Bayesian Modelling for Spatial Data

“Hierarchical Modeling and Analysis for Spatial Data”

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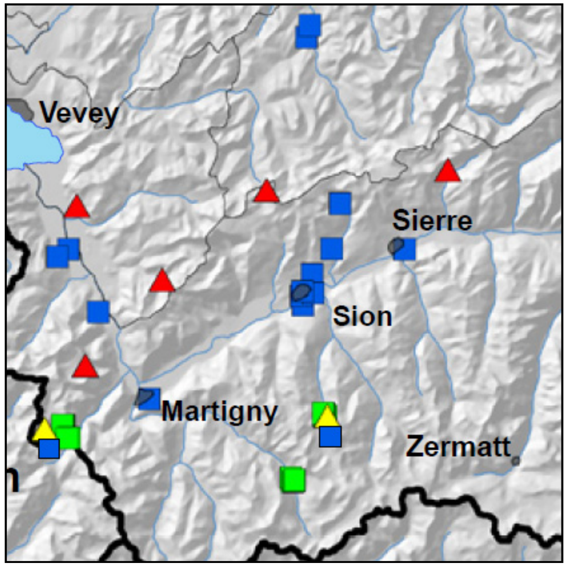
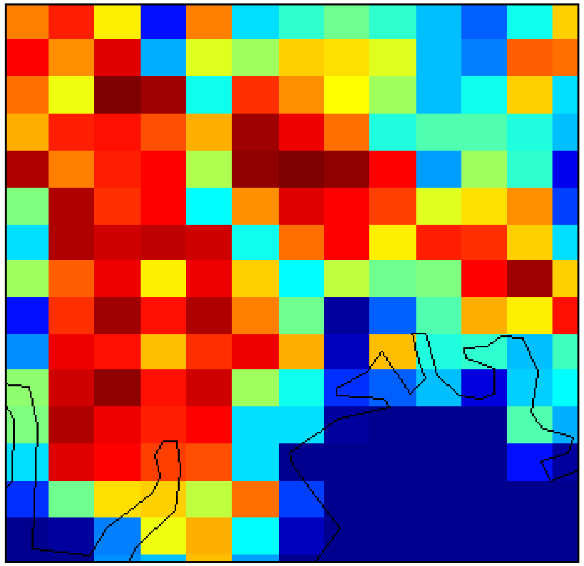
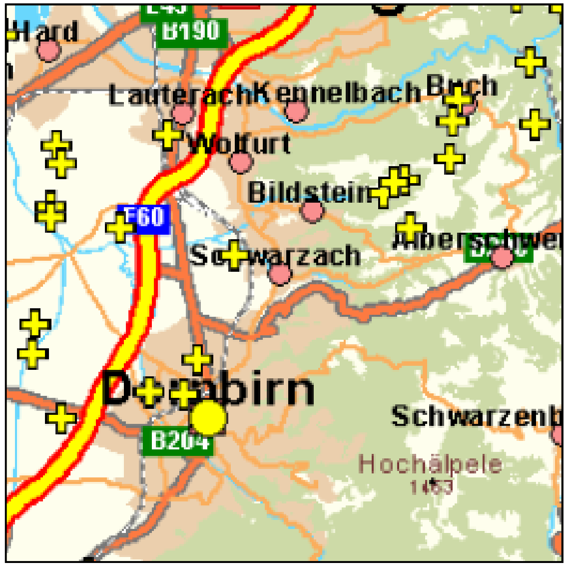
Introduction

- Types of spatial data, Problem setting
- Important properties of random fields
- Introduction to classical approach
- Basic model for homogenous random field
- Bayesian modelling: Inference and prediction
- Non-hierarchical vs. hierarchical approach
- Generalizations

3 Types of spatial data sets:







Point-referenced data

Known locations

$$Y(\mathbf{s}), \quad \mathbf{s} \in \Re^r$$

Aereal data

Aeral units ("blocks")

$$Y(B_i), \quad B_i \in D$$

Point pattern data

Random locations

$$Y(\mathbf{s}) = 1, \quad \mathbf{s} \in D$$

Objectives:

- Modelling the spatial variation of a random Variable $Y(\mathbf{s})$ at locations $\mathbf{s} \in D$.
- Description of trends and spatial correlation.
- Prediction for unobserved times / locations.

