponential x ≥ ε $f_X(x) = \lambda \exp(-\lambda(x-\varepsilon))$ $F_X(x) = 1 - e^{-\lambda(x-e)}$	ε $\lambda > 0$	$\mu = \varepsilon + \frac{1}{\lambda}$ $\sigma = \frac{1}{\lambda}$
Gamma x ≥ 0 $f_X(x) = \frac{b^p}{\Gamma(p)} \exp(-bx) x^{p-1}$ $F_X(x) = \frac{\Gamma(bx, p)}{\Gamma(p)}$	$p > 0$ $b > 0$	$\mu = \frac{p}{b}$ $\sigma = \frac{\sqrt{p}}{b}$
Beta a ≤ x ≤ b, r, t ≥ 1 $f_X(x) = \frac{\Gamma(r+t)}{\Gamma(r)\cdot\Gamma(t)} \frac{(x-a)^{r-1}(b-x)^{t-1}}{(b-a)^{r+t-1}}$ $F_X(x) = \frac{\Gamma(r+t)}{\Gamma(r)\cdot\Gamma(t)} \cdot \int_a^u \frac{(u-a)^{r-1}(b-u)^{t-1}}{(b-a)^{r+t-1}} du$	a b $r > 1$ $t > 1$	$\mu = a + (b-a)\frac{r}{r+1}$ $\sigma = \frac{b-a}{r+t}\sqrt{\frac{rt}{r+t+1}}$

Random Variables

- The Normal distribution

  The analytical form of the Normal distribution may be derived by repeated use of the result regarding the probability density function for the sum of two random variables

  The normal distribution is very frequently applied in engineering modelling when a random quantity can be assumed to be composed as a sum of a number of individual contributions.

  A linear combination S of n Normal distributed random variables $X_i, i = 1, 2, .., n$ is thus also a Normal distributed random variable

$$S = a_0 + \sum_{i=1}^{n} a_i X_i$$

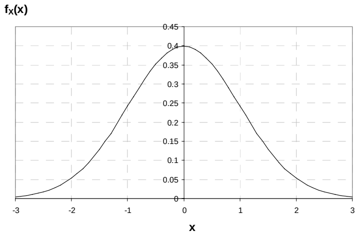
Random Variables

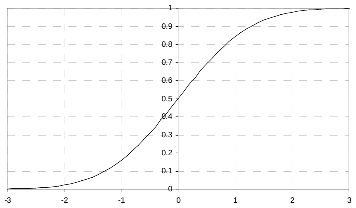
- **The Normal distribution:**

  In the case where the mean value is equal to zero and the standard deviation is equal to 1 the random variable is said to be *standardized*.

$$Z = \frac{X - \mu_X}{\sigma_X}$$ **Standardized random variable**

$$\left.\begin{array}{l} f_Z(z) = \varphi(z) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{1}{2} z^2 \right) \\[2em] F_Z(z) = \Phi(z) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{z} \exp\left(-\dfrac{1}{2} x^2 \right) dx \end{array}\right\} \textbf{Standard normal}$$





Stochastic Processes and Extremes

- Random quantities may be "time variant" in the sense that they take new values at different times or at new trials.

  - If the new realizations occur at discrete times and have discrete values the random quantity is called a random sequence

    failure events, traffic congestions,…

  - If the new realizations occur continuously in time and take continuous values the random quantity is called a random process or stochastic process

    wind velocity, wave heights,…

Stochastic Processes and Extremes

- **Random sequences**

  The Poisson counting process is one of the most commonly applied families of probability distributions applied in reliability theory

  The process *N(t)* denoting the number of events in a (time) interval *(t, t+Dt[* is called a Poisson process if the following conditions are fulfilled:

  1) the probability of one event in the interval *(t, t+Dt[* is asymptotically proportional to *Dt*.
  2) the probability of more than one event in the interval *(t, t+Dt[* is a function of higher order of *Dt for Dt→0*.
  3) events in disjoint intervals are mutually independent.

Stochastic Processes and Extremes

- **Random sequences**

The probability distribution function of the (waiting) time till the first event *T_1* is now easily derived recognizing that the probability of *$T_1 > t$* is equal to *$P_0(t)$* we get:

$$F_{T_1}(t_1) = 1 - P_0(t_1)$$

$$= 1 - exp(-\int_0^t \nu(\tau)d\tau)$$

$$F_{T_1}(t_1) = 1 - exp(-\nu t)$$

$$\uparrow$$

Homogeneous case !

Exponential probability distribution
Exponential probability density

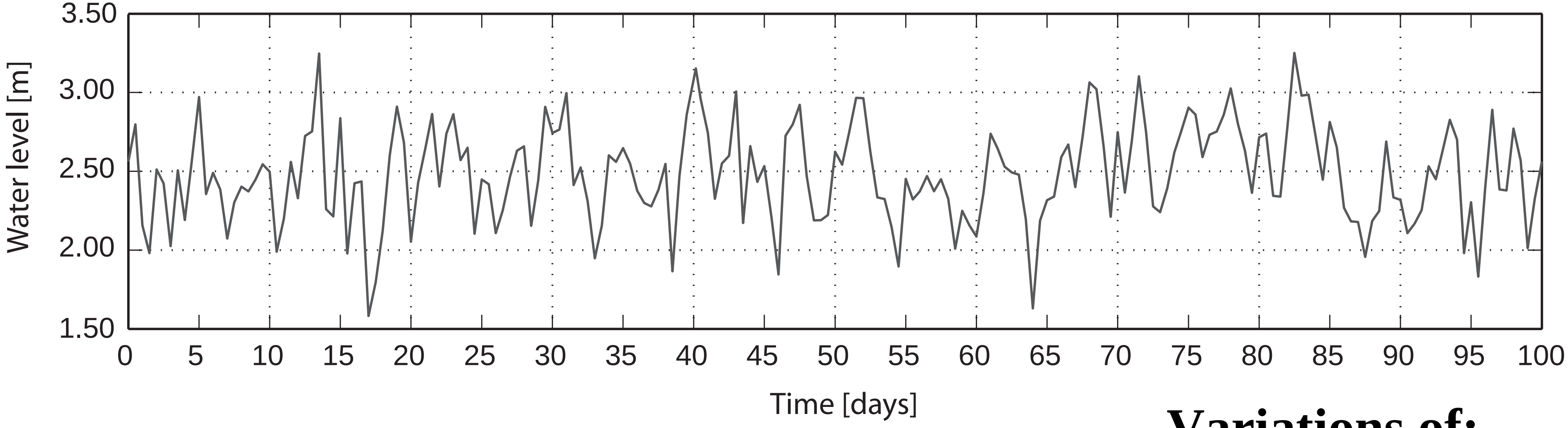
$$\downarrow$$

$$f_{T_1}(t_1) = \nu \cdot exp(-\nu t)$$

Stochastic Processes and Extremes

- **Continuous random processes**

  A continuous random process is a random process which has realizations continuously over time and for which the realizations belong to a continuous sample space.



Variations of:
water levels
wind speed
rain fall

Stochastic Processes and Extremes

- **Continuous random processes**

The mean value of the possible realizations of a random process is given as:

$$\mu_X(t) = E\left[X(t)\right] = \int_{-\infty}^{\infty} x \cdot f_X(x,t)dx$$

Function of time !

The correlation between realizations at any two points in time is given as:

$$R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 \cdot x_2 \cdot f_{XX}(x_1,x_2;t_1,t_2)dx_1dx_2$$

Auto-correlation function – refers to a scalar valued random process

Stochastic Processes and Extremes

Extreme Value Distributions

In engineering we are often interested in extreme values i.e. the smallest or the largest value of a certain quantity within a certain time interval e.g.:

The largest earthquake in 1 year

The highest wave in a winter season

The largest rainfall in 100 years

Stochastic Processes and Extremes

Extreme Value Distributions

We could also be interested in the smallest or the largest value of a certain quantity within a certain volume or area unit e.g.:

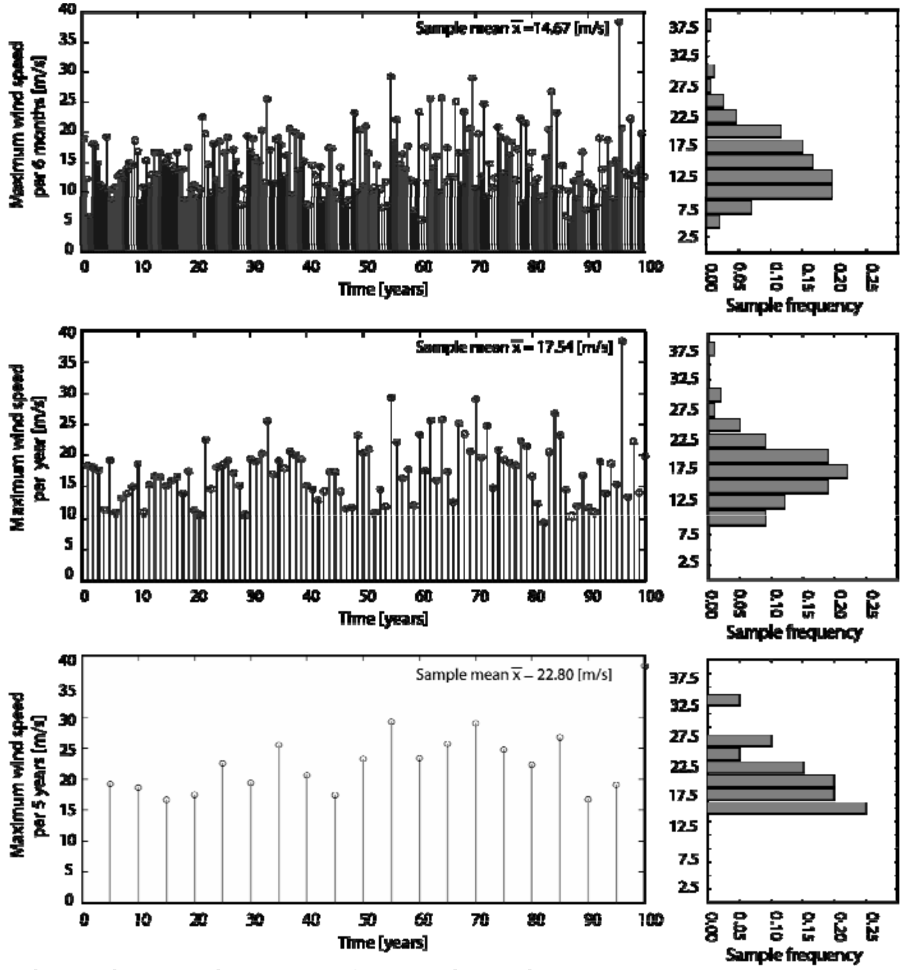
The largest concentration of pesticides in a volume of soil

