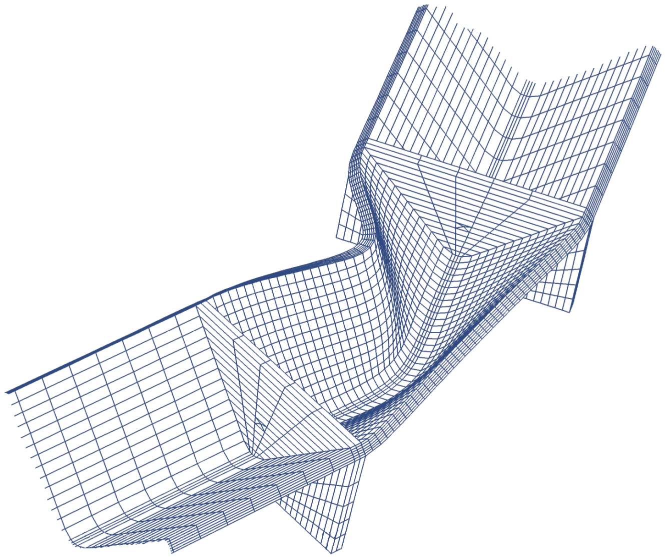
The Finite Element Method
for the Analysis of Non-Linear and Dynamic Systems



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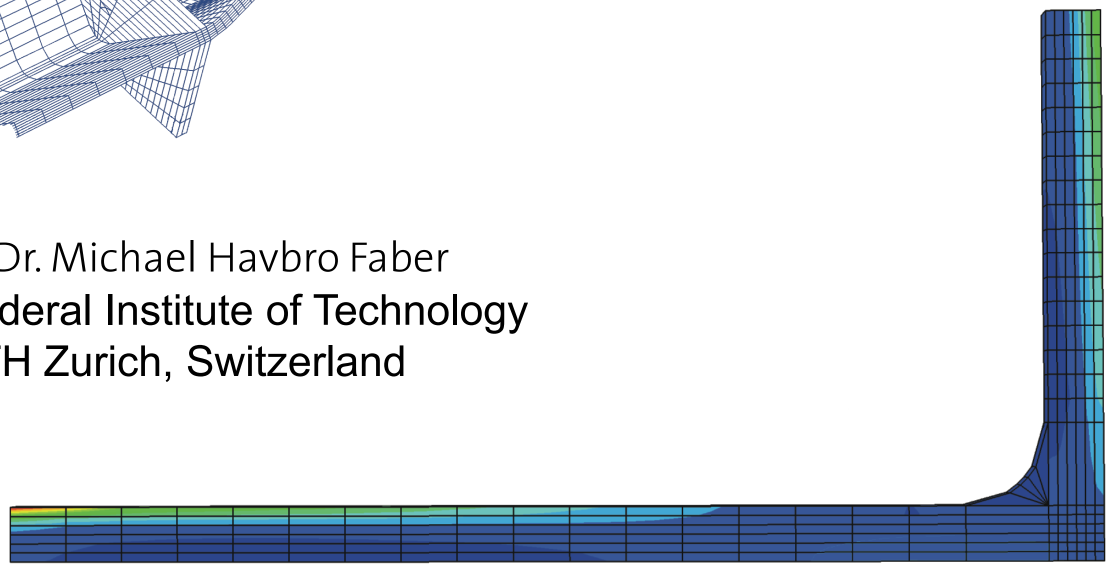


Contents of Today's Lecture

- **Solution of Equilibrium Equations in Dynamic Analysis**

  Direct Integration methods

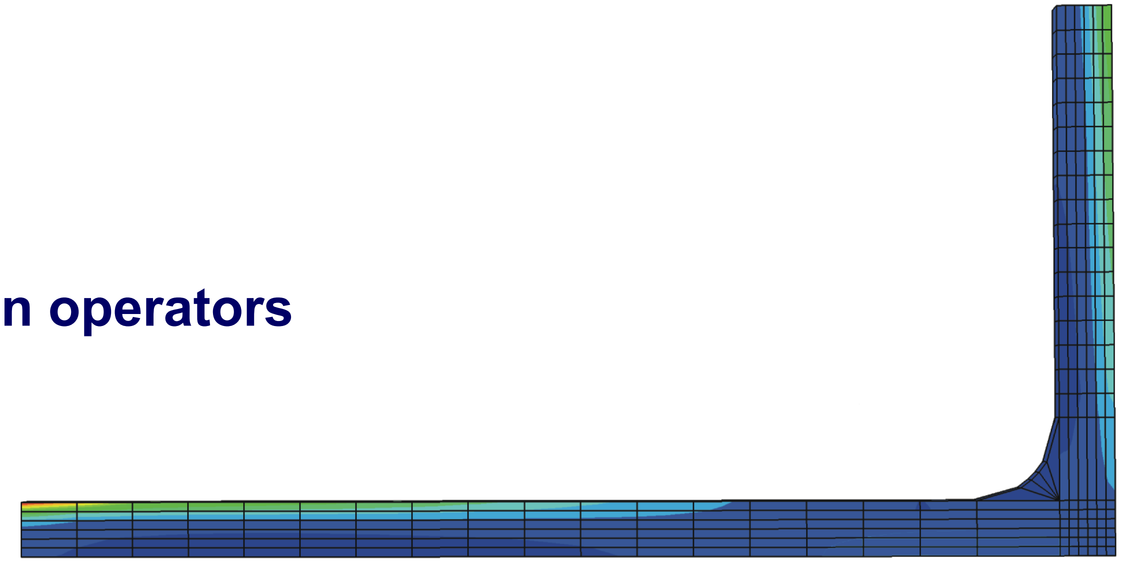
  - The central difference method

  - The Houbolt method

  - The Wilson θ method

  - The Newmark method

  - Coupling of integration operators



Introduction to Dynamic Analysis

- **Introduction**

The very basics ☺

Newtons 2'nd law of motion:

$$m\ddot{x} = w - k(\Delta + x)$$

$$\Downarrow$$

$$m\ddot{x} - w + k(\Delta + x) = 0$$

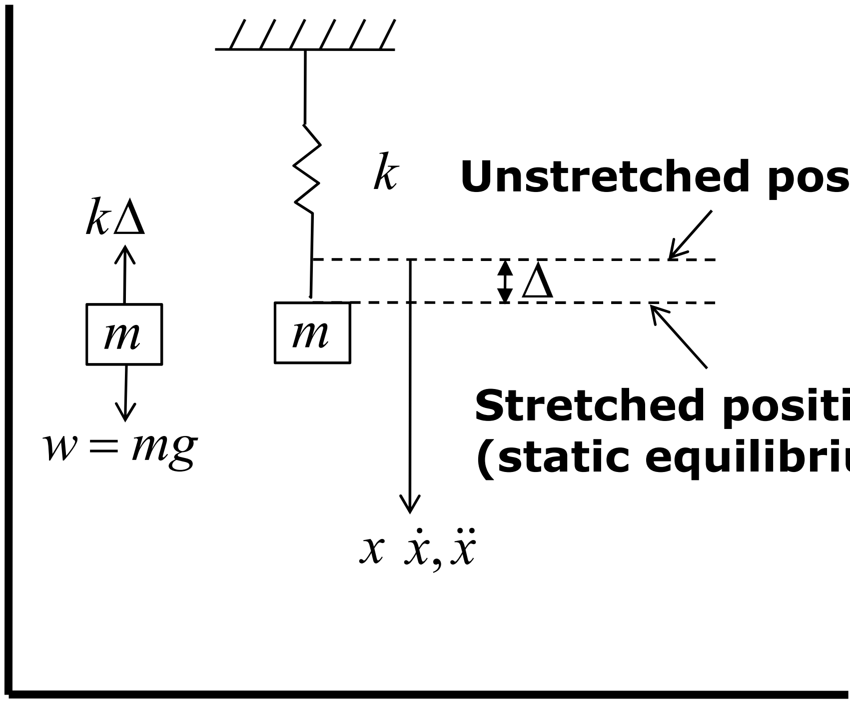
$$\Downarrow$$

$$m\ddot{x} - k\Delta + k(\Delta + x) = m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}x = \ddot{x} + \omega_n^2 x = 0$$