Examples:

- Soil properties on a construction site.
- Chloride concentration on a concrete surface.

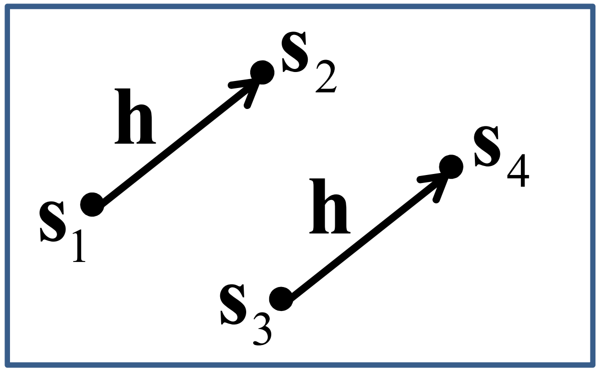
Stationarity

- Strict Stationarity:

$$dist(Y(\mathbf{s}_1),\ldots,Y(\mathbf{s}_n)) = dist(Y(\mathbf{s}_1+\mathbf{h}),\ldots,Y(\mathbf{s}_n+\mathbf{h})) \quad \forall\,\mathbf{h}\in\Re^r$$

- Weak Stationarity:

$$\mu(\mathbf{s}) = const. \qquad Cov(Y(\mathbf{s}),Y(\mathbf{s}+\mathbf{h})) = C(\mathbf{h})$$



Homogeneity

- Intrinsic Stationarity

$$E[Y(\mathbf{s}+\mathbf{h})-Y(\mathbf{s})]=0$$

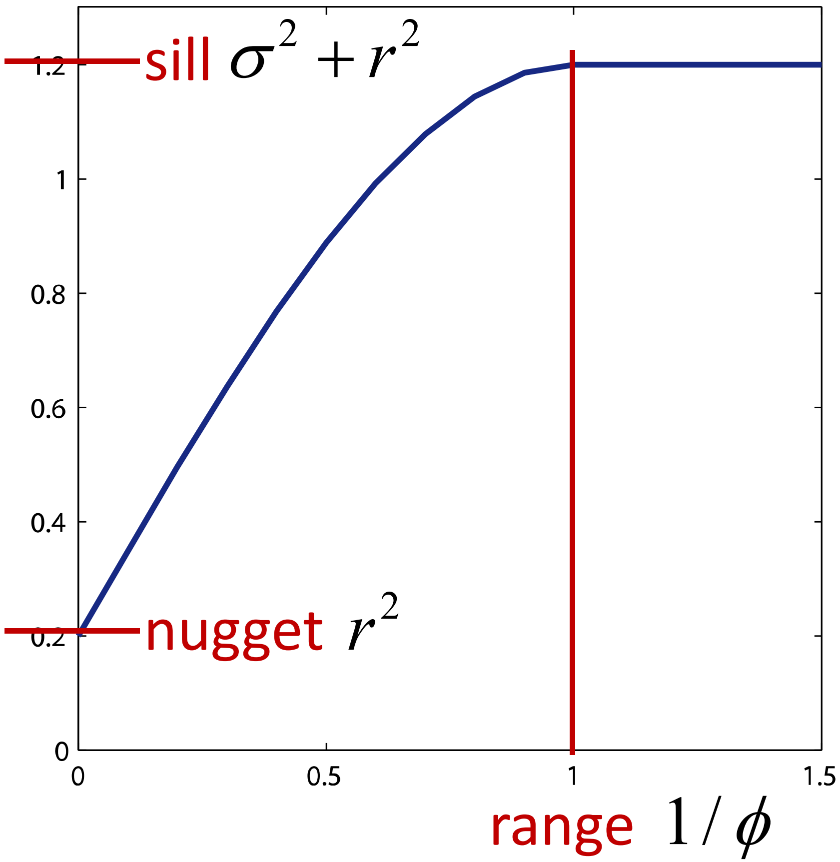
$$Var[Y(\mathbf{s}+\mathbf{h})-Y(\mathbf{s})]=2\gamma(\mathbf{h})$$ Variogram

