FEM Exercise 1
1-D Cantilever

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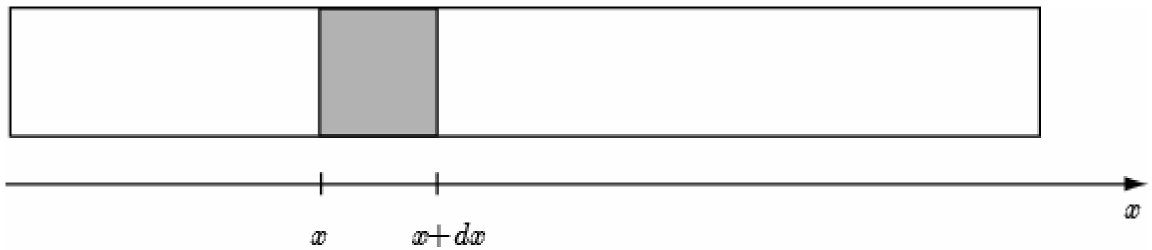
Flow of today's exercise

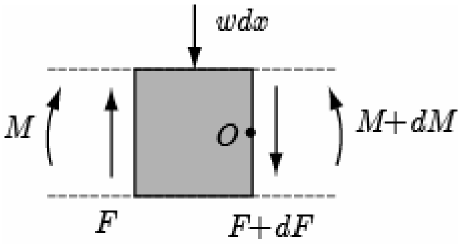
- Phenomena to model
- Strong form and weak form
- Matrix equation

Phenomena to model

Phenomena



Moment and shear force



Balance of force at point *O*.

$$F = (F + dF) + wdx$$

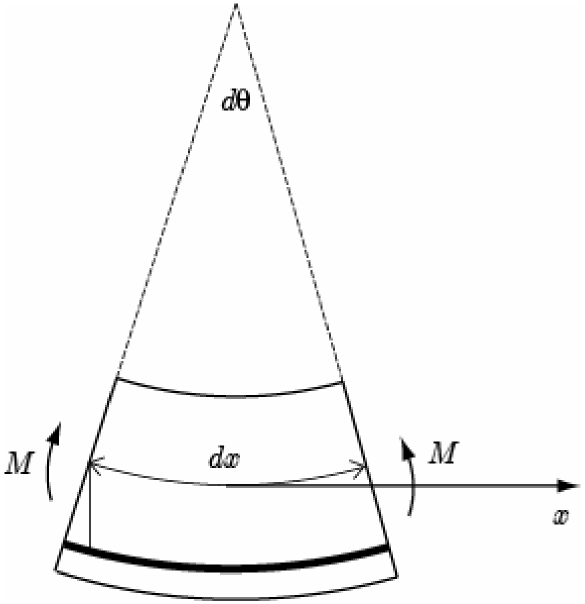
$$\Leftrightarrow \frac{dF}{dx} = -w$$

Balance of moment at point *O*.