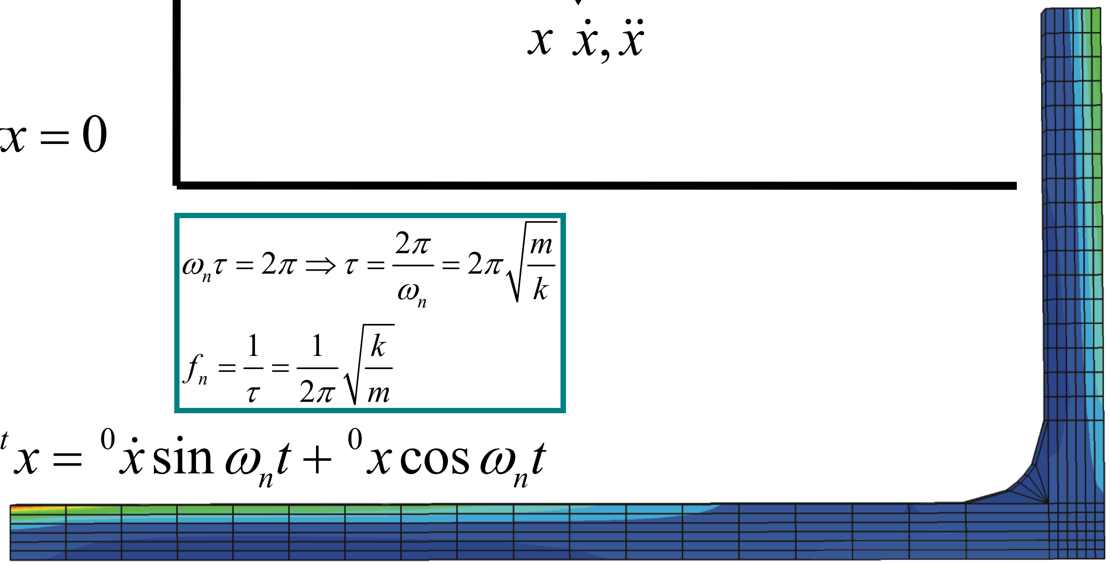
$$\Downarrow$$

$$x = A\sin\omega_n t + B\cos\omega_n t \Rightarrow {}^t x = {}^0\dot{x}\sin\omega_n t + {}^0 x\cos\omega_n t$$



$$\omega_n\tau = 2\pi \Rightarrow \tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}$$

$$f_n = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$



Introduction to Dynamic Analysis

- **Introduction**

We have previously considered the equilibrium equations governing the linear dynamic response of a system of finite elements:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$$

$\mathbf{M}$: **Mass matrix**

$\mathbf{C}$: **Damping matrix**

$\mathbf{K}$: **Stiffness matrix**

$\mathbf{U}$: **Displacements**

$\dot{\mathbf{U}}$: **Velocities**

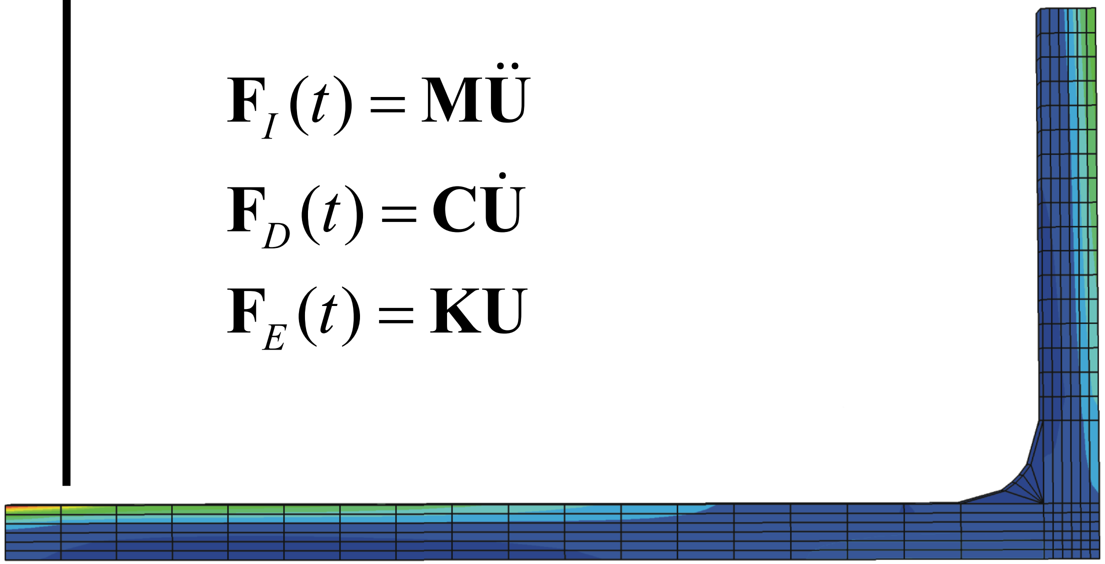
$\ddot{\mathbf{U}}$: **Accelerations**

$$\mathbf{F}_I(t) + \mathbf{F}_D(t) + \mathbf{F}_E(t) = \mathbf{R}(t)$$

$$\mathbf{F}_I(t) = \mathbf{M}\ddot{\mathbf{U}}$$

$$\mathbf{F}_D(t) = \mathbf{C}\dot{\mathbf{U}}$$

$$\mathbf{F}_E(t) = \mathbf{K}\mathbf{U}$$



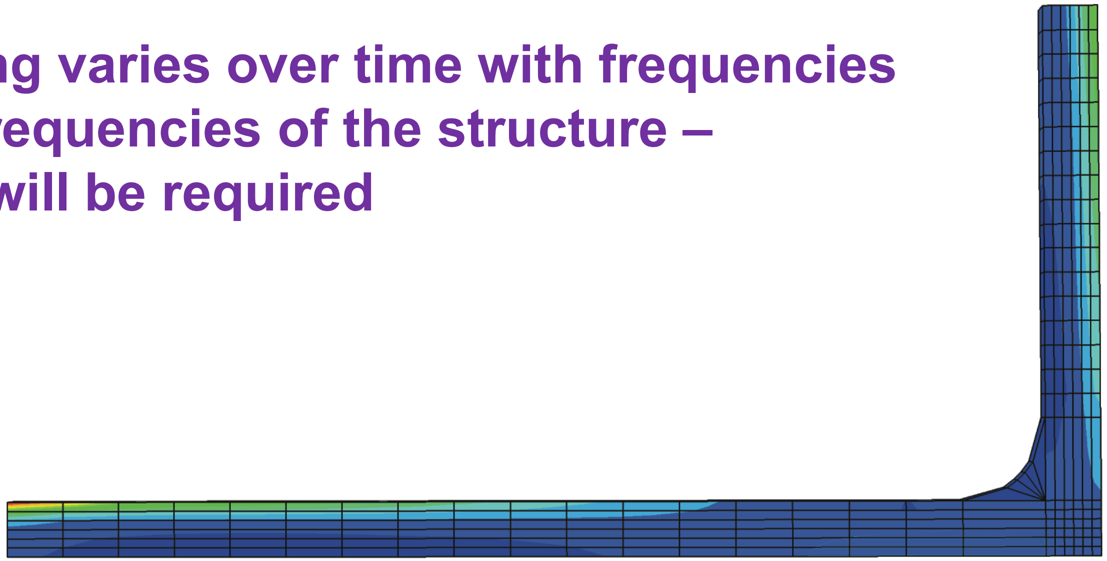
Introduction to Dynamic Analysis

- **Introduction**

  Whether a dynamic analysis is needed or not is generally up to engineering judgment –

  requires understanding of the interaction between loading and structural response!

