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Finites Element Procedure, Klaus-Jürgen Bathe

1. Mode Superposition (damping neglected)
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9.3 Mode Superposition (damping neglected)

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Direct Integration Methods

From last time:

$$M\ddot{U} + C\dot{U} + KU = R$$

M : Mass matrix

C : Damping matrix

K : Stiffness matrix

U : Displacements

$\dot{U}$: Velocities

$\ddot{U}$: Accelerations

Can be solved with Direct Integration Methods

Problem:

We have to do a lot of calculations at each time step

Generalized Displacements

$$\mathbf{U}(t) = \mathbf{P}\mathbf{X}(t)$$

U(t): Nodal displacement vector
P: nxn Matrix
X(t): Generalized displacement vector

Multiply the equilibrium equation with P^T

$$\tilde{\mathbf{M}}\ddot{\mathbf{X}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{X}}(t) + \tilde{\mathbf{K}}\mathbf{X}(t) = \tilde{\mathbf{R}}(t)$$

$$\tilde{\mathbf{M}} = \mathbf{P}^T\mathbf{M}\mathbf{P}; \qquad \tilde{\mathbf{C}} = \mathbf{P}^T\mathbf{C}\mathbf{P}; \qquad \tilde{\mathbf{K}} = \mathbf{P}^T\mathbf{K}\mathbf{P}; \qquad \tilde{\mathbf{R}} = \mathbf{P}^T\mathbf{R}$$

We now have to find a Matrix P which simplifies the problem

Neglecting damping

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0}$$

The solution of the problems becomes then:

$$\mathbf{U} = \boldsymbol{\phi} \sin \omega(t - t_0)$$

$$\ddot{U} = -\phi\omega^2 \sin(\omega(t - t_0))$$

The equation $(K - \omega^2 M)\phi = 0$ can only be solved if the inverse $(K - \omega^2 M)^{-1}$ exists

therefore the determinate $\left|\mathbf{K} - \omega_n^2 \mathbf{M}\right|$ has to be = 0

Eigenproblem

$$\left|\mathbf{K} - \omega_n^2 \mathbf{M}\right| = \left|\begin{bmatrix} k_{11} - \omega_n{}^2 m_1 & k_{21} \\ k_{12} & k_{22} - \omega_n{}^2 m_2 \end{bmatrix}\right| = 0$$

because M is diagonal the $\omega_n{}^2$ are equal the Eigen values and ϕ_n is equal the Eigen vector

There are n solutions for a system with n degrees of freedom

$$K\phi_n = \omega_n{}^2 M\phi_n$$

Matrix form

$$K\phi_n = \omega_n{}^2 M\phi_n$$

We can rewrite this n equations in matrix form:

$$\mathbf{K\Phi} = \mathbf{M\Phi\Omega}^2$$

$$\mathbf{\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n]; \qquad \mathbf{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix}$$

Multiplied with ϕ^T:

$$\mathbf{\Phi}^T\mathbf{K\Phi} = \mathbf{\Phi}^T\mathbf{M\Phi\Omega}^2$$

Diagonal matrix

$\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}$ and $\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}$ are diagonal, meaning:

$\phi_{\mathbf{n}}^{\mathbf{T}}\mathbf{K}\phi_{\mathbf{r}} = 0$ and $\phi_{\mathbf{n}}^{\mathbf{T}}\mathbf{M}\phi_{\mathbf{r}} = 0$ if $n \neq r$

Because:

Since $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $K^T = K$ (because K is symmetric)
we have $(\phi_n{}^T K\phi_r)^T = \phi_r{}^T K\phi_n$ (the same for $\phi_{\mathbf{n}}^{\mathbf{T}}\mathbf{M}\phi_{\mathbf{r}}$)

$$\phi_{\mathbf{r}}^{\mathbf{T}}\mathbf{K}\phi_{\mathbf{n}} = \omega_n^2\phi_{\mathbf{r}}^{\mathbf{T}}\mathbf{M}\phi_{\mathbf{n}} \longrightarrow \phi_{\mathbf{n}}^{T}\mathbf{K}\phi_{\mathbf{r}} = \omega_n^2\phi_{\mathbf{n}}^{T}\mathbf{M}\phi_{\mathbf{r}}$$

subtracting: $\phi_n^{\mathbf{T}}\mathbf{K}\phi_r = \omega_r^2\phi_n^{\mathbf{T}}\mathbf{M}\phi_r$

$$\downarrow$$

$$(\omega_n^2 - \omega_r^2)\phi_n^{\mathbf{T}}\mathbf{M}\phi_r = 0$$

Choosing ϕ_n

a multiple of ϕ_n still fulfils $K\phi_n = {\omega_n}^2 M\phi_n$

so we can choose ϕ_n such that $\mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} = \mathbf{I}$

therefore we get $\mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} = \mathbf{\Omega}^2$

Now we define P = Φ

$$\mathbf{U}(t) = \mathbf{\Phi} \mathbf{X}(t)$$

$$\ddot{\mathbf{X}}(t) + \mathbf{\Omega}^2 \mathbf{X}(t) = \mathbf{\Phi}^T \mathbf{R}(t)$$

equilibrium equation

$$\ddot{\mathbf{X}}(t) + \mathbf{\Omega}^2 \mathbf{X}(t) = \mathbf{\Phi}^T \mathbf{R}(t)$$

since Ω^2 is diagonal the components of the vector X can be analytical calculated

$$\left.\begin{aligned} \ddot{x}_i(t) + \omega_i^2 x_i(t) &= r_i(t) \\ r_i(t) &= \boldsymbol{\phi}_i^T \mathbf{R}(t) \end{aligned}\right\} \qquad i = 1, 2, \ldots, n$$

$$\mathbf{U}(t) = \sum_{i=1}^{n} \boldsymbol{\phi}_i x_i(t)$$

using $\mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} = \mathbf{I}$ and $\mathbf{U}(t) = \mathbf{\Phi} \mathbf{X}(t)$ we get the initial conditions

$$x_i \big|_{t=0} = \boldsymbol{\phi}_i^T \mathbf{M}\, {}^0\mathbf{U}$$

$$\dot{x}_i \big|_{t=0} = \boldsymbol{\phi}_i^T \mathbf{M}\, {}^0\dot{\mathbf{U}}$$

dynamic response

$$\ddot{x}(t) + \omega^2 x(t) = R \sin \hat{\omega} t$$

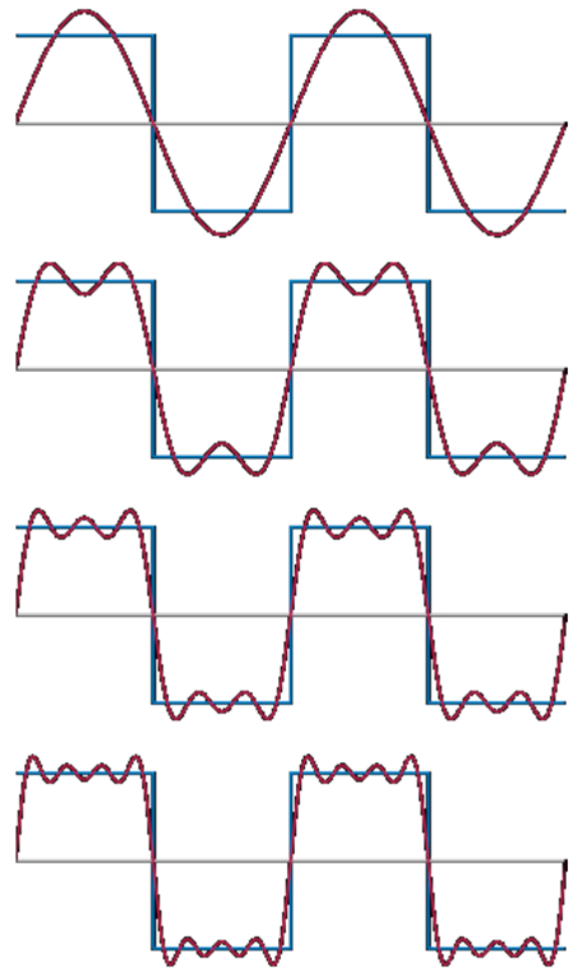
The analysis of the response of the single degree of freedom system considered in this example showed that the complete response is the sum of two contributions:

1. A dynamic response obtained by multiplying the static response by a dynamic load factor (this is the particular solution of the governing differential equation), and
2. An additional dynamic response which we called the transient response.

$$x(t) = \frac{R}{\omega} \int_0^t \sin \hat{\omega}\tau \sin \omega(t - \tau)\, d\tau + \alpha \sin \omega t + \beta \cos \omega t$$

Superposition

A dynamic load can be designed as Fourier series



The total solution of such a problem is equal to the superposition of solution of the Fourier terms.

$$\mathbf{U}^p(t) = \sum_{i=1}^{p} \boldsymbol{\phi}_i x_i(t)$$

Problems with neglected damping

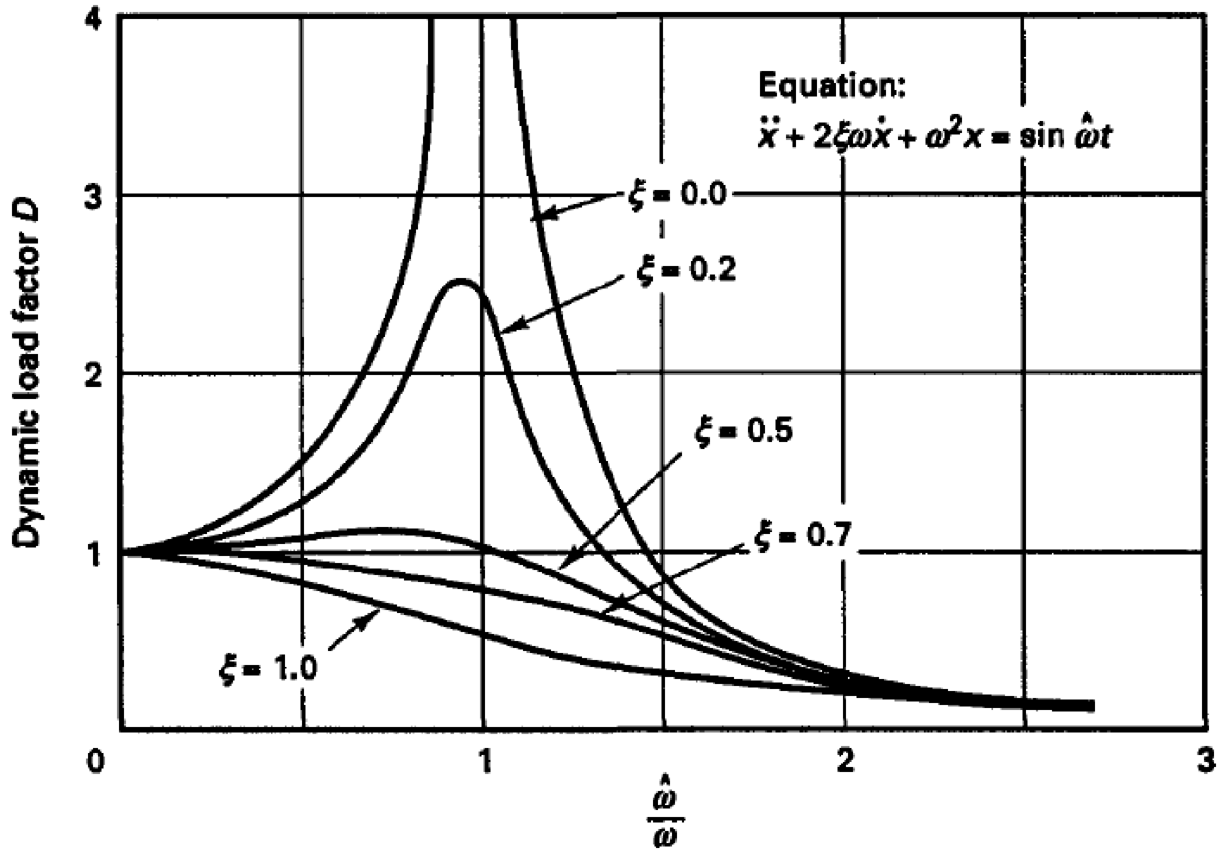
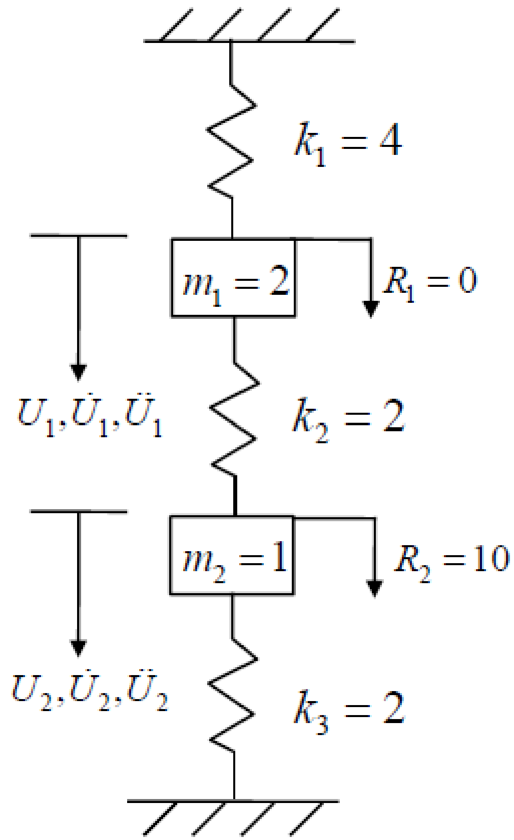