- Isotropy

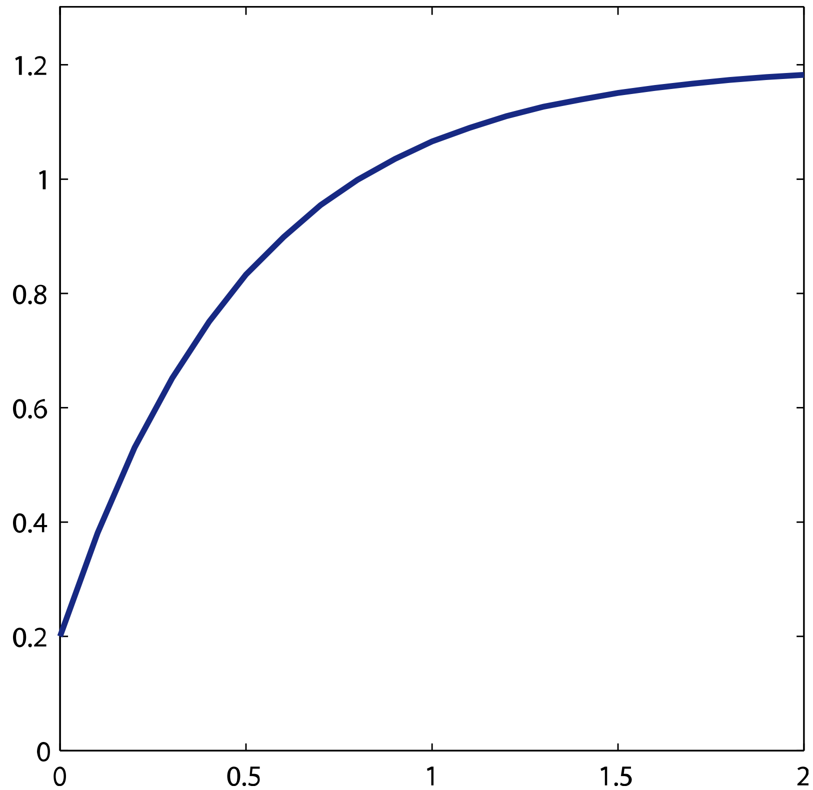
$$\gamma(\mathbf{h})=\gamma(\|\mathbf{h}\|)$$

⇨ Simple parametric variograms can be used to model spatial correlation structure