  In general – if the loading varies over time with frequencies higher than the Eigen-frequencies of the structure – then dynamic analysis will be required



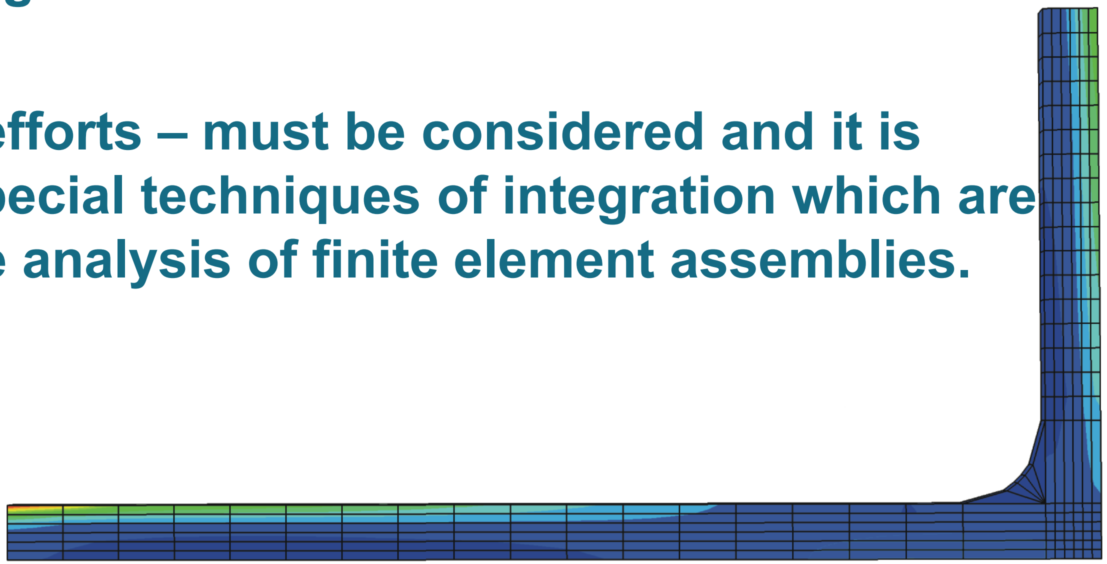
Introduction to Dynamic Analysis

- **Introduction**

$$\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}$$

In principle the equilibrium equations may be solved by any standard numerical integration scheme – BUT !

Efficiency – numerical efforts – must be considered and it is worthwhile to look at special techniques of integration which are especially suited for the analysis of finite element assemblies.



Direct Integration Methods

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$$

- **Direct Integration Methods**

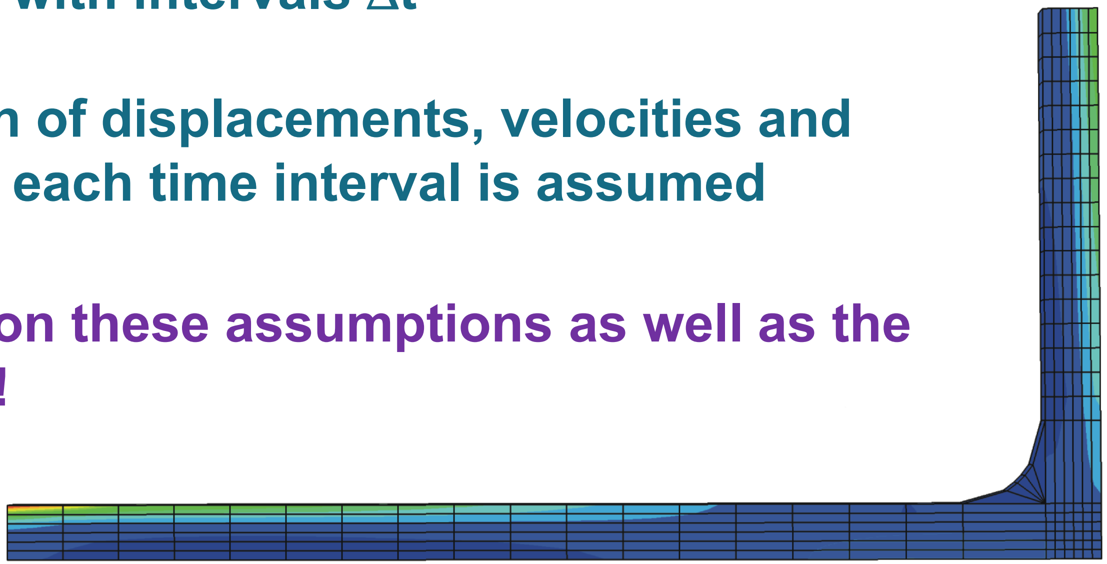
Direct means: The equations are solved in their original form!

Two ideas are utilized

1) **The equilibrium equations are satisfied only at time steps, i.e. at discrete times with intervals Δt**

2) **A particular variation of displacements, velocities and accelerations within each time interval is assumed**

The accuracy depends on these assumptions as well as the choice of time intervals!



Direct Integration Methods

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$$

- **Direct Integration Methods**

The displacements, velocities and accelerations

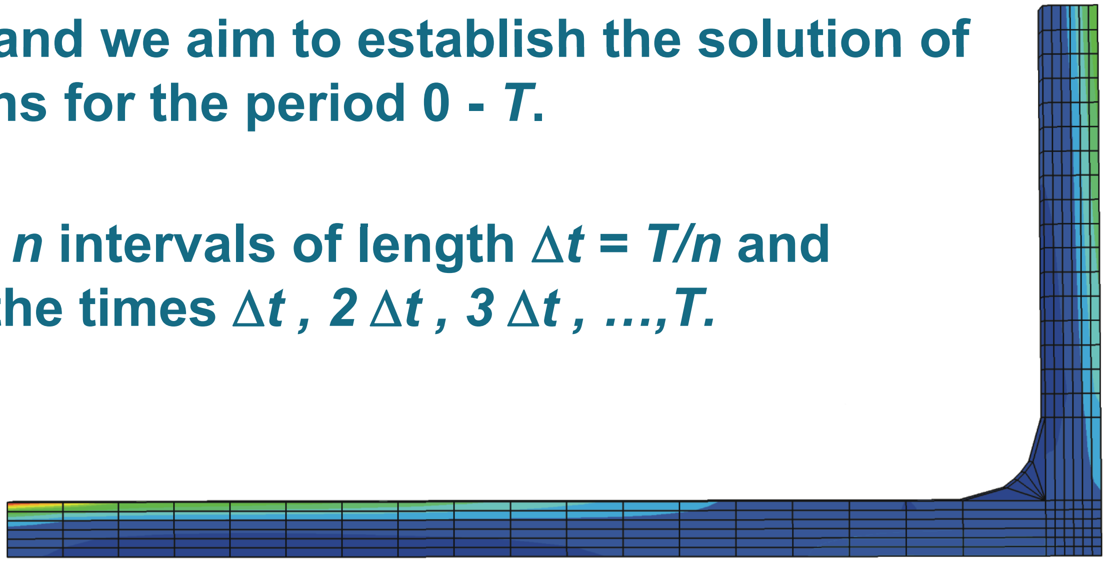
$^{0}\mathbf{U}$: **Displacement vector at time** $t = 0$

$^{0}\dot{\mathbf{U}}$: **Velocity vector at time** $t = 0$

$^{0}\ddot{\mathbf{U}}$: **Acceleration vector at time** $t = 0$

are assumed to known and we aim to establish the solution of the equilibrium equations for the period 0 - ***T*****.**

So we sub-divide ***T*** **into** ***n*** **intervals of length** $\Delta t = T/n$ **and establish solutions for the times** Δt **,** $2\,\Delta t$ **,** $3\,\Delta t$ **, ...,** ***T*****.**