The weakest link in a chain

The smallest thickness of concrete cover

Stochastic Processes and Extremes

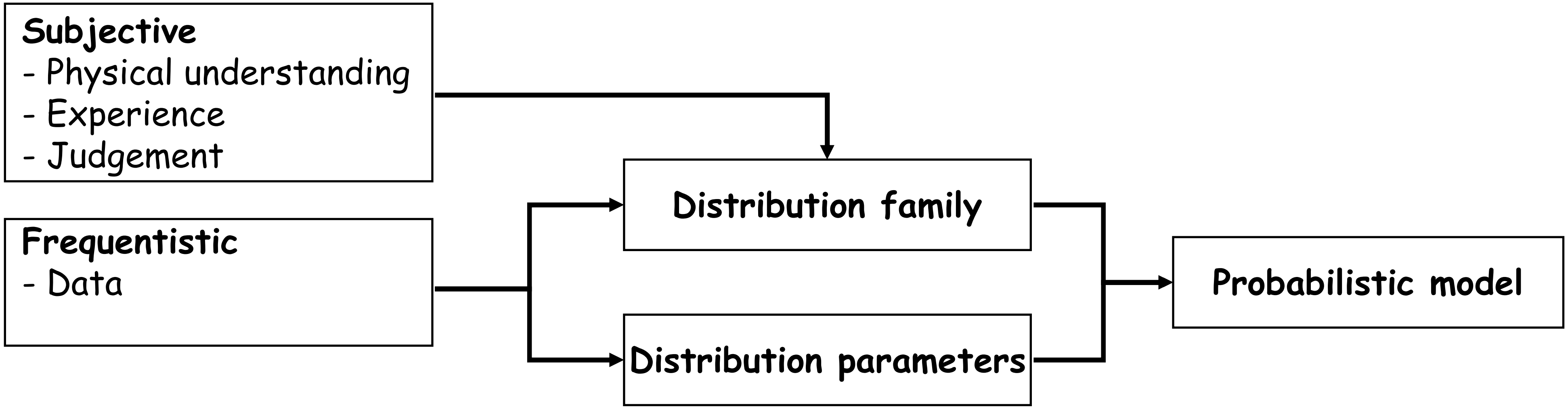
Extremes of a random process:



Overview of Estimation and Model Building

Different types of information is used when developing engineering models

- **- subjective information**
- **- frequentististic information**



Subjective
- Physical understanding
- Experience
- Judgement

Frequentistic
- Data

Distribution family

Distribution parameters

Probabilistic model

Structural Reliability Analysis

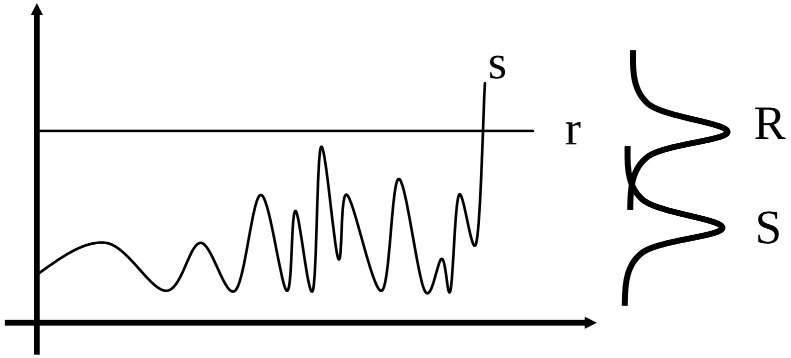
Reliability of structures cannot be assessed through failure rates because

- Structures are unique in nature

- Structural failures normally take place due to extreme loads exceeding the residual strength

Therefore in structural reliability, models are established for resistances R and loads S individually and the structural reliability is assessed through:

$$P_f = P(R - S \leq 0)$$



Structural Reliability Analysis

If only the resistance is uncertain the failure probability may be assessed by

$$P_f = P(R \le s) = F_R(s) = P(R/s \le 1)$$

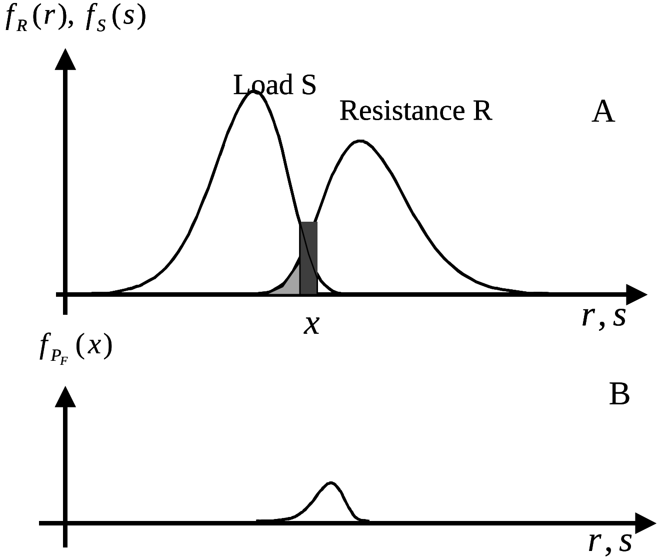
If also the load is uncertain we have

$$P_f = P(R \le S) = P(R - S \le 1) = \int_{-\infty}^{\infty} F_R(x) f_S(x) dx$$

where it is assumed that the load and the resistance are independent

This is called the

„Fundamental Case“



Structural Reliability Analysis

In the case where *R* and *S* are normal distributed the safety margin *M* is also normal distributed

$$M = R - S$$

Then the failure probability is

$$P_F = P(R - S \leq 0) = P(M \leq 0)$$

with the mean value of M

$$\mu_M = \mu_R - \mu_S$$

and standard deviation of M

$$\sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2}$$

The failure probability is then

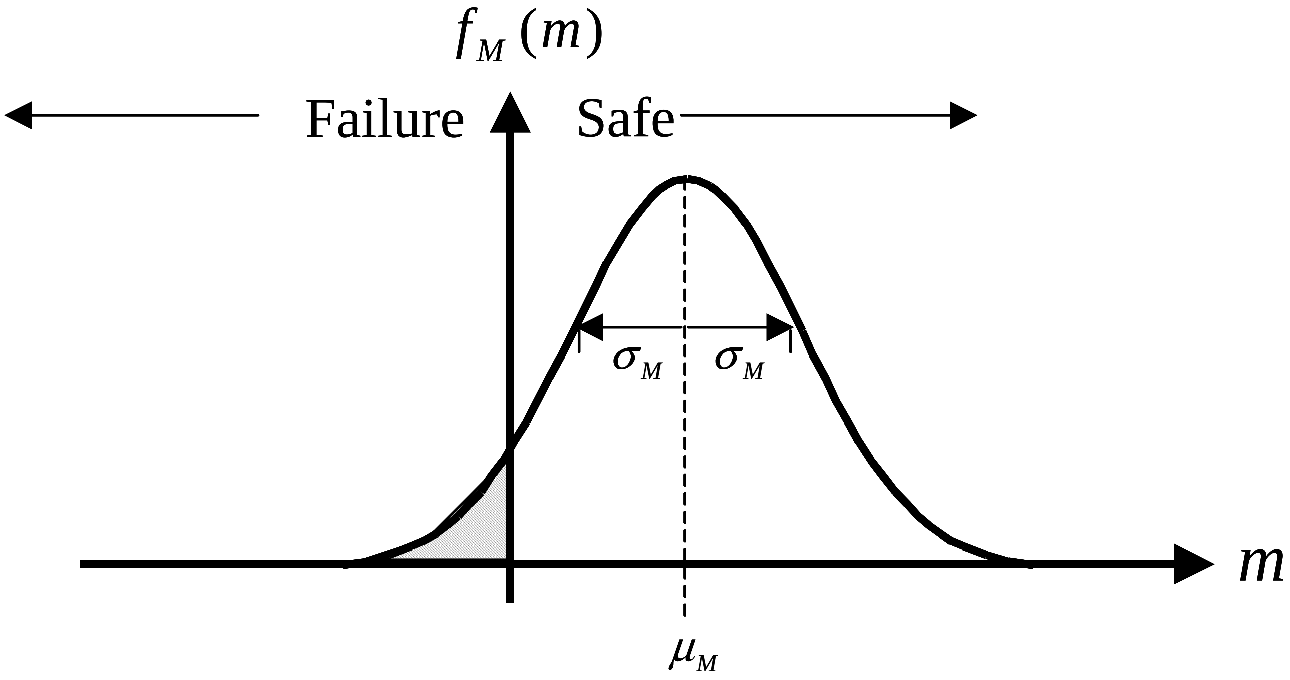
$$P_F = \Phi(\frac{0 - \mu_M}{\sigma_M}) = \Phi(-\beta)$$