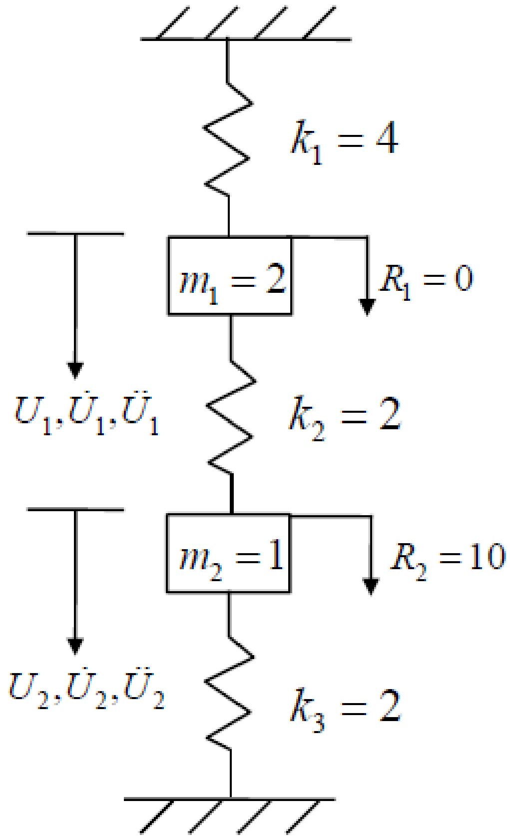
Figure 9.3 The dynamic load factor

Example 9.7

Calculate the displacement response of the system



From Example 9.6



$$\mathbf{K} = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}; \qquad \mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \mathbf{R} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

The eigenproblem: $K\phi_n = \omega_n{}^2 M\phi_n$

$$\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\boldsymbol{\phi} = \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{\phi}$$