Direct Integration Methods

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$$

- **Direct Integration Methods**

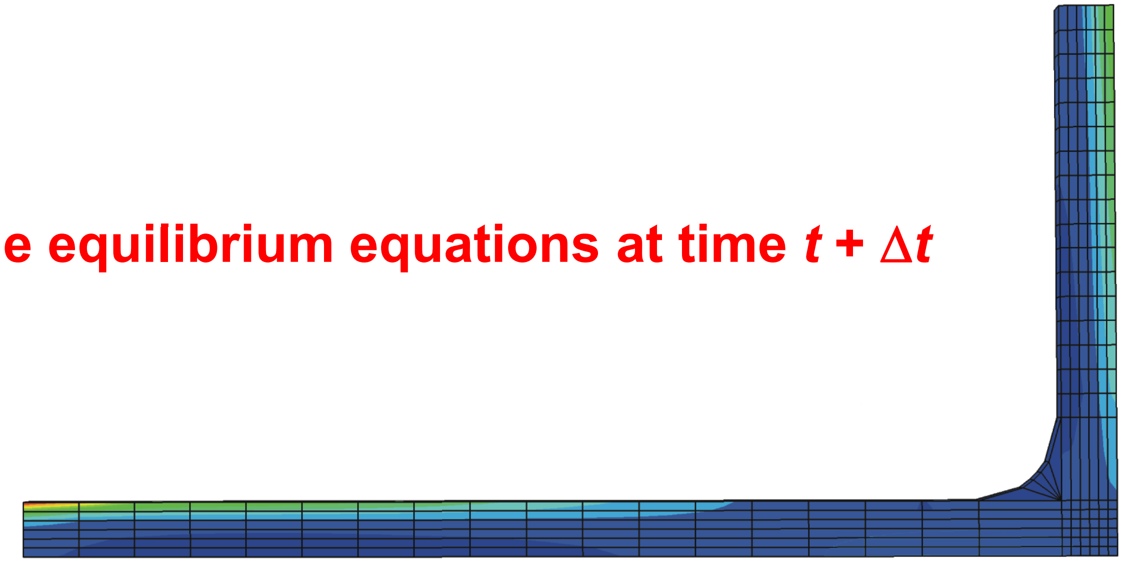
We distinguish principally between

Explicit methods:

Solution is based on the equilibrium equations at time *t*

Implicit methods:

Solution is based on the equilibrium equations at time $t + \Delta t$



Direct Integration Methods

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} \qquad \mathbf{c}$$

- **The Central difference method**

$$ {}^{t}\ddot{\mathbf{U}} = \frac{1}{\Delta t^2}\left({}^{t-\Delta t}\mathbf{U} - 2\,{}^{t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right) \qquad \mathbf{a}$$

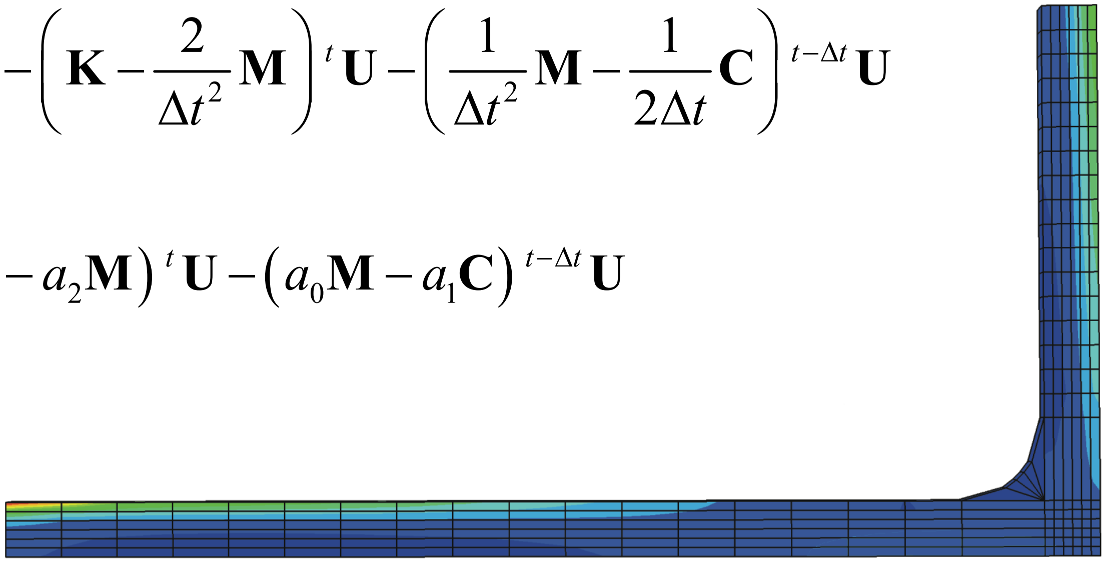
$$ {}^{t}\dot{\mathbf{U}} = \frac{1}{2\Delta t}\left(-{}^{t-\Delta t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right) \qquad \mathbf{b}$$

a and b inserted in c

$$\left(\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}\right){}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{R} - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right){}^{t}\mathbf{U} - \left(\frac{1}{\Delta t^2}\mathbf{M} - \frac{1}{2\Delta t}\mathbf{C}\right){}^{t-\Delta t}\mathbf{U}$$

$$\Downarrow$$

$$\left(a_0\mathbf{M} + a_1\mathbf{C}\right){}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{R} - \left(\mathbf{K} - a_2\mathbf{M}\right){}^{t}\mathbf{U} - \left(a_0\mathbf{M} - a_1\mathbf{C}\right){}^{t-\Delta t}\mathbf{U}$$



Direct Integration Methods

$$\left(a_0\mathbf{M}+a_1\mathbf{C}\right)\,{}^{t+\Delta t}\mathbf{U}=$$
$${}^{t}\mathbf{R}-\left(\mathbf{K}-a_2\mathbf{M}\right){}^{t}\mathbf{U}-\left(a_0\mathbf{M}-a_1\mathbf{C}\right){}^{t-\Delta t}\mathbf{U}$$

- **The Central difference method**

We see that we do not need to factorize the stiffness matrix ☺

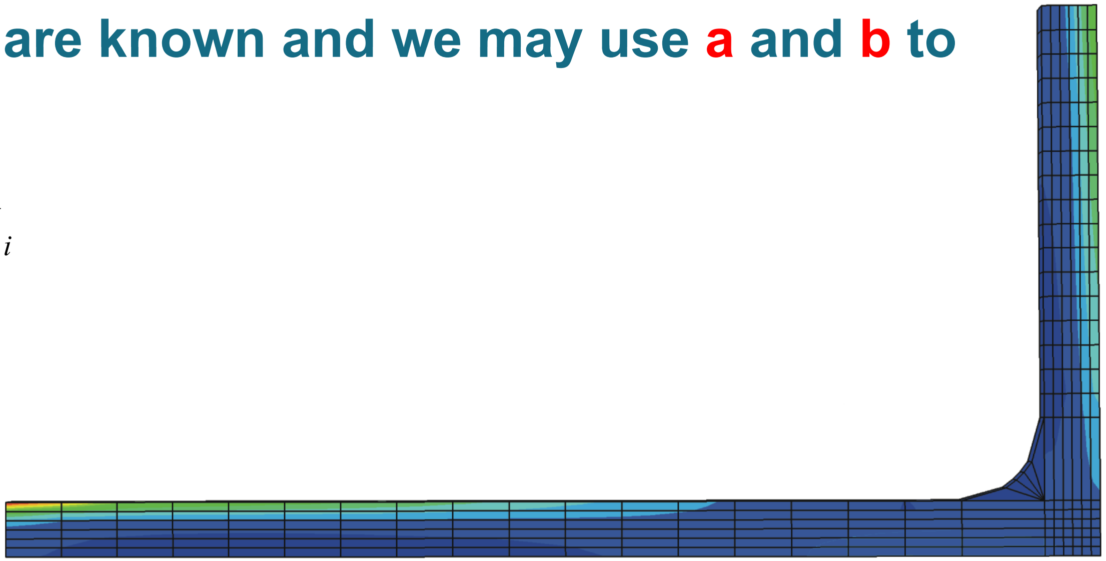
We also see that in order to calculate the displacements at time Δt we need to know the displacements at time 0 and $-\Delta t$