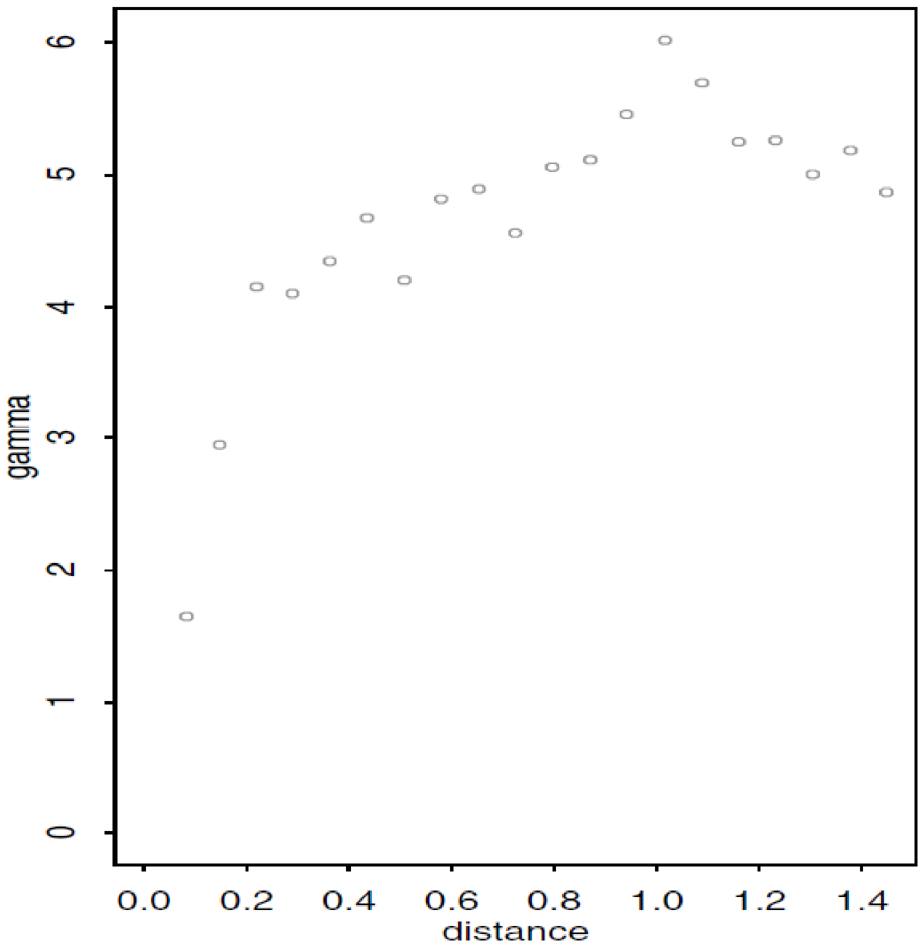
Spherical semivariogram



Exponential semivariogram



Empirical semivariograms



Empirical semivariogram:

$$\hat{\gamma}(t) = \frac{1}{2N(t)} \sum_{(\mathbf{s}_i, \mathbf{s}_j \notin N(t))} [Y(\mathbf{s}_i) - Y(\mathbf{s}_j)]^2$$

- Similar to histogram construction.
- Sensitive to outliers.
- $N(t)$ varies across bins.

Objective:

- Spatial prediction given data:

  Predict $Y(\mathbf{s}_0)$ given $\mathbf{Y} = (Y(\mathbf{s}_1), \ldots, Y(\mathbf{s}_1))$

Approach:

- Classical Kriging: Minimum mean-squared error estimate.
- Bayesian Kriging: Specify prior distribution for model parameters, Bayesian updating .

Example of a simple spatial process

Basic Model:

$$Y(\mathbf{s}) = \mu(\mathbf{s}) + \underbrace{\omega(\mathbf{s}) + \varepsilon(\mathbf{s})}$$

Spatial & nonspatial variation

Assumptions:

- Gaussian spatial process: $\mathbf{Y} \sim MVN$
- Homogeneity (stationarity and isotropy).
- No covariates.

Example of a simple spatial process

Basic Model:

$$Y(\mathbf{s}) = \mu(\mathbf{s}) + \omega(\mathbf{s}) + \varepsilon(\mathbf{s})$$

$$\mu(\mathbf{s}) = \mathbf{x}^T(\mathbf{s})\boldsymbol{\beta} \equiv \mu \quad const.$$

$$\omega(\mathbf{s}) = f(\sigma^2, \phi)$$ Partial sill, range

$$\varepsilon(\mathbf{s}) = f(r^2)$$ Nugget

$$\mathbf{Y}\,|\,\boldsymbol{\theta} \sim MVN(\mu, \underbrace{\sigma^2 H(\phi)} + \underbrace{r^2 I})$$