Eigen values

$$|K - \omega^2 M| = \left|\begin{bmatrix} 6 - 2{\omega_n}^2 & -2 \\ -2 & 4 - 1{\omega_n}^2 \end{bmatrix}\right| = 2{\omega_n}^4 - 14{\omega_n}^2 + 20 = 0$$

$${\omega_1}^2 = \frac{14 - \sqrt{196 - 160}}{4} = 2 \qquad {\omega_2}^2 = \frac{14 + \sqrt{196 - 160}}{4} = 5$$

Eigen vectors

$$(K - \omega_1{}^2 M)\phi_1 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 0 \longrightarrow \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = \lambda * \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We defined: $\phi_1{}^T M \phi_1 = 1$

Therefore: $\lambda^2 * (m_1 + m_2) = 1 \longrightarrow \lambda = \frac{1}{\sqrt{3}}$

$$\boldsymbol{\phi}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Same for: $\boldsymbol{\phi}_2 = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}$

Two equilibrium equations

$$\ddot{\mathbf{X}}(t) + \mathbf{\Omega}^2 \mathbf{X}(t) = \mathbf{\Phi}^T \mathbf{R}(t)$$

$$\ddot{\mathbf{X}}(t) + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \mathbf{X}(t) = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2}\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$\ddot{x}_1 + 2x_1 = \frac{10}{\sqrt{3}} \qquad \ddot{x}_2 + 5x_2 = -10\sqrt{\frac{2}{3}}$$

analytic solution

$$\ddot{x}_1 + 2x_1 = \frac{10}{\sqrt{3}} \qquad \ddot{x}_2 + 5x_2 = -10\sqrt{\frac{2}{3}}$$

Initial conditions: $\mathbf{U}|_{t=0} = \mathbf{0},\ \dot{\mathbf{U}}|_{t=0} = \mathbf{0},$

$$x_i|_{t=0} = \boldsymbol{\phi}_i^T \mathbf{M}\ {}^0\mathbf{U} \qquad \dot{x}_i|_{t=0} = \boldsymbol{\phi}_i^T \mathbf{M}\ {}^0\dot{\mathbf{U}} \longrightarrow \begin{matrix} x_1|_{t=0} = 0 & \dot{x}_1|_{t=0} = 0 \\ x_2|_{t=0} = 0 & \dot{x}_2|_{t=0} = 0 \end{matrix}$$

$$x_1 = \frac{5}{\sqrt{3}}(1 - \cos\sqrt{2}\,t) \qquad x_2 = 2\sqrt{\frac{2}{3}}(-1 + \cos\sqrt{5}\,t)$$

$$\mathbf{U}(t) = \sum_{i=1}^{n} \boldsymbol{\phi}_i \, x_i(t)$$

$$\mathbf{U}(t) = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{3}}(1 - \cos\sqrt{2}t) \\ 2\sqrt{\frac{2}{3}}(-1 + \cos\sqrt{5}t) \end{bmatrix}$$

$\Delta t = 0.28$

Time	Δt	$2\Delta t$	$3\Delta t$	$4\Delta t$	$5\Delta t$	$6\Delta t$	$7\Delta t$	$8\Delta t$	$9\Delta t$	$10\Delta t$	$11\Delta t$	$12\Delta t$
${}^t\mathbf{U}$	0.003 0.382	0.038 1.41	0.176 2.78	0.486 4.09	0.996 5.00	1.66 5.29	2.338 4.986	2.861 4.277	3.052 3.457	2.806 2.806	2.131 2.484	1.157 2.489

Newark method

The Newmark method : **Solution procedure:**

A: Initial calculations

1) Form stiffness matrix, mass matrix and damping matrix
2) Initialize ${}^{0}\mathbf{U}$, ${}^{0}\dot{\mathbf{U}}$ **and** ${}^{0}\ddot{\mathbf{U}}$
3) Select time step Δt and parameters α and δ

$$a_0 = \frac{1}{\alpha \Delta t^2}, \quad a_1 = \frac{\delta}{\alpha \Delta t}, \quad a_2 = \frac{1}{\alpha \Delta t}, \quad a_3 = \frac{1}{2\alpha} - 1, \quad a_4 = \frac{\delta}{2\alpha} - 1$$

$$a_5 - \frac{\delta}{\alpha} - 1, \quad a_6 - \frac{\Delta t}{2}(\frac{\delta}{\alpha} - 2), \quad a_7 - \Delta t(1-\delta), \quad a_8 - \delta \Delta t$$