In general ${}^{0}\mathbf{U},\ {}^{0}\dot{\mathbf{U}}$ and ${}^{0}\ddot{\mathbf{U}}$ are known and we may use a and b to obtain ${}^{-\Delta t}\mathbf{U}_i$

$${}^{-\Delta t}\mathbf{U}_i={}^{0}\mathbf{U}_i-\Delta t\,{}^{0}\dot{\mathbf{U}}_i+\frac{\Delta t^2}{2}\,{}^{0}\ddot{\mathbf{U}}_i$$

$$\Downarrow$$

$${}^{-\Delta t}\mathbf{U}_i={}^{0}\mathbf{U}_i-\Delta t\,{}^{0}\dot{\mathbf{U}}_i+a_3\,{}^{0}\ddot{\mathbf{U}}_i$$



Direct Integration Methods

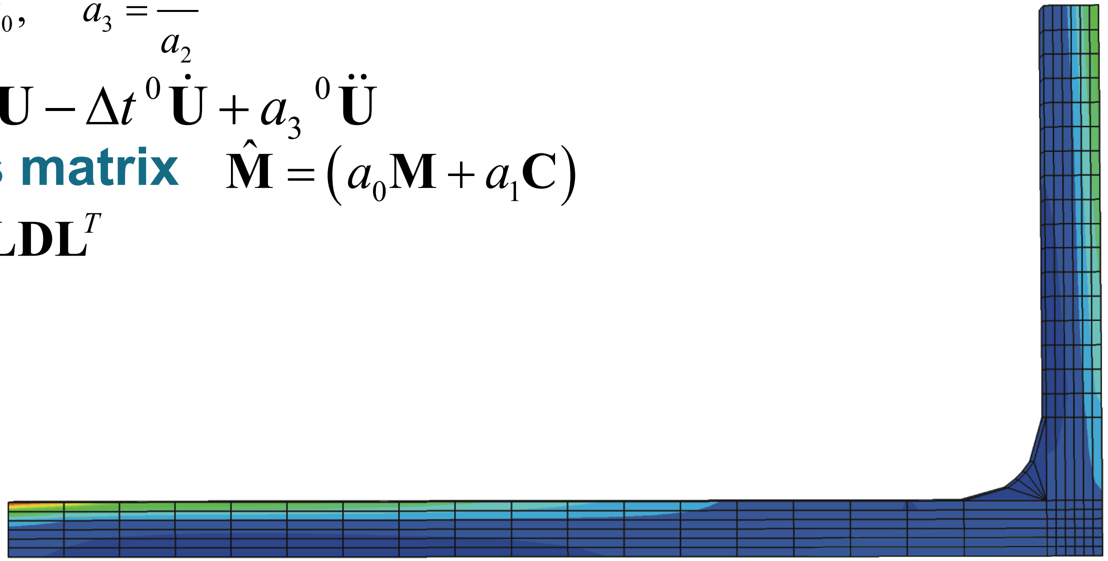
- The Central difference method: **Solution procedure:**

  A: Initial calculations

  1) Form stiffness matrix, mass matrix and damping matrix
  2) Initialize ${}^{0}\mathbf{U}$, ${}^{0}\dot{\mathbf{U}}$ **and** ${}^{0}\ddot{\mathbf{U}}$
  3) Select time step Δt, $\Delta t \le \Delta t_{cr}$ and calculate integration constants

  $$a_0 = \frac{1}{\Delta t^2}, \quad a_1 = \frac{1}{2\Delta t}, \quad a_2 = 2a_0, \quad a_3 = \frac{1}{a_2}$$

  4) Calculate ${}^{-\Delta t}\mathbf{U} = {}^{0}\mathbf{U} - \Delta t\,{}^{0}\dot{\mathbf{U}} + a_3\,{}^{0}\ddot{\mathbf{U}}$
  5) Form effective mass matrix $\hat{\mathbf{M}} = \left(a_0\mathbf{M} + a_1\mathbf{C}\right)$
  6) Triangularize $\hat{\mathbf{M}} = \mathbf{L}\mathbf{D}\mathbf{L}^T$



Direct Integration Methods

- The Central difference method: **Solution procedure:**

B: For each time step

1) Calculate effective loads at time *t*:

$$ {}^{t}\hat{\mathbf{R}} = {}^{t}\mathbf{R} - \left(\mathbf{K} - a_2\mathbf{M}\right){}^{t}\mathbf{U} - \left(a_0\mathbf{M} - a_1\mathbf{C}\right){}^{t-\Delta t}\mathbf{U} $$

2) Solve for the displacements U at time $t + \Delta t$

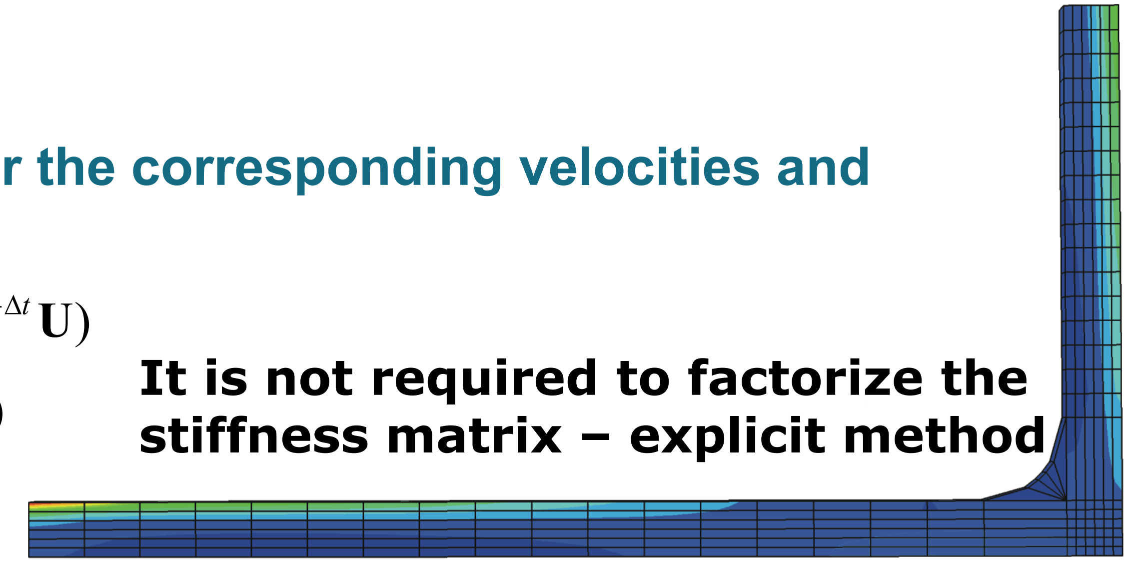
$$ \mathbf{LDL}^{T}\,{}^{t+\Delta t}\mathbf{U} = {}^{t}\hat{\mathbf{R}} $$

3) If required, solve for the corresponding velocities and accelerations

$$ {}^{t}\ddot{\mathbf{U}} = a_0\left({}^{t-\Delta t}\mathbf{U} - 2\,{}^{t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right) $$

$$ {}^{t}\dot{\mathbf{U}} = a_1\left(-{}^{t-\Delta t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right) $$

It is not required to factorize the stiffness matrix – explicit method



Direct Integration Methods

- **The Central difference method**

The effectiveness of the central difference method depends on the efficiency of the time step solution – because we need a lot of them☹
For this reason the method is usually only applied when a lumped mass matrix can be assumed and when the velocity dependent damping can be neglected, i.e.:

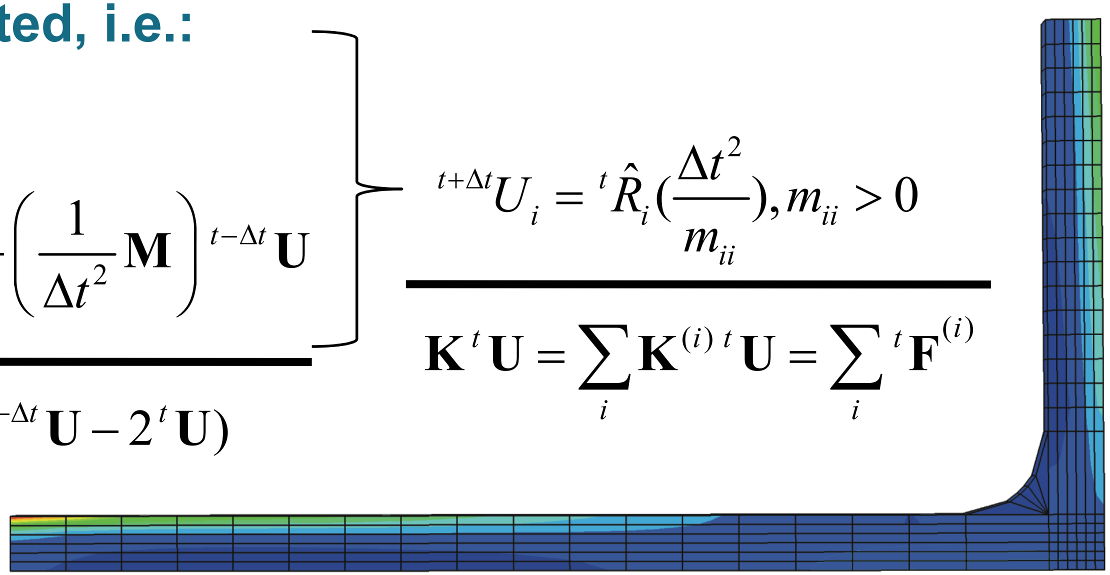
$$\frac{1}{\Delta t^2}\mathbf{M}\,{}^{t+\Delta t}\mathbf{U} = {}^{t}\hat{\mathbf{R}}$$

$${}^{t}\hat{\mathbf{R}} = {}^{t}\mathbf{R} - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right){}^{t}\mathbf{U} - \left(\frac{1}{\Delta t^2}\mathbf{M}\right){}^{t-\Delta t}\mathbf{U}$$

$${}^{t+\Delta t}U_i = {}^{t}\hat{R}_i\left(\frac{\Delta t^2}{m_{ii}}\right), m_{ii} > 0$$

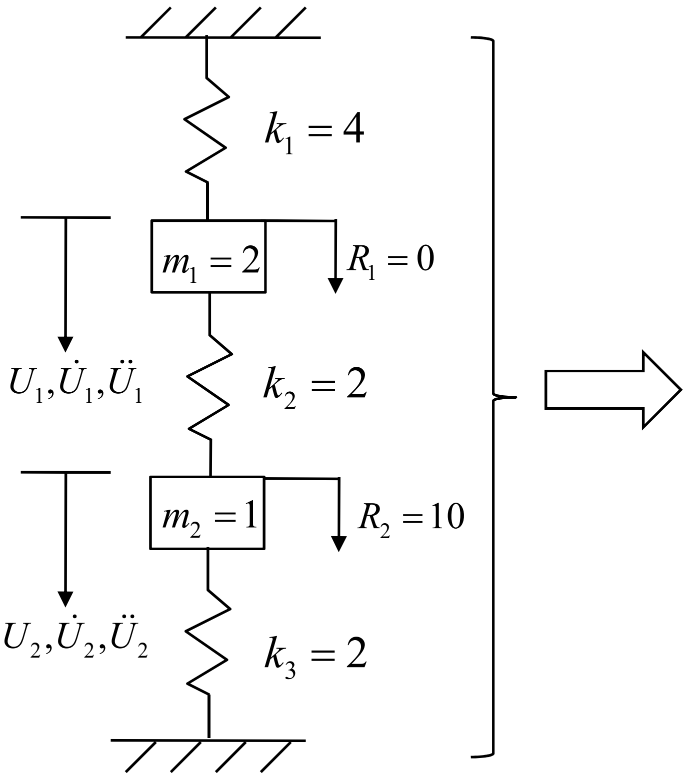
$$\mathbf{K}\,{}^{t}\mathbf{U} = \sum_i \mathbf{K}^{(i)}\,{}^{t}\mathbf{U} = \sum_i {}^{t}\mathbf{F}^{(i)}$$

$${}^{t}\hat{\mathbf{R}} = {}^{t}\mathbf{R} - \sum_i {}^{t}\mathbf{F}^{(i)} - \frac{1}{\Delta t^2}\mathbf{M}\left({}^{t-\Delta t}\mathbf{U} - 2\,{}^{t}\mathbf{U}\right)$$



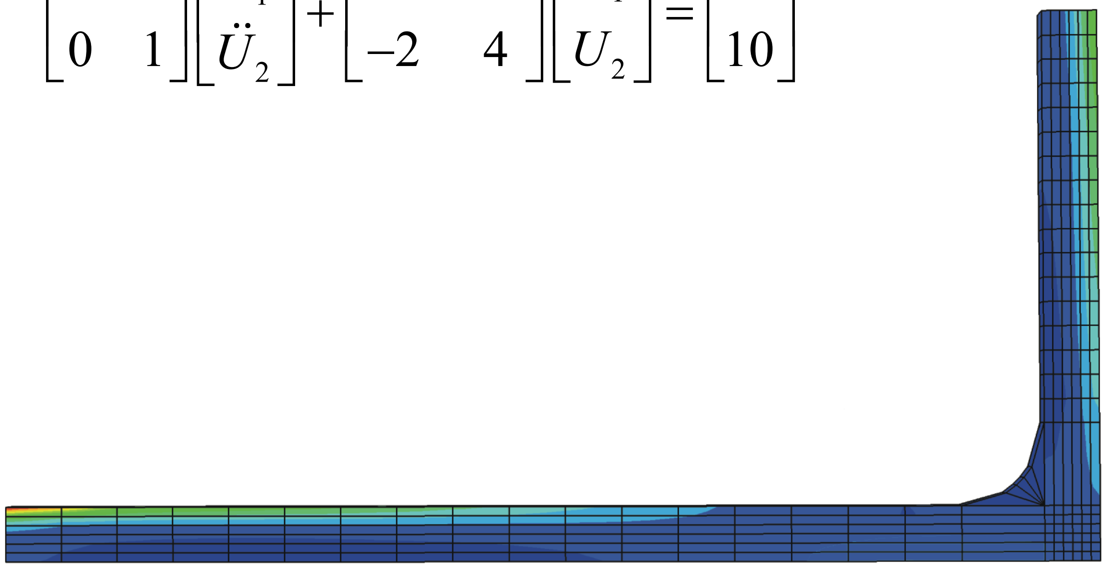
Direct Integration Methods

- ## The Central difference method: **Example:**



With natural periods:
***T_1* = 4.45**
***T_2* = 2.8**

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$



Direct Integration Methods

$$^{-\Delta t}\mathbf{U}_i = {}^{0}\mathbf{U}_i - \Delta t\,{}^{0}\dot{\mathbf{U}}_i + \frac{\Delta t^2}{2}\,{}^{0}\ddot{\mathbf{U}}_i$$

$$\Downarrow$$

$$^{-\Delta t}\mathbf{U}_i = {}^{0}\mathbf{U}_i - \Delta t\,{}^{0}\dot{\mathbf{U}}_i + a_3\,{}^{0}\ddot{\mathbf{U}}_i$$

- **The Central difference method: Example:**

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

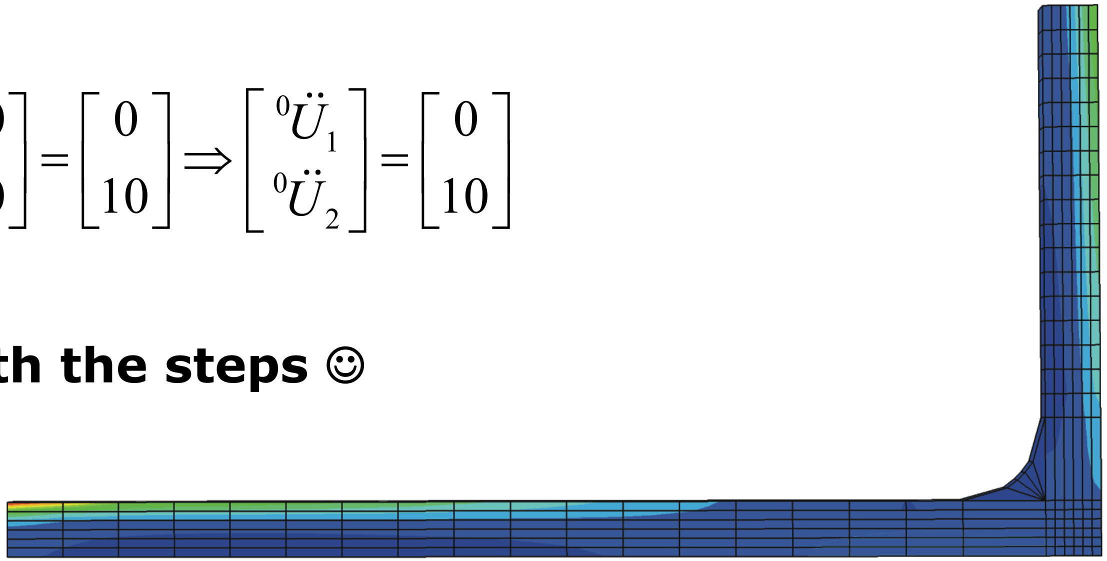
With Eigen periods:
T_1 = 4.45
T_2 = 2.8