Spatial & nonspatial variation

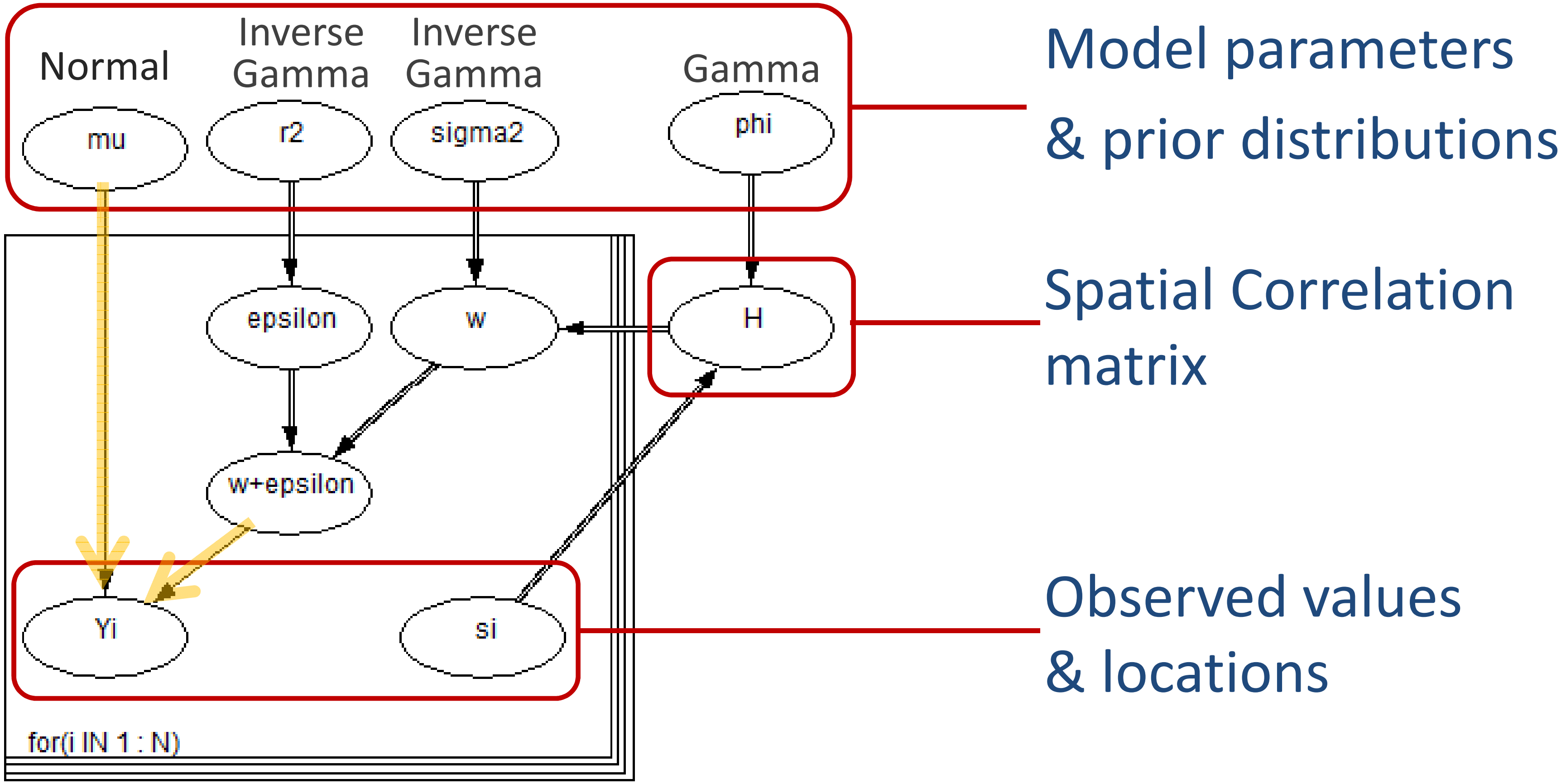
with correlation matrix

$$H_{ij} = \rho(\mathbf{s}_i - \mathbf{s}_j; \phi) = \exp(-\phi \|\mathbf{s}_i - \mathbf{s}_j\|)$$