4) Form effective stiffness matrix $\hat{\mathbf{K}} = \mathbf{K} + a_0 \mathbf{M} + a_1 \mathbf{C}$
5) Triangularize $\hat{\mathbf{K}} = \mathbf{LDL}^T$

B: For each time step

1) Calculate effective loads at time t:

$${}^{t+\Delta t}\hat{\mathbf{R}} = {}^{t+\Delta t}\mathbf{R} + \mathbf{M}(a_0\,{}^{t}\mathbf{U} + a_2\,{}^{t}\dot{\mathbf{U}} + a_3\,{}^{t}\ddot{\mathbf{U}})$$

$$+\mathbf{C}(a_1\,{}^{t}\mathbf{U} + a_4\,{}^{t}\dot{\mathbf{U}} + a_5\,{}^{t}\ddot{\mathbf{U}})$$

2) Solve for the displacements U at time $t + \Delta t$

$$\mathbf{LDL}^T\,{}^{t+\Delta t}\mathbf{U} = {}^{t+\Delta t}\hat{\mathbf{R}}$$

3) Solve for the corresponding velocities and accelerations

$${}^{t+\Delta t}\ddot{\mathbf{U}} = a_0({}^{t+\Delta t}\mathbf{U} - {}^{t}\mathbf{U}) - a_2\,{}^{t}\dot{\mathbf{U}} - a_3\,{}^{t}\ddot{\mathbf{U}}$$

$${}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + a_7\,{}^{t+\Delta t}\ddot{\mathbf{U}} + a_6\,{}^{t}\ddot{\mathbf{U}}$$

9.3 Mode Superposition (damping included)

(Page 785 – 795)

$$\ddot{\mathbf{X}}(t) + \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}\dot{\mathbf{X}}(t) + \mathbf{\Omega}^2\mathbf{X}(t) = \mathbf{\Phi}^T\mathbf{R}(t)$$

Proportional damping

$$\ddot{\mathbf{X}}(t) + \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}\dot{\mathbf{X}}(t) + \mathbf{\Omega}^2\mathbf{X}(t) = \mathbf{\Phi}^T\mathbf{R}(t)$$

Assume C proportional, so the solution can still be calculated independently for each x_i

$$\boldsymbol{\phi}_i^T\mathbf{C}\boldsymbol{\phi}_j = 2\omega_i\xi_i\delta_{ij}$$

ξ_i is a modal damping parameter

$\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$ Kronecker delta

$$\ddot{x}_i(t) + 2\omega_i\xi_i\dot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t)$$

$$x_i(t) = \frac{1}{\overline{\omega}_i}\int_0^t r_i(\tau)e^{-\xi_i\omega_i(t-\tau)}\sin\overline{\omega}_i(t-\tau)\,d\tau + e^{-\xi_i\omega_i t}(\alpha_i\sin\overline{\omega}_i t + \beta_i\cos\overline{\omega}_i t)$$

Rayleigh damping

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

If there are only two different damping ratios ξ_i Rayleigh damping can be used

$$\boldsymbol{\phi}_i^T(\alpha\mathbf{M} + \beta\mathbf{K})\boldsymbol{\phi}_i = 2\omega_i\xi_i$$

$$\alpha + \beta\omega_i^2 = 2\omega_i\xi_i$$

In actual analysis it may well be that the damping ratios are known for many more than two frequencies. In that case two average values, say $\bar{\xi}_1$ and $\bar{\xi}_2$, are used to evaluate α and β. Consider the following example.

Problem with Rayleigh damping

$$\alpha + \beta\omega_i^2 = 2\omega_i\xi_i$$

A problem with Rayleigh damping is that higher modes are much more damped than lower modes.

But in general Rayleigh damping is a good assumption