Exercise 1: Identity between strong form and weak form

In what follows an "informal proof" for the identity of the strong form and the weak form is given.

(W)

$EIu_{xxxx} = 0, \quad x \in [0,1]$ (1)

$EIu_{xxx}(0) = Q$ (2)

$EIu_{xx}(0) = 0$ (3)

$u(1) = 0$ (4)

$u_x(1) = 0$ (5)

(S)

$\int_0^1 w_{xx} EIu_{xx} dx = w(0)Q$ (6)

$u(1) = 0$ (4)

$u_x(1) = 0$ (5)

(W)→(S)

Let w be a proper function which satisfies:

$w(1) = 0$ (7)

$w_x(1) = 0$ (8)

From equation (1) if follows:

$$
\begin{aligned}
0 &= \int_0^1 wEIu_{xxxx} dx \\
&= wEIu_{xxx}\Big|_0^1 - \int_0^1 w_x EIu_{xxx} dx \\
&= -w(0)Q - w_x EIu_{xx}\Big|_0^1 + \int_0^1 w_{xx} EIu_{xx} dx \quad (\because (2),(7)) \\
&= -w(0)Q + \int_0^1 w_{xx} EIu_{xx} dx \quad (\because (3),(8))
\end{aligned}
$$

Therefore, equation (6) follows.

(S)→(W)

From equation (6), it follows:

$$
\begin{aligned}
&\int_0^1 w_{xx} EIu_{xx} dx = w(0)Q \\
&\Rightarrow w_x EIu_{xx}\Big|_0^1 - \int_0^1 w_x EIu_{xxx} dx = w(0)Q \\
&\Rightarrow -w_x(0)EIu_{xx}(0) - wEIu_{xxx}\Big|_0^1 + \int_0^1 wEIu_{xxxx} dx = w(0)Q \quad (\because (8)) \\
&\Rightarrow \int_0^1 wEIu_{xxxx} dx + w(0)\left(EIu_{xxx}(0) - Q\right) - w_x(0)EIu_{xx}(0) = 0 \quad (\because (7))
\end{aligned}
$$

Since w is an almost arbitrary function, equation (1) to (3) must hold.