$$M + Fdx - wdx \cdot \frac{dx}{2} = M + dM$$

$$\Leftrightarrow \frac{dM}{dx} = F$$

Moment and curvature



Strain

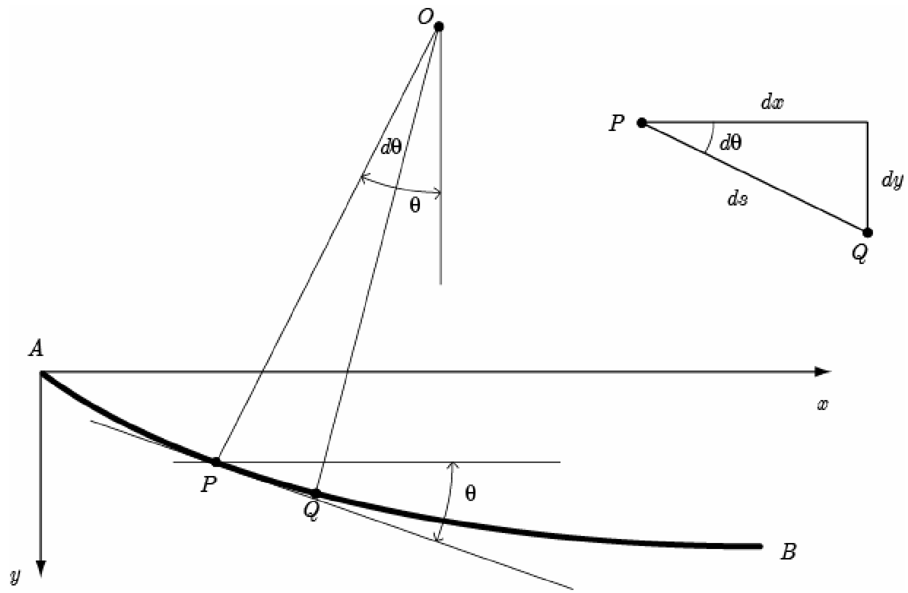
$$\varepsilon = \frac{y d\theta}{dx} = \frac{y}{\rho}, \quad \text{where } \rho = \frac{dx}{d\theta}$$

Stress

$$\sigma = E\varepsilon = E\frac{y}{\rho}$$

$$M = \int_A y\sigma dA = \frac{E}{\rho}\int_A y^2 dA = \frac{EI}{\rho}, \quad \textit{where } I = \int_A y^2 dA$$

Deflection curve



$$ds = -\rho d\theta \Leftrightarrow \frac{1}{\rho} = -\frac{d\theta}{ds}$$

$$\frac{d\theta}{dx} = \frac{d}{dx}\left\{\tan^{-1}\left(\frac{dy}{dx}\right)\right\} = \frac{d^2y/dx^2}{1+(dy/dx)^2}$$

$$\frac{dx}{ds} = \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+(dy/dx)^2}}$$

$$\frac{1}{\rho} = -\frac{d\theta}{dx}\cdot\frac{dx}{ds} = \frac{-d^2y/dx^2}{\left\{1+(dy/dx)^2\right\}^{3/2}} \approx -\frac{d^2y}{dx^2}$$

Model

$$\left(-\frac{1}{\rho}=\right)\frac{d^2y}{dx^2}=-\frac{M}{EI}\Leftrightarrow EI\frac{d^2y}{dx^2}=-M\Leftrightarrow EI\frac{d^3y}{dx^3}=-F\Leftrightarrow EI\frac{d^4y}{dx^4}=w$$

Strong form and weak form

Strong form

(S)

Given constant Q, find $u:[0,1] \to \mathbb{R}$ such that

$$EIu_{xxxx} = 0 \text{ on } [0,1]$$

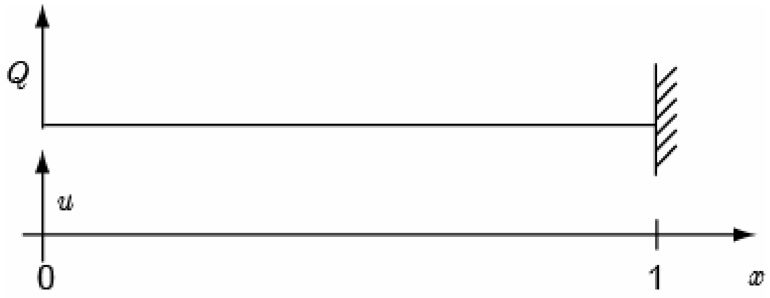
$$u(1) = 0$$

$$u_x(1) = 0$$

$$EIu_{xx}(0) = 0$$

$$EIu_x(0) = Q$$

Weak form (Galerkin method)



(W)