(here: exponential model)

and prior distributions for μ , σ^2, ϕ and r^2

$$\mathbf{Y}|\boldsymbol{\theta} \sim MVN(\mu, \sigma^2 H(\phi) + r^2 I)$$



Bayesian modelling - Inference

likelihood prior

posterior
$$\pi_1(\boldsymbol{\theta}|\mathbf{y}) = \frac{f(\mathbf{y}|\boldsymbol{\theta})\pi_0(\boldsymbol{\theta})}{\int f(\mathbf{y}|\boldsymbol{\theta})\pi_0(\boldsymbol{\theta})\,d\boldsymbol{\theta}}$$

$$\begin{aligned}
\pi_1(\mu,\sigma^2,\phi,r^2|\mathbf{y}) & \\
&\propto f(\mathbf{y}|\mu,\sigma^2,\phi,r^2)\pi_0(\mu,\sigma^2,\phi,r^2) \\
&= f(\mathbf{y}|\mu,\sigma^2,\phi,r^2)\pi_0(\mu)\pi_0(\sigma^2)\pi_0(\phi)\pi_0(r^2)
\end{aligned}$$

With the posterior for $\boldsymbol{\theta}$, predictions $Y(\mathbf{s}_0)$ at new locations $\mathbf{s}_0$ are straight-forward:

$$f(y_0|\mathbf{y}) = \int f(y_0, \boldsymbol{\theta}|\mathbf{y})d\boldsymbol{\theta} = \int f(y_0|\mathbf{y}, \boldsymbol{\theta})\pi_1(\boldsymbol{\theta}|\mathbf{y})d\boldsymbol{\theta}$$

The spatial correlation matrix needs to be extended to include the new locations

$$\mathbf{Y}, \mathbf{Y_0}|\boldsymbol{\theta} \sim MVN(\mu, \sigma^2 H(\phi) + r^2 I)$$

$$\mathbf{Y} \big| \boldsymbol{\theta}, \mathbf{W} \sim MVN(\mu + \mathbf{W}, r^2 I)$$

with

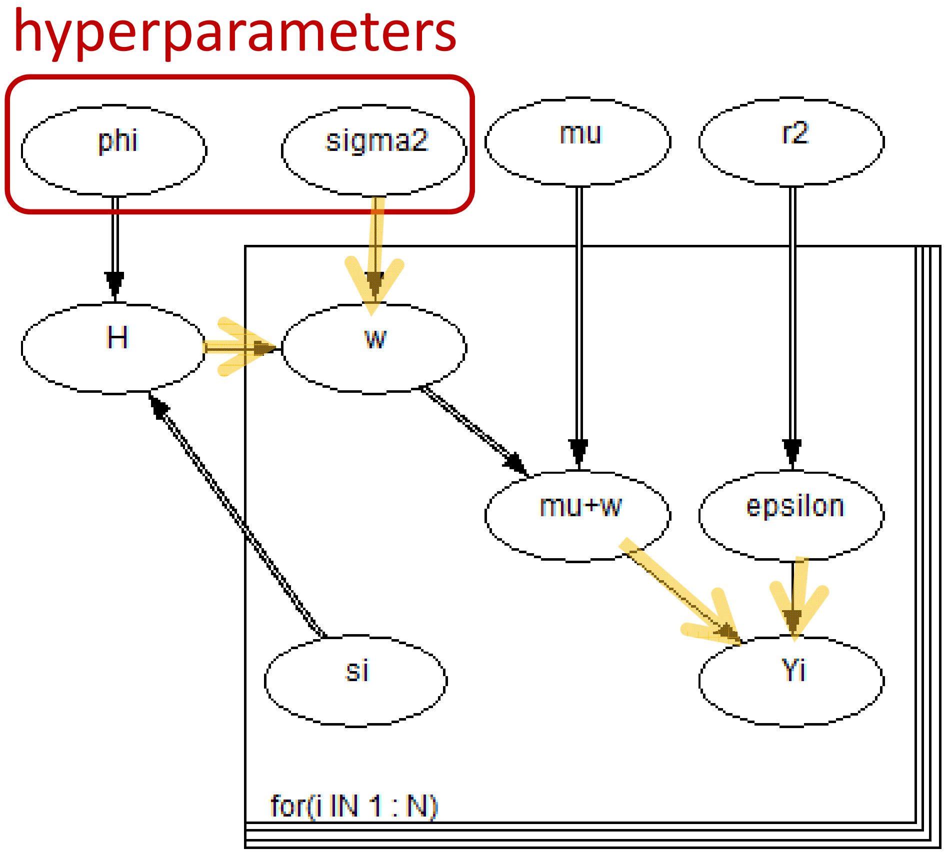
$$\mathbf{W} \big| \sigma^2, \phi \sim MVN(\mathbf{0}, \sigma^2 H(\phi))$$