where the reliability index is

$$\beta = \mu_M / \sigma_M$$

Structural Reliability Analysis

The normal distributed safety margin *M*



Structural Reliability Analysis

In the general case the resistance and the load may be defined in terms of functions

$$R = f_1(\mathbf{X})$$
$$S = f_2(\mathbf{X})$$

where $\mathbf{X}$ are basic random variables

and the safety margin as

$$M = R - S = f_1(\mathbf{X}) - f_2(\mathbf{X}) = g(\mathbf{X})$$

where $g(\mathbf{x}) \leq 0$ is called the

limit state function

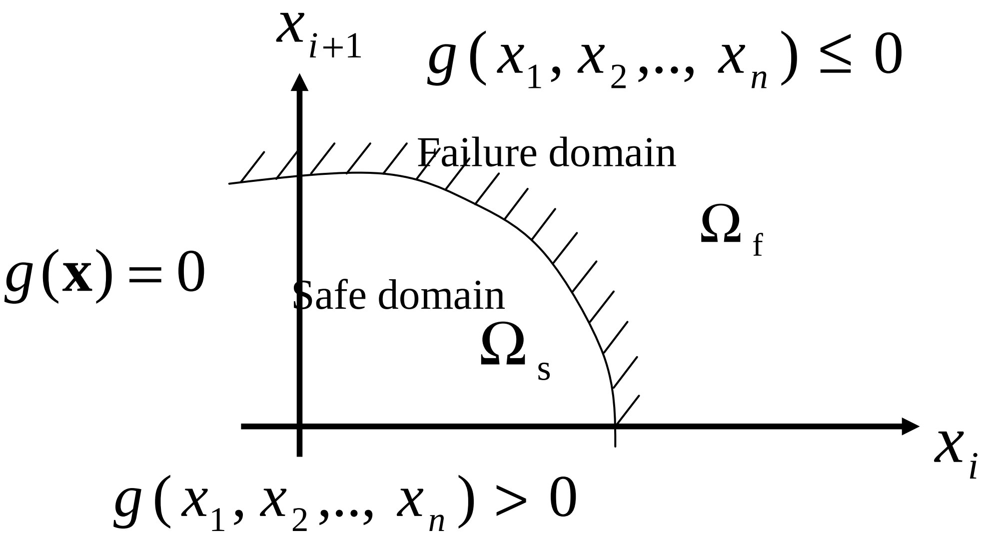
failure occurs when

$$g(\mathbf{x}) \leq 0$$

Structural Reliability Analysis

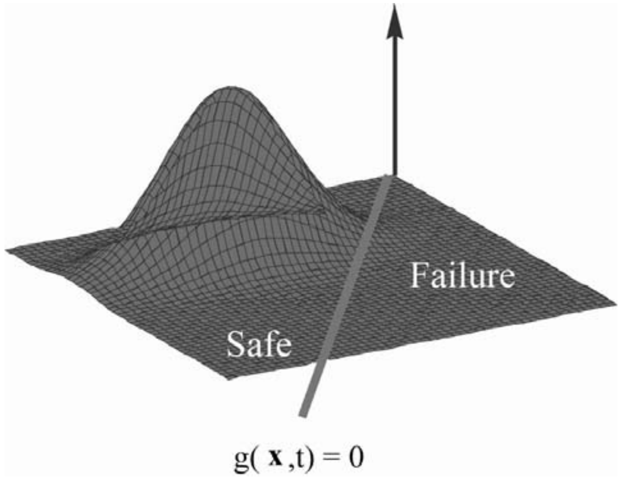
Setting $g(\mathbf{x}) = 0$ defines a (n-1) dimensional surface in the space spanned by the *n* basic variables X

This is the failure surface separating the sample space of X into a safe domain and a failure domain



The failure probability may in general terms be written as

$$P_f = \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$



Failure event

$$\mathbf{F} = \{g(\mathbf{x}) \leq 0\}$$

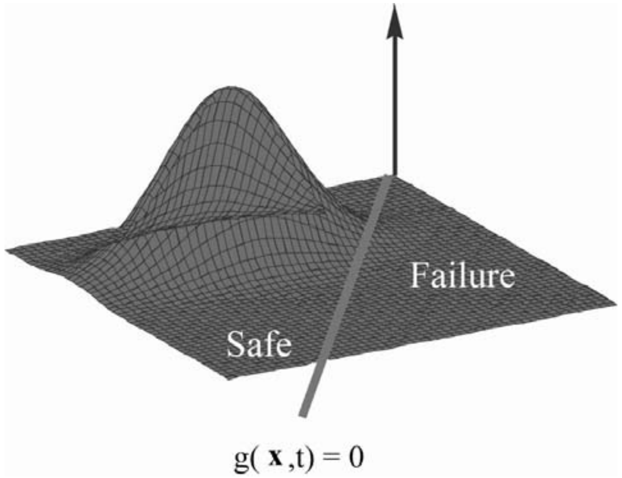
Basics of Structural Reliability Methods

The probability of failure can be assessed by

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \leq 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

where $f_{\mathbf{X}}(\mathbf{x})$ **is the joint probability density function for the basic random variables X**

For the 2-dimensional case the failure probability simply corresponds to the integral under the joint probability density function in the area of failure



Basics of Structural Reliability Methods

The probability of failure can be calculated using

- numerical integration
  (Simpson, Gauss, Tchebyschev, etc.)

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \leq 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

but for problems involving dimensions higher than say 6 the numerical integration becomes cumbersome

Other methods are necessary !

Basics of Structural Reliability Methods

When the limit state function is linear

$$g(\mathbf{x}) = a_0 + \sum_{i=1}^{n} a_i \cdot x_i$$

the saftey margin M is defined through

$$M = a_0 + \sum_{i=1}^{n} a_i \cdot X_i$$

with

mean value

$$\mu_M = a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}$$

and

variance

$$\sigma_M^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \rho_{ij} a_i a_j \sigma_i \sigma_j$$

Basics of Structural Reliability Methods

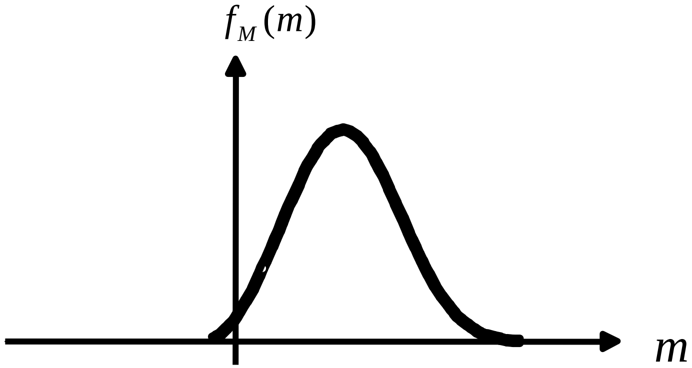
The failure probability can then be written as

$$P_F = P(g(\mathbf{X}) \leq 0) = P(M \leq 0)$$

The reliability index is defined as

$$\beta = \frac{\mu_M}{\sigma_M}$$

Basler and Cornell



Provided that the safety margin is normal distributed

the failure probability is determined as

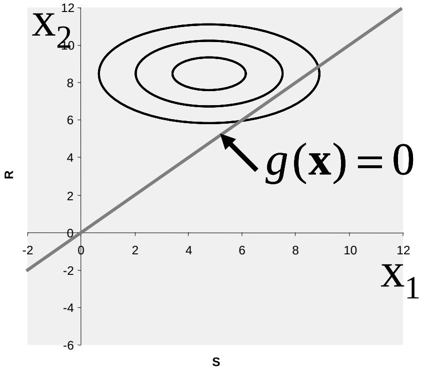
$$P_F = \Phi(-\beta)$$

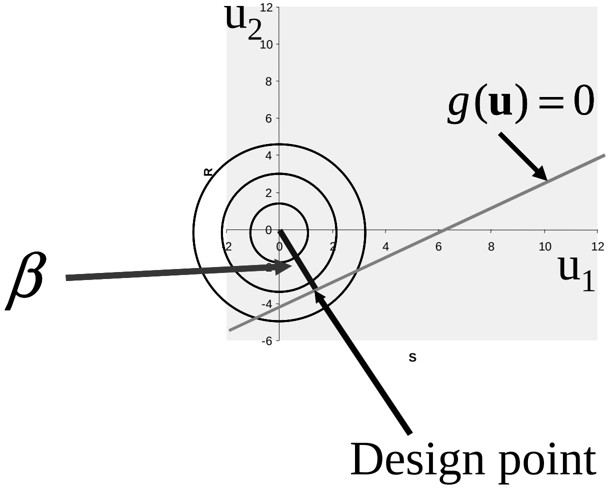
Basics of Structural Reliability Methods

The reliability index β has the geometrical interpretation of being the shortest distance between the failure surface and the origin in standard normal distributed space
u

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}$$

in which case the components of U have zero means and variances equal to 1





Basics of Structural Reliability Methods

Example:

Consider a steel rod with resistance *r* subjected to a tension force *s*

$$g(\mathbf{X}) = R - S$$

r* and *s* are modeled by the random variables *R* and *S

$$\mu_R = 350, \sigma_R = 35$$
$$\mu_S = 200, \sigma_S = 40$$

The probability of failure is wanted

$$P(R - S \leq 0)$$

Basics of Structural Reliability Methods

Example:

Consider a steel rod with resistance *r* subjected to a tension force *s*

$$g(\mathbf{X}) = R - S$$

r* and *s* are modeled by the random variables *R* and *S

$$\mu_R = 350, \sigma_R = 35$$
$$\mu_S = 200, \sigma_S = 40$$

The probability of failure is wanted

$$P(R - S \leq 0)$$

The safety margin is given as

$$M = R - S \begin{cases} \mu_M = 350 - 200 = 150 \\ \sigma_M = \sqrt{35^2 + 40^2} = 53.15 \end{cases}$$

The reliability index is then

$$\beta = \frac{150}{53.15} = 2.84$$

and the probability of failure

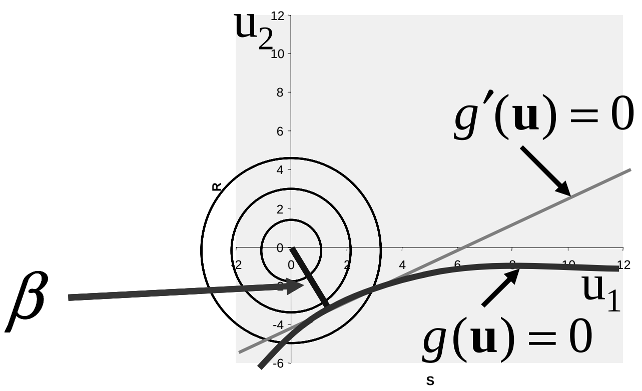
$$P_F = \Phi(-2.84) = 2.4 \cdot 10^{-3}$$

Basics of Structural Reliability Methods

Usually the limit state function is non-linear

- this small phenomenon caused the so-called invariance problem

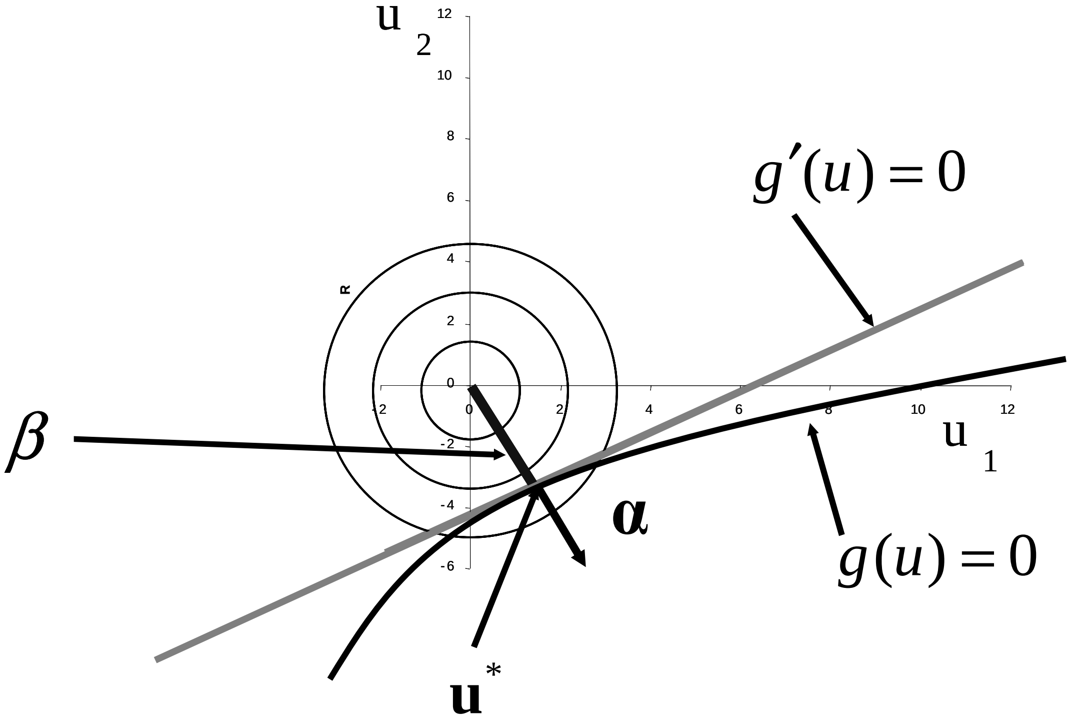
Can however easily be linearized !



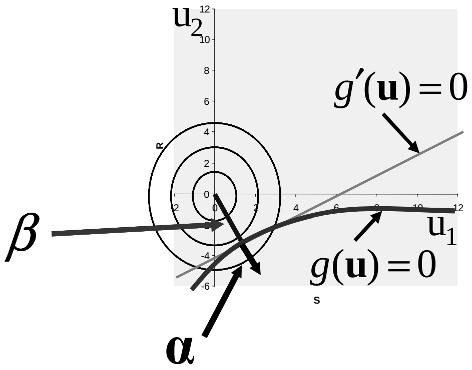
Hasofer & Lind suggested to linearize the limit state function in the design point
- this solved the invariance problem

The reliability index may then be determined by the following optimization problem

$$\beta = \min_{\mathbf{u}\in\{g(\mathbf{u})=0\}} \sqrt{\sum_{i=1}^{n} u_i^2}$$


u_2
12
10
8
6
4
2
0
-2
-4
-6
R
-2
0
2
4
6
8
10
12
u_1
$g'(u) = 0$
β
$\boldsymbol{\alpha}$
$g(u) = 0$
$\mathbf{u}^*$

Basics of Structural Reliability Methods



The optimization problem can be formulated as an iteration problem

1) the design point is determined as

$$\mathbf{u}^* = \beta \cdot \boldsymbol{\alpha}$$

2) the normal vector to the limit state function is determined as

$$\alpha_i = \frac{-\frac{\partial g}{du_i}(\beta \cdot \boldsymbol{\alpha})}{\left[\sum_{j=1}^{n} \frac{\partial g}{du_i}(\beta \cdot \boldsymbol{\alpha})^2\right]^{1/2}}, \quad i = 1,2,..n$$

3) the safety index is determined as

$$g(\beta \cdot \alpha_1, \beta \cdot \alpha_2, ...\beta \cdot \alpha_n) = 0$$

4) a new design point is determined as

$$u^* = \left(\beta \cdot \alpha_1, \beta \cdot \alpha_2, ...\beta \cdot \alpha_n\right)^T$$

5) continue the above steps until convergence in β

Basics of Structural Reliability Methods

Example :

Consider the steel rod with cross-sectional area *a* and yield stress *r*

$$h = a \cdot r$$

The rod is loaded with the tension force s

The limit state function can then be written as

$$g(\mathbf{x}) = r \cdot a - s$$

***r*, *a* and *s* are uncertain and modeled by normal distributed random variables**

$$\mu_R = 350, \sigma_R = 35 \qquad \mu_S = 1500, \sigma_R = 300$$

$$\mu_A = 10, \sigma_A = 1$$

we would like to calculate the probability of failure

Basics of Structural Reliability Methods

The first step is to transform the basic random variables into standardized normal distributed space

$$U_R = \frac{R-\mu_R}{\sigma_R}$$

$$U_A = \frac{A-\mu_A}{\sigma_A}$$

$$U_S = \frac{S-\mu_S}{\sigma_S}$$

Then we write the limit state function in terms of the realizations of the standardized normal distributed random variables

$$g(u) = (u_R\sigma_R + \mu_R)(u_A\sigma_A + \mu_A) - (u_S\sigma_S + \mu_S)$$

$$= (35u_R + 350)(u_A + 10) - (300u_S + 1500)$$

$$= 350\mathrm{u}_\mathrm{R} + 350u_A - 300u_S + 35u_Ru_A + 2000$$

Basics of Structural Reliability Methods

The reliability index is calculated as

$$\beta = \frac{-2000}{350\alpha_R + 350\alpha_A - 300\alpha_S + 35\beta\alpha_R\alpha_A}$$

the components of the α-vector are then calculate as

$$\begin{cases} \alpha_R = -\frac{1}{k}(350 + 35\beta\alpha_A) \\ \alpha_A = -\frac{1}{k}(350 + 35\beta\alpha_R) \\ \alpha_S = \frac{300}{k} \end{cases}$$

where

$$k = \sqrt{\alpha_R^2 + \alpha_A^2 + \alpha_S^2}$$

Basics of Structural Reliability Methods

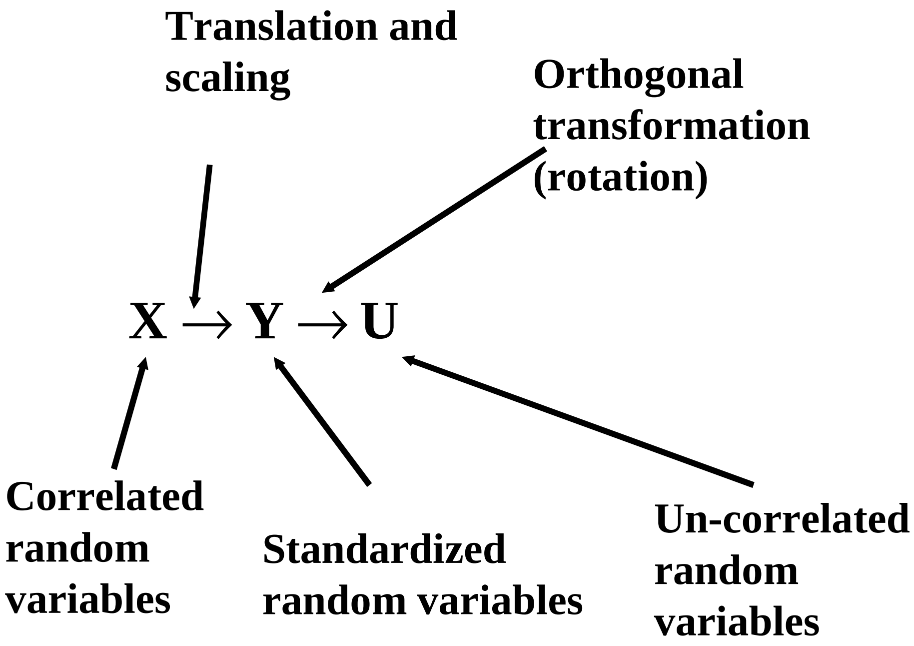
following the iteration scheme we get the following iteration history