Given constant Q, find u^h such that

for all $w^h \in \left\{ w^h \mid w^h(1) = w^h{}_x(1) = 0 \right\}$

$$\int_0^1 w^h{}_{xx} EI u^h{}_{xx} dx = w^h(0) Q$$

$$u^h(1) = 0$$

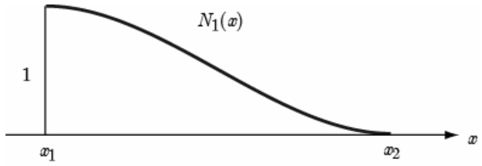
$$u^h{}_x(1) = 0$$

Strong form → weak form (exercise)

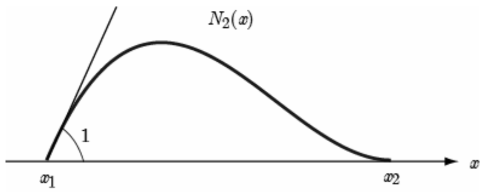
Strong form ← weak form (exercise)

Matrix equation

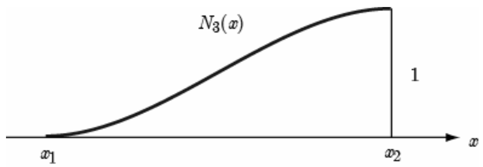
Test functions (or weighting functions, interpolation functions)



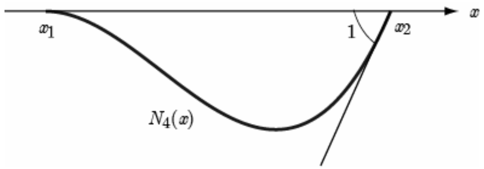
$$N_1(x) = \frac{-(x-x_2)\left[-h+2(x_1-x)\right]}{h^3}$$



$$N_2(x) = \frac{(x-x_1)(x-x_2)^2}{h^2}$$



$$N_3(x) = \frac{(x-x_1)^2\left[h+2(x_2-x)\right]}{h^3}$$



$$N_4(x) = \frac{(x-x_1)^2(x-x_2)}{h^2}$$

How to construct test functions? (exercise)

Superposition

$$w^h = \mathbf{c}^t \cdot \mathbf{N}$$
$$u^h = \mathbf{d}^t \cdot \mathbf{N}$$
where
$$\mathbf{c} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}^t$$
$$\mathbf{d} = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \end{bmatrix}^t$$
$$\mathbf{N} = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix}^t$$

Matrix equation (Number of element = 1)

$$\int_0^1 w^h{}_{xx} EI u^h{}_{xx} dx = w^h(0)Q$$

$$\Leftrightarrow \int_0^1 \mathbf{c}^t \cdot \mathbf{N}_{xx} EI \mathbf{d}^t \cdot \mathbf{N}_{xx} dx = \mathbf{c}^t \cdot \mathbf{N}(0)Q$$

$$\Leftrightarrow \int_0^1 \mathbf{c}^t \cdot \mathbf{N}_{xx} EI \mathbf{N}^t{}_{xx} \cdot \mathbf{d} dx = \mathbf{c}^t \cdot \mathbf{N}(0)Q$$

$$\Leftrightarrow \mathbf{c}^t \int_0^1 \mathbf{N}_{xx} EI \mathbf{N}^t{}_{xx} dx \mathbf{d} = \mathbf{c}^t \cdot \mathbf{N}(0)Q$$

$$\Leftrightarrow \mathbf{c}^t \left[\int_0^1 \mathbf{N}_{xx} EI \mathbf{N}^t{}_{xx} dx \cdot \mathbf{d} - \mathbf{N}(0)Q \right] = 0$$

$$\underbrace{\int_0^1 \mathbf{N}_{xx} EI \mathbf{N}^t{}_{xx} dx}_{\mathbf{K}} \cdot \underbrace{\mathbf{d}}_{\mathbf{d}} = \underbrace{\mathbf{N}(0)Q}_{\mathbf{f}}$$