We will calculate the the response of the system for $\Delta t = T_2/10$ and for $\Delta t = 10\,T_2$ over 12 time steps

First we calculate $^{0}\ddot{\mathbf{U}}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} {}^{0}\ddot{U}_1 \\ {}^{0}\ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} {}^{0}\ddot{U}_1 \\ {}^{0}\ddot{U}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

Then we continue with the steps ☺



Direct Integration Methods

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

- **The Central difference method: Example:**

$$a_0 = \frac{1}{\Delta t^2}, \quad a_1 = \frac{1}{2\Delta t}, \quad a_2 = 2a_0, \quad a_3 = \frac{1}{a_2}$$

$$a_0 = \frac{1}{(0.28)^2} = 12.8, \quad a_1 = \frac{1}{2\cdot 0.28} = 1.79,$$

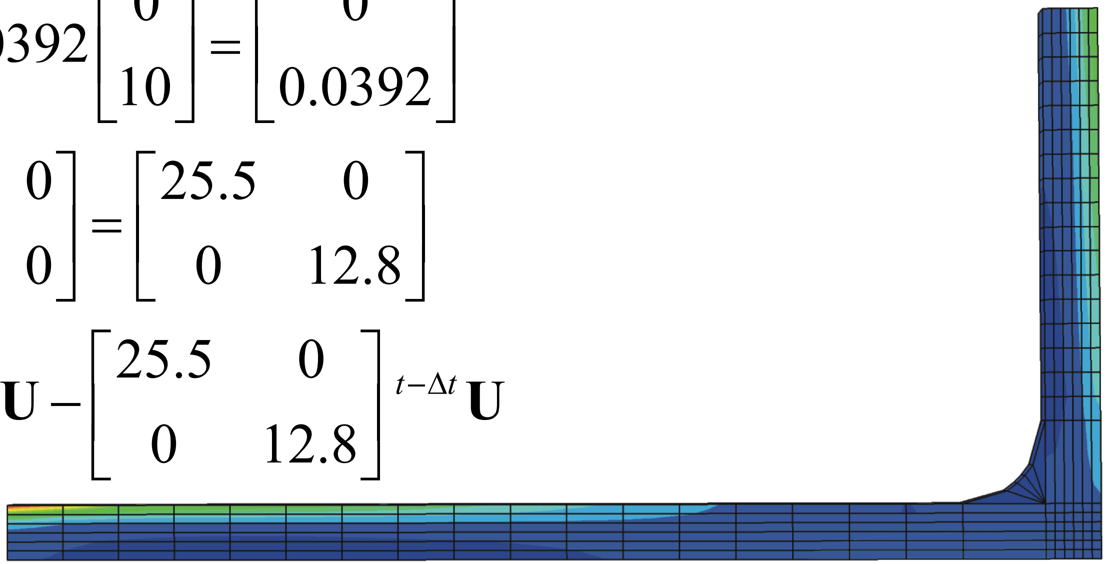
For Δt = 0.28

$$a_2 = 2\cdot\frac{1}{(0.28)^2} = 25.5, \quad a_3 = \frac{1}{25.5} = 0.0392$$

$$^{-\Delta t}\mathbf{U}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 0.28\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.0392\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.0392 \end{bmatrix}$$

$$\hat{\mathbf{M}} = 12.8\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 1.79\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix}$$

$$^{t}\hat{\mathbf{R}} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} + \begin{bmatrix} 45.0 & 2 \\ 2 & 21.5 \end{bmatrix} {}^{t}\mathbf{U} - \begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix} {}^{t-\Delta t}\mathbf{U}$$



Direct Integration Methods

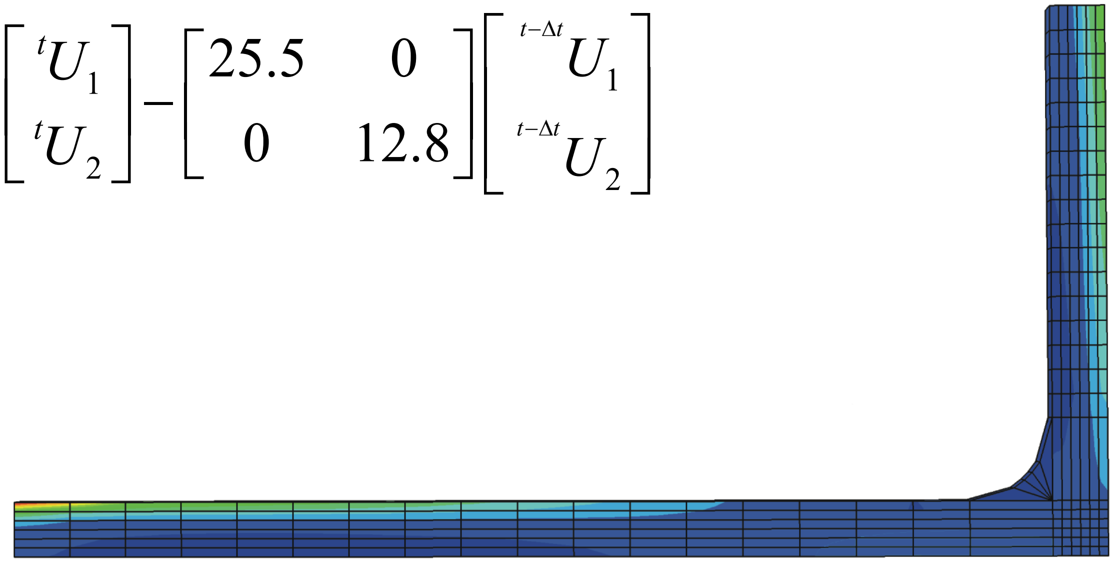
$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

- **The Central difference method: Example:**

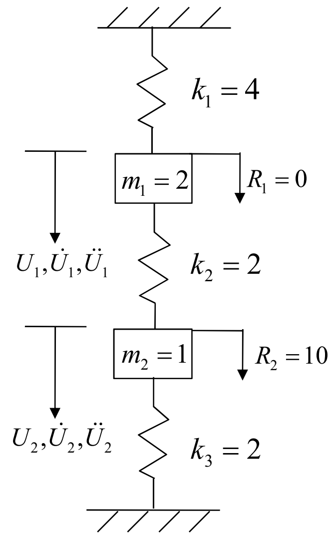
The equation which must be solved for each time step is:

$$\begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix}\begin{bmatrix} {}^{t+\Delta t}U_1 \\ {}^{t+\Delta t}U_2 \end{bmatrix} = {}^{t}\hat{\mathbf{R}}$$

$${}^{t}\hat{\mathbf{R}} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} + \begin{bmatrix} 45.0 & 2 \\ 2 & 21.5 \end{bmatrix}\begin{bmatrix} {}^{t}U_1 \\ {}^{t}U_2 \end{bmatrix} - \begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix}\begin{bmatrix} {}^{t-\Delta t}U_1 \\ {}^{t-\Delta t}U_2 \end{bmatrix}$$