Iteration	Start	1	2	3	4	5
β	3.0000	3.6719	3.7399	3.7444	3.7448	3.7448
α_R	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_A	-0.5800	-0.5701	-0.5612	-0.5611	-0.5610	-0.5610
α_S	0.5800	0.5916	0.6084	0.6086	0.6087	0.6087

Basics of Structural Reliability Methods

The procedure can be extended to consider

Correlated random variables



Translation and scaling

Orthogonal transformation (rotation)

$$\mathbf{X} \rightarrow \mathbf{Y} \rightarrow \mathbf{U}$$

Correlated random variables

Standardized random variables

Un-correlated random variables

Basics of Structural Reliability Methods

Correlated random variables

The covariance matrix for the random variables is given as

$$\mathbf{C_X} = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] ... & Cov[X_1, X_n] \\ \vdots & \vdots & \vdots \\ Cov[X_n, X_1] & \cdots & Var[X_n] \end{bmatrix}$$

and the correlation coefficient matrix is

$$\boldsymbol{\rho}_\mathbf{X} = \begin{bmatrix} 1 & \cdots & \rho_{1n} \\ \vdots & 1 & \vdots \\ \rho_{n1} & \cdots & 1 \end{bmatrix}$$

The first step is the standardization

$$Y_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}, i = 1,2,..n$$

Basics of Structural Reliability Methods

Correlated random variables

The transformation of the correlated random variables into non-correlated random variables can be written as

$$\mathbf{Y} = \mathbf{T}\mathbf{U}$$

where $\mathbf{T}$ is a lower triangular matrix

then we can write

$$\mathbf{C}_{\mathbf{Y}} = E\left[\mathbf{Y} \cdot \mathbf{Y}^T\right] = E\left[\mathbf{T} \cdot \mathbf{U} \cdot \mathbf{U}^T \cdot \mathbf{T}^T\right] = \mathbf{T} \cdot E\left[\mathbf{U} \cdot \mathbf{U}^T\right] \cdot \mathbf{T}^T = \mathbf{T} \times \mathbf{T}^T = \boldsymbol{\rho}_{\mathbf{X}}$$

with *T* ***standing for transpose matrix***

Basics of Structural Reliability Methods

Correlated random variables

In the case of 3 random variables we have

$$\mathbf{T}\cdot\mathbf{T}^T = \boldsymbol{\rho}_{\mathbf{X}} = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ & \rho_{22} & \rho_{23} \\ sym. & & \rho_{33} \end{bmatrix}$$

As $\mathbf{T}$ is a lower triangular matrix we have

$$T_{11} = \sqrt{1}$$
$$T_{21} = \rho_{12}$$
$$T_{31} = \rho_{13}$$
$$T_{22} = \sqrt{1 - T_{21}^2}$$
$$T_{32} = \frac{\rho_{23} - T_{31}\cdot T_{21}}{T_{22}}$$
$$T_{33} = \sqrt{1 - T_{31}^2 - T_{32}^2}$$
$$\vdots$$

$$\mathbf{T}\cdot\mathbf{T}^T = \begin{bmatrix} T_{11} & 0 & 0 \\ T_{21} & T_{22} & 0 \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ & \rho_{22} & \rho_{23} \\ sym. & & \rho_{33} \end{bmatrix}$$

Basics of Structural Reliability Methods

The normal-tail approximation

$$F_{X_{ii}}(x_i^*) = \Phi(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}})$$

$$f_{X_{ii}}(x_i^*) = \frac{1}{\sigma_{X_i}}\varphi(\frac{x_i^* - \mu'_{X_i}}{\sigma'_{X_i}})$$

$$\sigma'_{X_i} = \frac{\varphi(\Phi^{-1}(F_{X_i}(x_i^*)))}{f_{X_i}(x_i^*)}$$

$$\mu'_{X_i} = x_i^* - \Phi^{-1}(F_{X_i}(x_i^*))\sigma'_{X_i}$$

Basics of Structural Reliability Methods

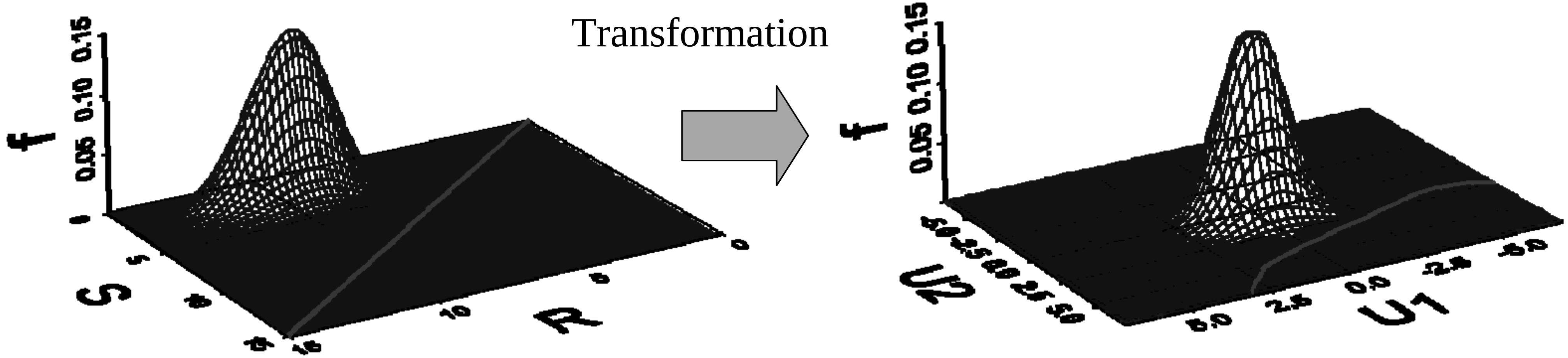
Non-normal distributed random variables

$$F_X(x) = F_{X_n}(x_n \mid x_1, x_2, \ldots x_{n-1}) \cdot F_{X_{n-1}}(x_{n-1} \mid x_1, x_2, \ldots x_{n-2}) \ldots F_{X_1}(x_1)$$

Rosenblatt Transformation

$$\begin{aligned}
\Phi(u_1) &= F_{X_1}(x_1) \\
\Phi(u_2) &= F_{X_2}(x_2 \mid x_1) \\
&\vdots \\
\Phi(u_n) &= F_{X_n}(x_n \mid x_1, x_2, \ldots x_{n-1})
\end{aligned}$$