Writing down elements of matrix

$$\mathbf{K} = EI \begin{bmatrix} \int_0^1 N_{1xx} N_{1xx} dx & \int_0^1 N_{1xx} N_{2xx} dx & \int_0^1 N_{1xx} N_{3xx} dx & \int_0^1 N_{1xx} N_{4xx} dx \\ \int_0^1 N_{2xx} N_{1xx} dx & \int_0^1 N_{2xx} N_{2xx} dx & \int_0^1 N_{2xx} N_{3xx} dx & \int_0^1 N_{2xx} N_{4xx} dx \\ \int_0^1 N_{3xx} N_{1xx} dx & \int_0^1 N_{3xx} N_{2xx} dx & \int_0^1 N_{3xx} N_{3xx} dx & \int_0^1 N_{3xx} N_{4xx} dx \\ \int_0^1 N_{4xx} N_{1xx} dx & \int_0^1 N_{4xx} N_{2xx} dx & \int_0^1 N_{4xx} N_{3xx} dx & \int_0^1 N_{4xx} N_{4xx} dx \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \qquad \mathbf{f} = Q \begin{bmatrix} N_1(0) \\ N_2(0) \\ N_3(0) \\ N_4(0) \end{bmatrix}$$

Essential boundary conditions

$$\left[\begin{array}{cc|cc}
\int_0^1 N_{1xx}N_{1xx}dx & \int_0^1 N_{1xx}N_{2xx}dx & \int_0^1 N_{1xx}N_{3xx}dx & \int_0^1 N_{1xx}N_{4xx}dx \\
\int_0^1 N_{2xx}N_{1xx}dx & \int_0^1 N_{2xx}N_{2xx}dx & \int_0^1 N_{2xx}N_{3xx}dx & \int_0^1 N_{2xx}N_{4xx}dx \\
\hline
\int_0^1 N_{3xx}N_{1xx}dx & \int_0^1 N_{3xx}N_{2xx}dx & \int_0^1 N_{3xx}N_{3xx}dx & \int_0^1 N_{3xx}N_{4xx}dx \\
\int_0^1 N_{4xx}N_{1xx}dx & \int_0^1 N_{4xx}N_{2xx}dx & \int_0^1 N_{4xx}N_{3xx}dx & \int_0^1 N_{4xx}N_{4xx}dx
\end{array}\right]
\left[\begin{array}{c} d_1 \\ d_2 \\ \hline 0 \\ 0 \end{array}\right]
= \frac{Q}{EI}\left[\begin{array}{c} N_1(0) \\ N_2(0) \\ \hline N_3(0) \\ N_4(0) \end{array}\right]$$

Solving equation

$$\begin{bmatrix} \int_0^1 N_{1xx}N_{1xx}dx & \int_0^1 N_{1xx}N_{2xx}dx \\ \int_0^1 N_{2xx}N_{1xx}dx & \int_0^1 N_{2xx}N_{2xx}dx \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{Q}{EI} \begin{bmatrix} N_1(0) \\ N_2(0) \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{Q}{EI} \begin{bmatrix} \int_0^1 N_{1xx}N_{1xx}dx & \int_0^1 N_{1xx}N_{2xx}dx \\ \int_0^1 N_{2xx}N_{1xx}dx & \int_0^1 N_{2xx}N_{2xx}dx \end{bmatrix}^{-1} \begin{bmatrix} N_1(0) \\ N_2(0) \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{Q}{EI} \begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{Q}{6EI} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$u^h(x) = [d_1 \quad d_2] \begin{bmatrix} N_1(x) \\ N_2(x) \end{bmatrix} = \frac{Q}{6EI} [2 \quad -3] \begin{bmatrix} 2x^3 - 3x^2 + 1 \\ x^3 - 2x^2 + x \end{bmatrix} = \frac{Q}{6EI} \left(x^3 - 3x + 2 \right)$$

Further consideration (No. of element = 2)