with prior distributions for μ and r^2

and for the hyperparameters σ^2 and ϕ

$$\mathbf{Y} \mid \boldsymbol{\theta}, \mathbf{W} \sim MVN(\mu + \mathbf{W}, r^2 I)$$

$$\mathbf{W} \mid \sigma^2, \phi \sim$$
$$MVN(\mathbf{0}, \sigma^2 H(\phi))$$



Spatial dependence is modelled on the second hierarchical level

Hierarchical bayesian modelling

Hierarchical modelling:

Specification of priors in two stages

hyperparameters

$$\pi_1(\mu, r^2, \sigma^2, \phi \mid \mathbf{y})$$

$$\propto \int \underbrace{f(\mathbf{y} \mid \mu, r^2, \mathbf{W})}\, \underbrace{\pi_0(\mu, r^2)}_{\text{"normal" prior}}\, \underbrace{f(\mathbf{W} \mid \sigma^2, \phi)}_{\text{first-stage prior}}\, \underbrace{\pi_0(\sigma^2, \phi)}_{\text{hyper-prior}}\, d\mathbf{W}$$

Modelling in WinBUGS

- Model can be directly implemented using stochastic and logical links.
- Faster: GeoBUGS function “spatial”: Bayesian Gaussian kriging model

Try yourself! Code will be provided by email

WinBUGS:

http://www.mrc-bsu.cam.ac.uk/bugs/winbugs/contents.shtml

http://www.mrc-bsu.cam.ac.uk/bugs/winbugs/winbugsthemovie.html

What is the difference?

- **Mathematically:** No difference.

- **Computationally:** Hierarchical model is more demanding.

- **Conceptually:** Direct (first-stage) modelling of spatial dependence in the error term versus spatial modelling in the mean of the process

Generalizations

- Include covariates in the mean term (no constant mean).
- Model anisotrophy with the aid of more complex Correlation functions.
- Transform linear predictor / data for more general use.

- Generalized linear spatial process modelling.

Generalized Linear Models

Extend linear modelling to non-normal setting.

Idea:

- The outcomes Y_i are conditionally independent given the distribution parameters μ_i and ψ.
- $E(Y_i \mid \mathbf{X}_i) = \mu_i$ is dependent on a set of explanatory variables, $\mathbf{X}_i$.

$$p(\mathbf{y} \mid \mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\psi}) = \prod_{i=1}^{n} p(y_i \mid (\mathbf{X}\boldsymbol{\beta})_i, \psi)$$

Specification in 3 stages:

- Probability distribution for the outcome variable Y with mean $E[y|X]=\mu$.
- Link function: $g(\cdot)$; $\mu = g^{-1}(\eta) = g^{-1}(X\beta)$
- Linear predictor: $\eta = X\beta$

$$p(\mathbf{y}|\mathbf{X},\boldsymbol{\beta},\boldsymbol{\psi}) = \prod_{i=1}^{n} p(y_i|\mu_i,\psi)$$

$$= \prod_{i=1}^{n} p(y_i|g^{-1}((\mathbf{X}\boldsymbol{\beta})_i),\psi)$$

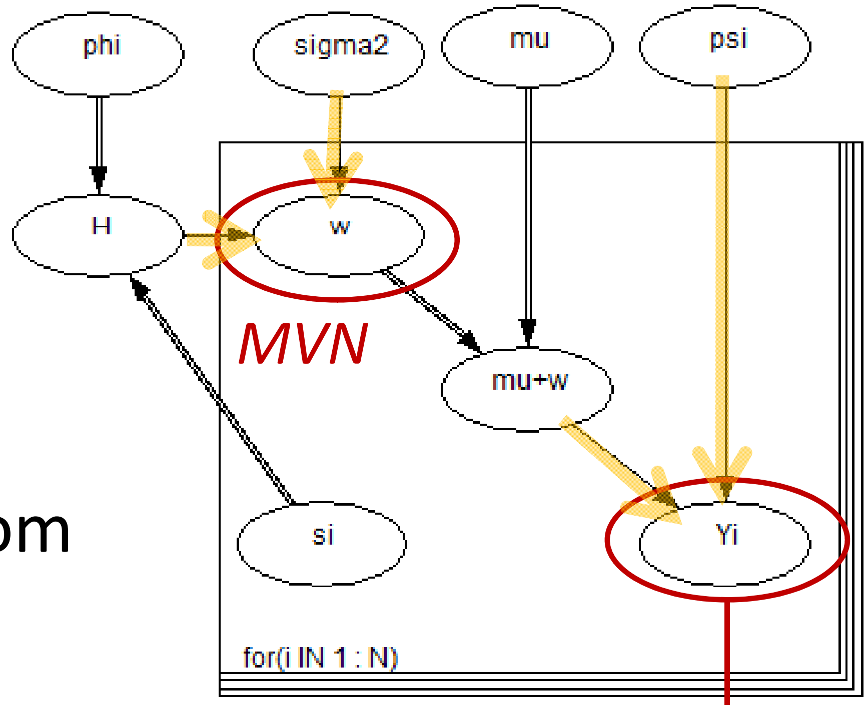
Generalized linear spatial process modelling

Model spatial dependence in the linear predictor:

$$\eta(\mathbf{s}_i) = \mathbf{x}^T(\mathbf{s})\boldsymbol{\beta} + w(\mathbf{s}_i)$$

$$\mathbf{W} \sim MVN(0, \sigma^2 H(\phi))$$

(here for homogenous random field without covariates, i.e. $\mathbf{x}^T(\mathbf{s})\boldsymbol{\beta} \equiv \mu$)



Any probability model in the exponential family

Binary data: $Y(\mathbf{s}_i) \in \{0;1\}$

$$\underbrace{P(Y(\mathbf{s}_i)=1 \mid p(\mathbf{s}_i)) = p(\mathbf{s}_i)}_{\text{probability model}} = \underbrace{\operatorname{logit}^{-1}}_{\text{link}}\underbrace{(\mathbf{x}^T(\mathbf{s})\boldsymbol{\beta} + w(\mathbf{s}_i))}_{\text{linear predictor}}$$

$$\mathbf{W} \sim MVN(0, \sigma^2 H(\phi))$$

$$\underbrace{\operatorname{logit}^{-1}(u) = \frac{e^u}{1+e^u}}_{\text{transform continuous variable to (0;1)}}$$

⇨Spatial dependence is modelled in the mean of the process.

Summary

- Random fields as a means to model spatial variation and correlation.
- Modelling with a Bayesian approach: Direct vs. hierarchical.
- Hierarchical modelling not necessary with a Gaussian spatial process.
- Hierarchical approach facilitates further generalizations

THANKS!