Basics of Structural Reliability Methods



g(Z): linear

$\mu_{Z1}, \mu_{Z2} \in R$

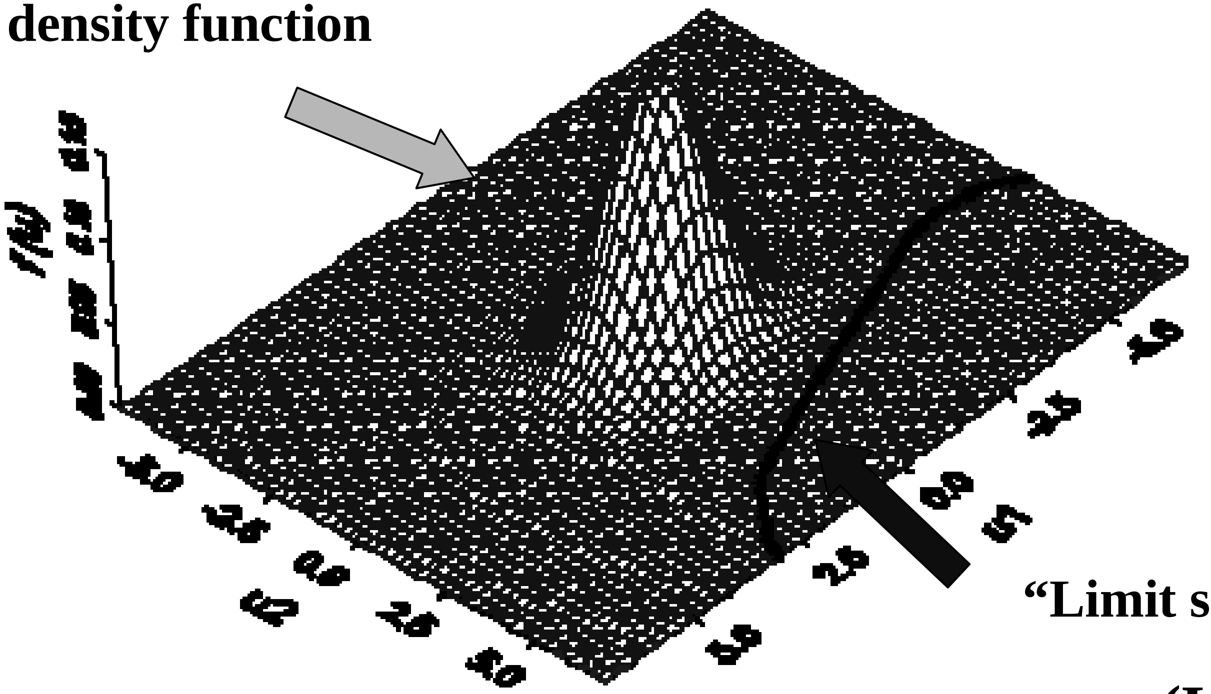
$\sigma_{Z1}, \sigma_{Z2} \in R$

g(U): non linear

$\mu_{U1} = \mu_{U2} = 0$

$\sigma_{U1} = \sigma_{U2} = 1$

Basics of Structural Reliability Methods

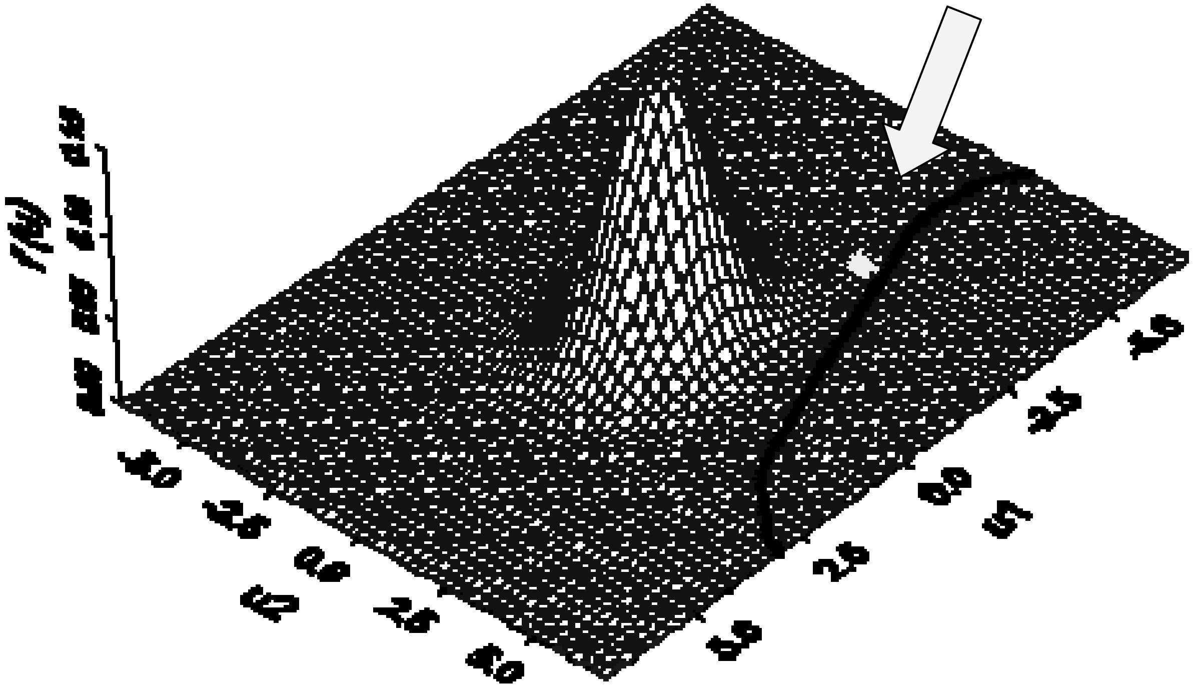


joint probability density function

“Limit state function”

$$g(U) = R-S$$

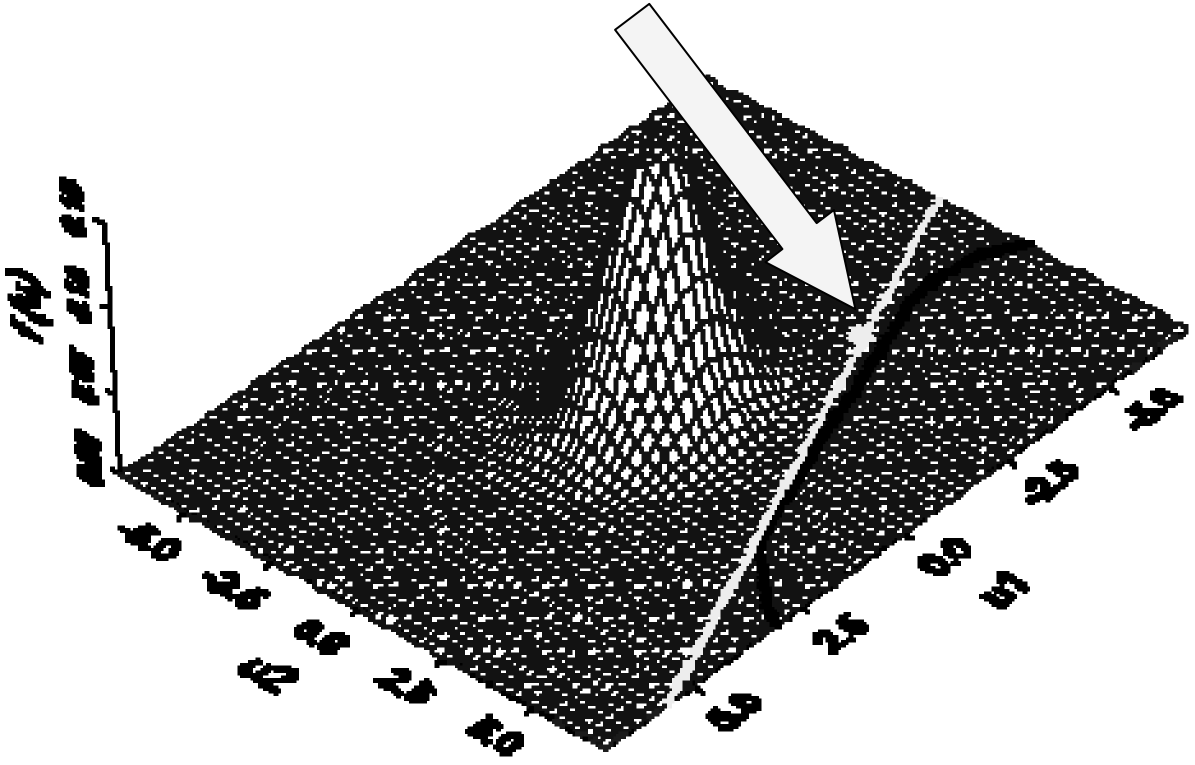
Basics of Structural Reliability Methods



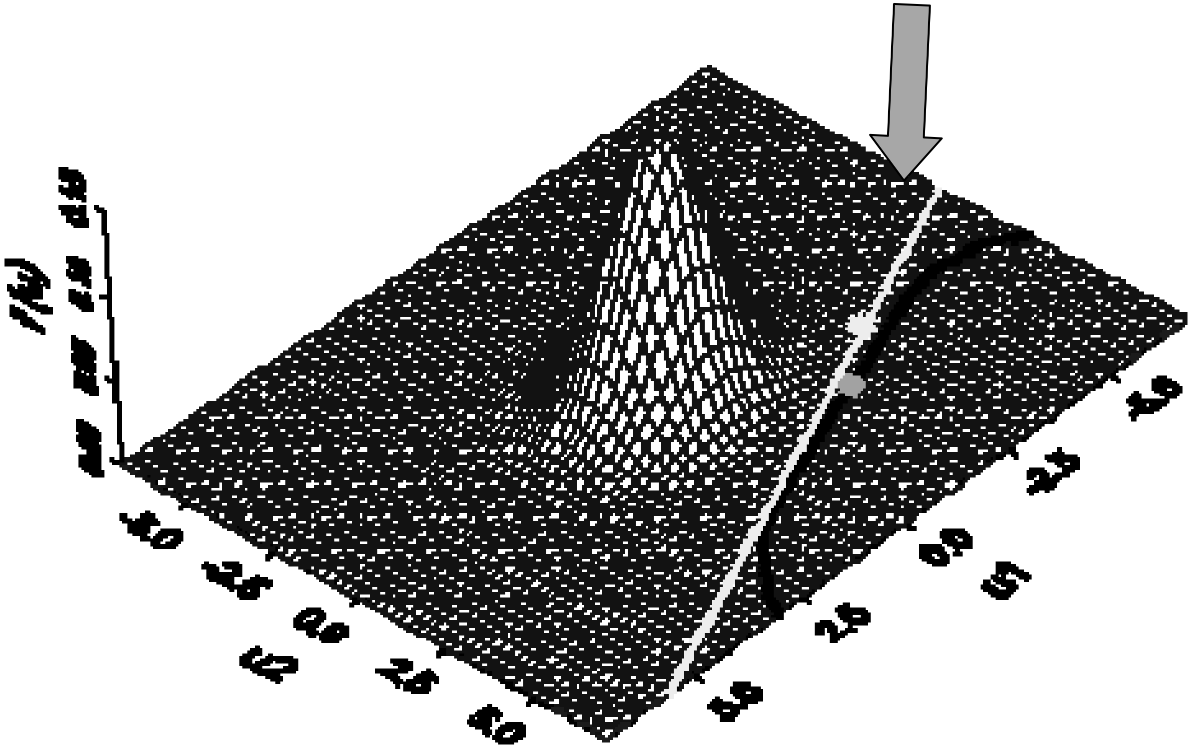
Start point X^1

Basics of Structural Reliability Methods

Linearization of Limit state function in starting point

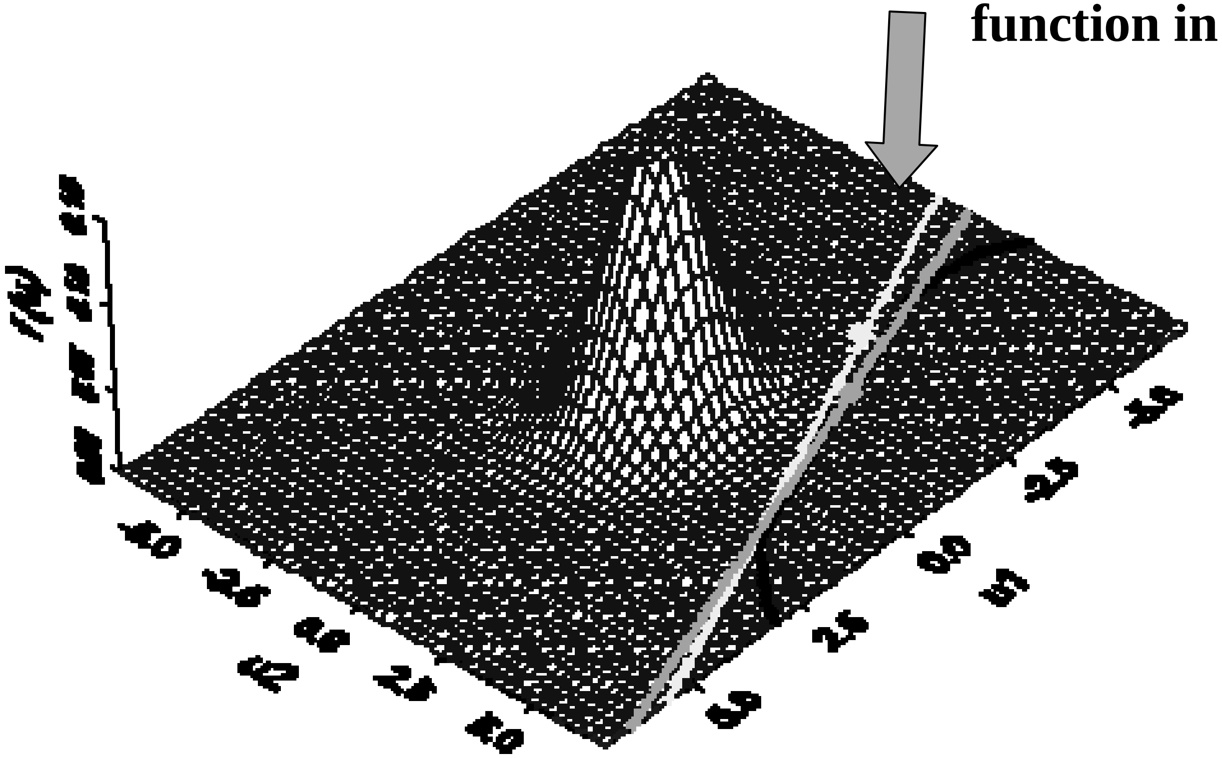


Basics of Structural Reliability Methods



Calculation of new design point X^2

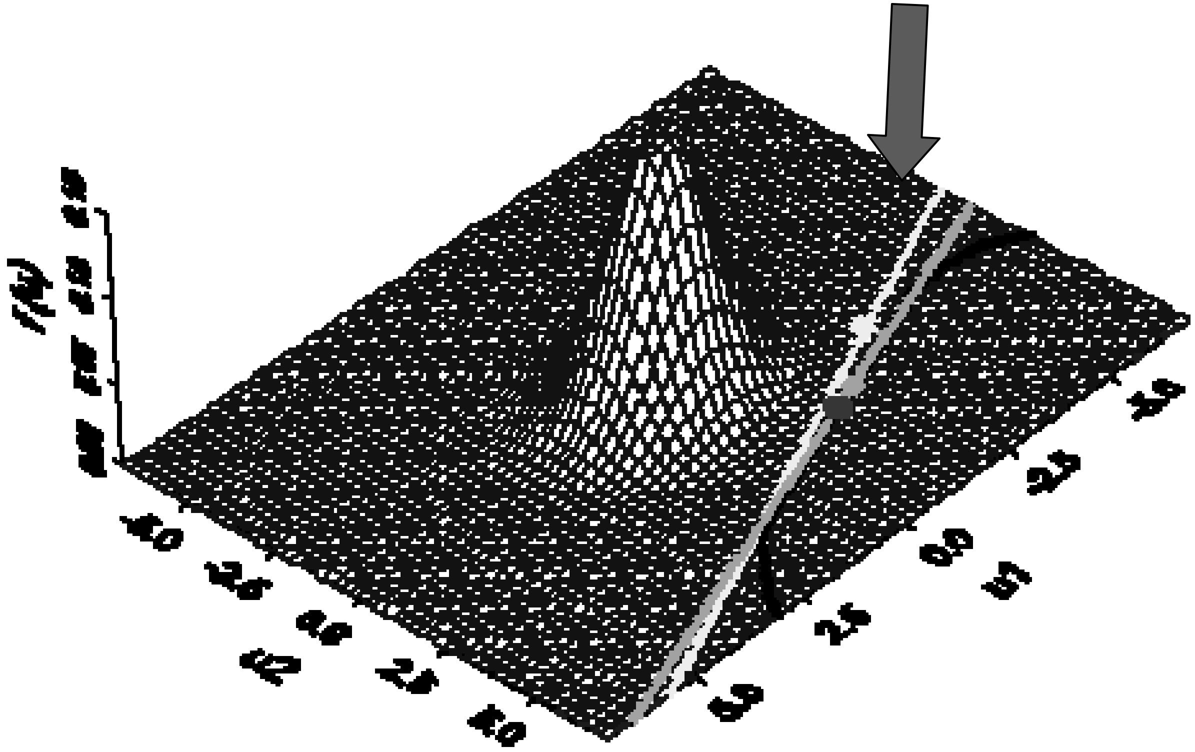
Basics of Structural Reliability Methods



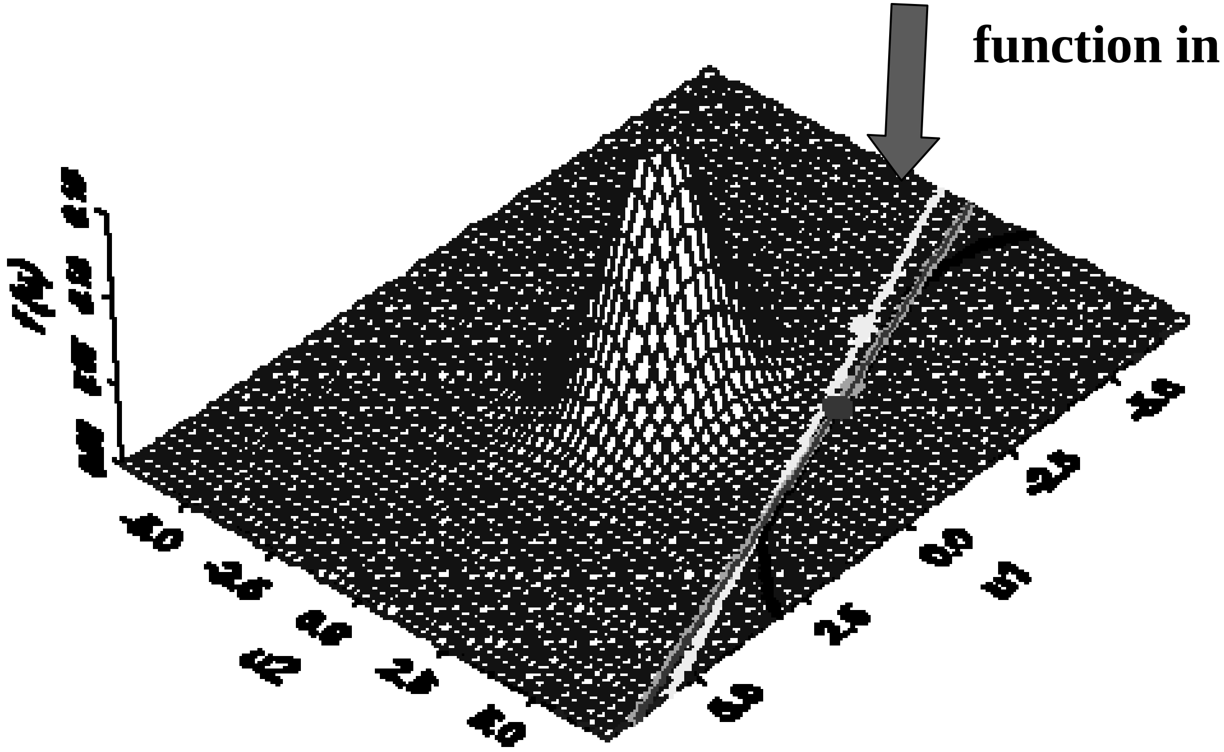
Linearisation of Limit state function in X^2

Basics of Structural Reliability Methods

Calculation of new design point X^3

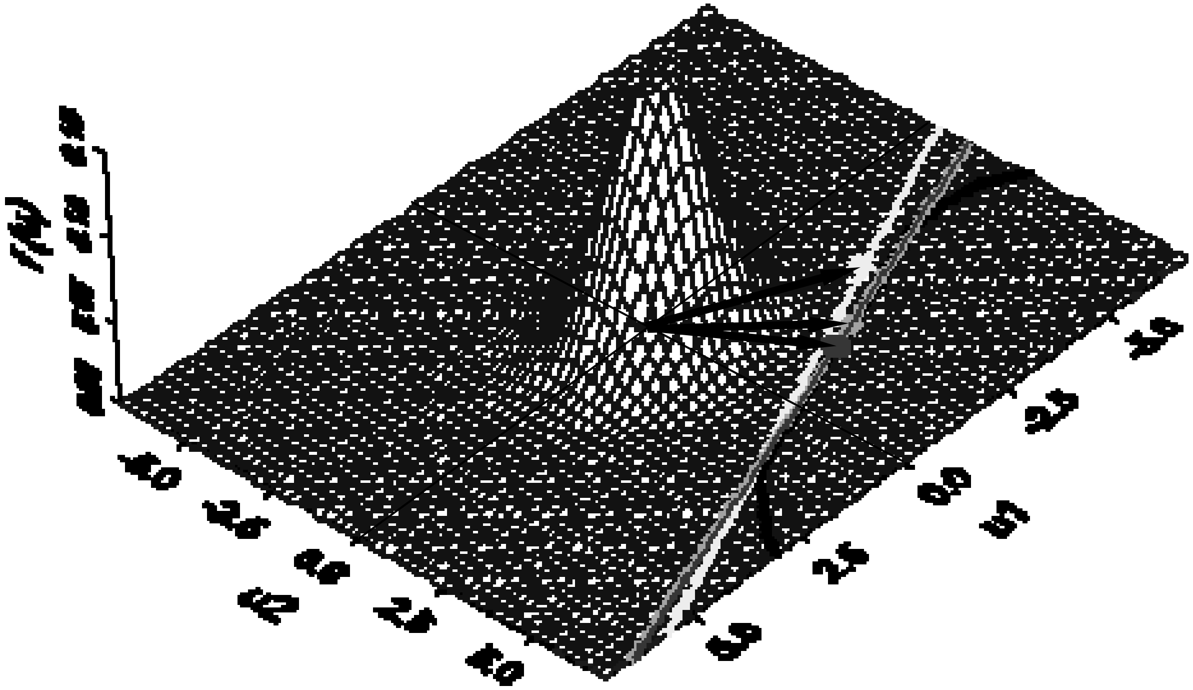


Basics of Structural Reliability Methods



Linearization of Limit state function in X^3

Basics of Structural Reliability Methods



β^1=3.556

β^2=3.607

β^3=3.608

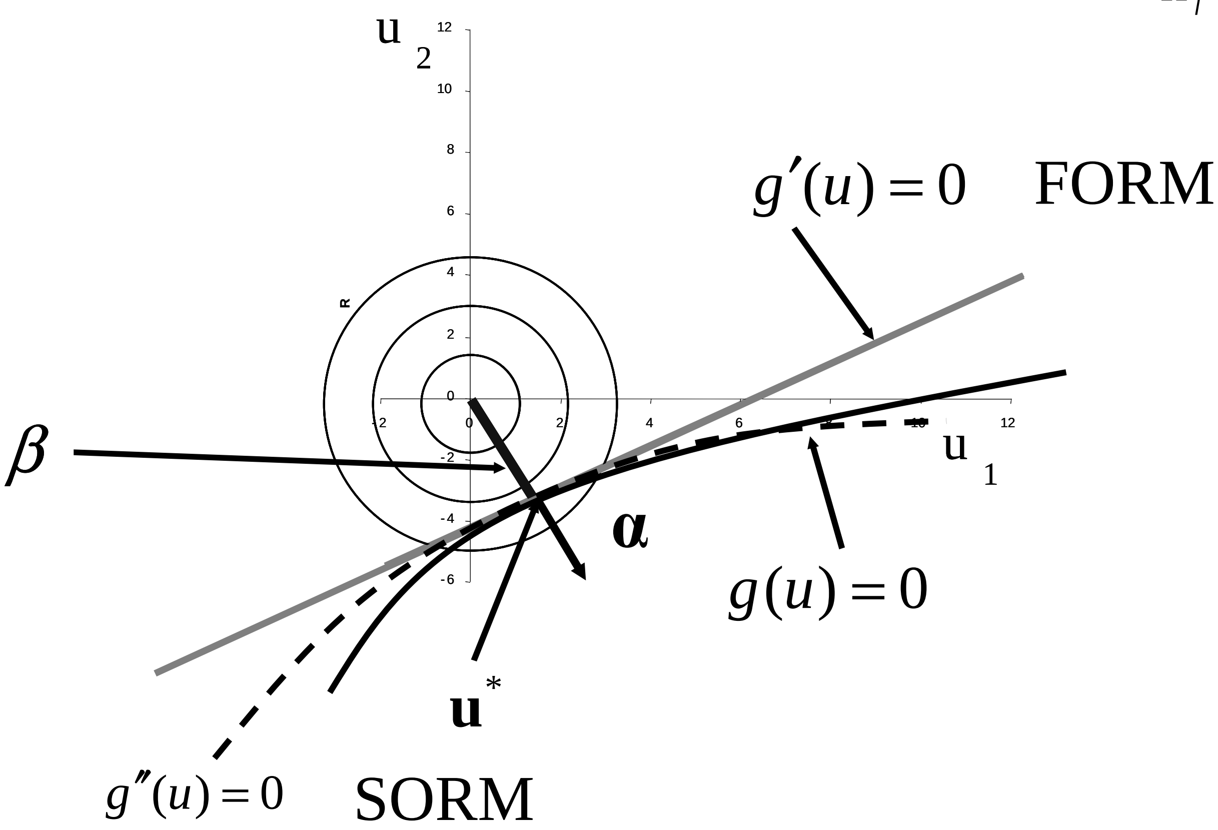
β^4=3.608

Convergency Criteria: $\Delta\beta = \left|\beta^{n+1} - \beta^{n}\right| \leq \varepsilon$

Basics of Structural Reliability Methods

SORM Improvements

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \le 0\}} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$$

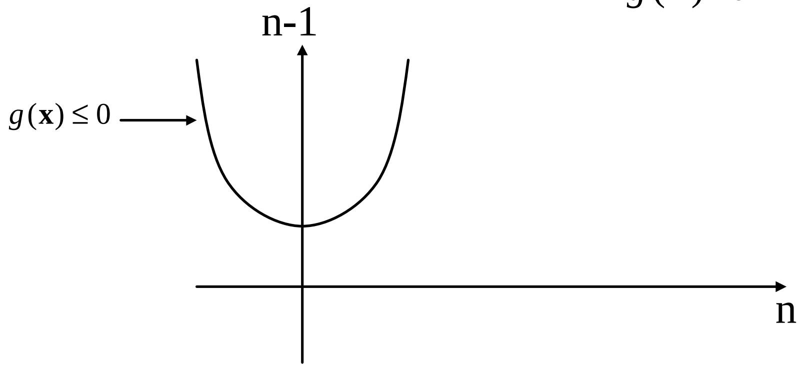


Basics of Structural Reliability Methods

$$I = \int_{g(\mathbf{x})\le 0} e^{\lambda h(\mathbf{x})} d\mathbf{x}$$

SORM Improvements

Asymptotic Laplace integral solutions



$$I = \int_{g(\mathbf{x})\le 0} e^{\lambda h(\mathbf{x})} d\mathbf{x} \approx \frac{(2\pi)^{(n-1)/2}\lambda^{-(n+1)}}{\lambda^{(n+1)}\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}}$$

Main curvatures

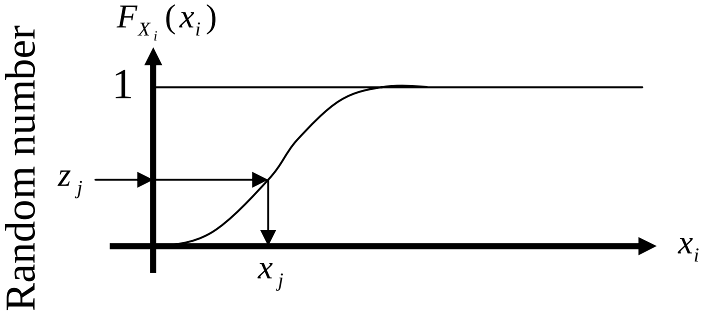
$$P_f = \int_{g(\mathbf{x})\le 0} \frac{e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}}{\left(\sqrt{2\pi}\right)^n} d\mathbf{x} \approx \frac{e^{-\beta^2/2}}{\sqrt{2\pi}\beta\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}} = \frac{\varphi(-\beta)}{\beta\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}} \cong \frac{\Phi(-\beta)}{\sqrt{\prod_{i=1}^{n-1}(1-\kappa_i)}}$$

Basics of Structural Reliability Methods

Simulation methods may also be used to solve the integration problem

$$P_f = \int_{\Omega_f = \{g(\mathbf{x}) \le 0\}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

1) *m* realizations of the vector X are generated



2) for each realization the value of the limit state function is evaluated

3) the realizations where the limit state function is zero or negative are counted

$$n_f$$

4) The failure probability is estimated as

$$p_f = \frac{n_f}{m}$$

Basics of Structural Reliability Methods

- **Estimation of failure probabilities using Monte Carlo Simulation**

  - ***m* random outcomes of R und S are generated and the number of outcomes n_f in the failure domain are recorded and summed**
  - **The failure probability p_f is then**

$$p_f = \frac{n_f}{m}$$