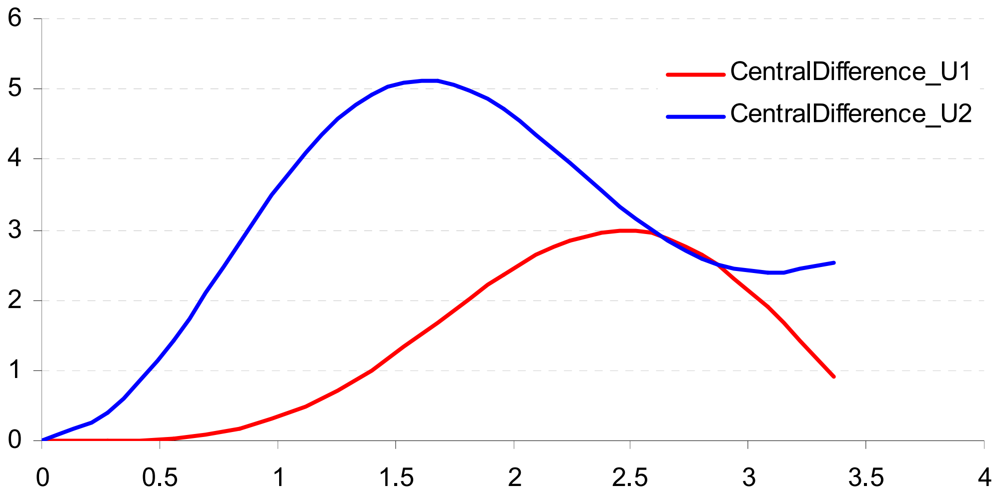


Direct Integration Methods

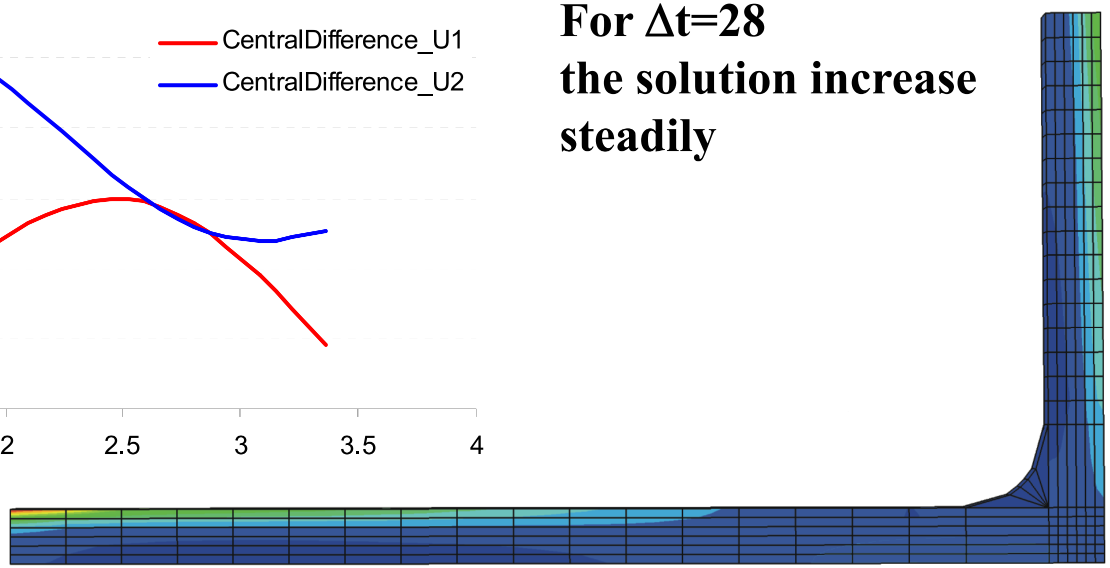


- **The Central difference method: Example:**
  The results are:

t	0	0.28	0.56	0.84	1.12	1.4	1.68	1.96	2.24	2.52	2.8	3.08	3.36
U_1	0	0	0.030529	0.16643	0.483346	1.007119	1.679429	2.35714	2.849397	2.980892	2.658201	1.913392	0.90657
U_2	0	0.38925	1.435068	2.807237	4.087467	4.915203	5.10715	4.706874	3.948481	3.151808	2.592586	2.39952	2.518075
R_1	0	0.7785	4.24396	12.32531	25.68153	42.82543	60.10707	72.65961	76.01274	67.78413	48.79149	23.11753	
R_2	4.9824	18.36888	35.93263	52.31957	62.91459	65.37152	60.24799	50.54056	40.34314	33.1851	30.71386	32.23136	



For Δt=28 the solution increase steadily



Direct Integration Methods

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} \qquad \mathbf{c}$$

- **The Houbolt method**

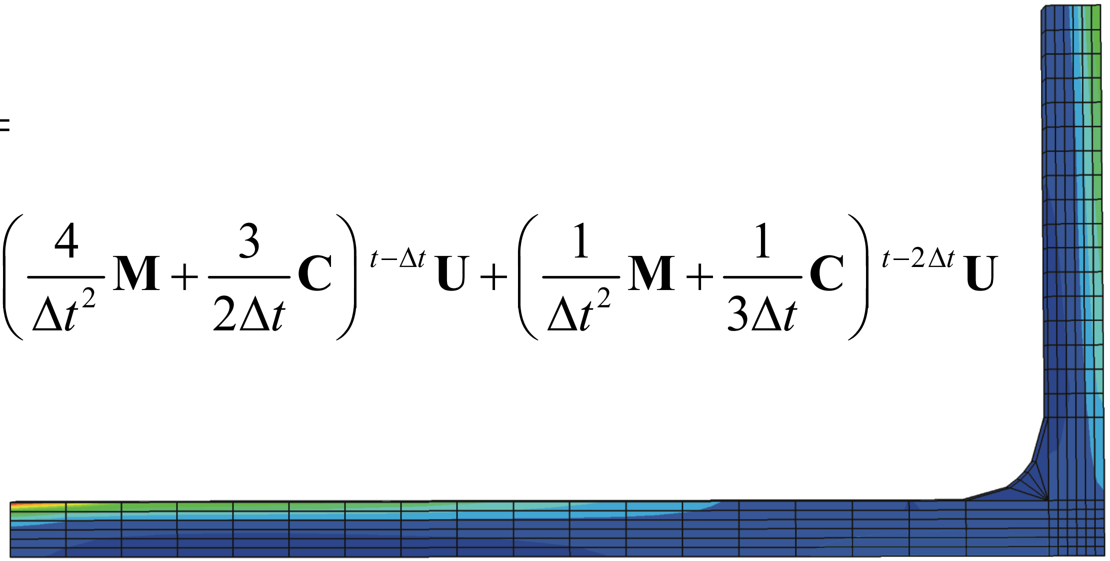
$$^{t+\Delta t}\ddot{\mathbf{U}} = \frac{1}{\Delta t^2}\left(2\,^{t+\Delta t}\mathbf{U} - 5\,^{t}\mathbf{U} + 4\,^{t-\Delta t}\mathbf{U} - \,^{t-2\Delta t}\mathbf{U}\right) \qquad \mathbf{a}$$

$$^{t+\Delta t}\dot{\mathbf{U}} = \frac{1}{6\Delta t}\left(11\,^{t+\Delta t}\mathbf{U} - 18\,^{t}\mathbf{U} + 9\,^{t-\Delta t}\mathbf{U} - 2\,^{t-2\Delta t}\mathbf{U}\right) \qquad \mathbf{b}$$

a and b inserted in c

$$\left(\frac{2}{\Delta t^2}\mathbf{M} + \frac{12}{6\Delta t}\mathbf{C} + \mathbf{K}\right)\,^{t+\Delta t}\mathbf{U} =$$

$$^{t+\Delta t}\mathbf{R} + \left(\frac{5}{\Delta t^2}\mathbf{M} + \frac{3}{\Delta t}\mathbf{C}\right)\,^{t}\mathbf{U} - \left(\frac{4}{\Delta t^2}\mathbf{M} + \frac{3}{2\Delta t}\mathbf{C}\right)\,^{t-\Delta t}\mathbf{U} + \left(\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{3\Delta t}\mathbf{C}\right)\,^{t-2\Delta t}\mathbf{U}$$



Direct Integration Methods

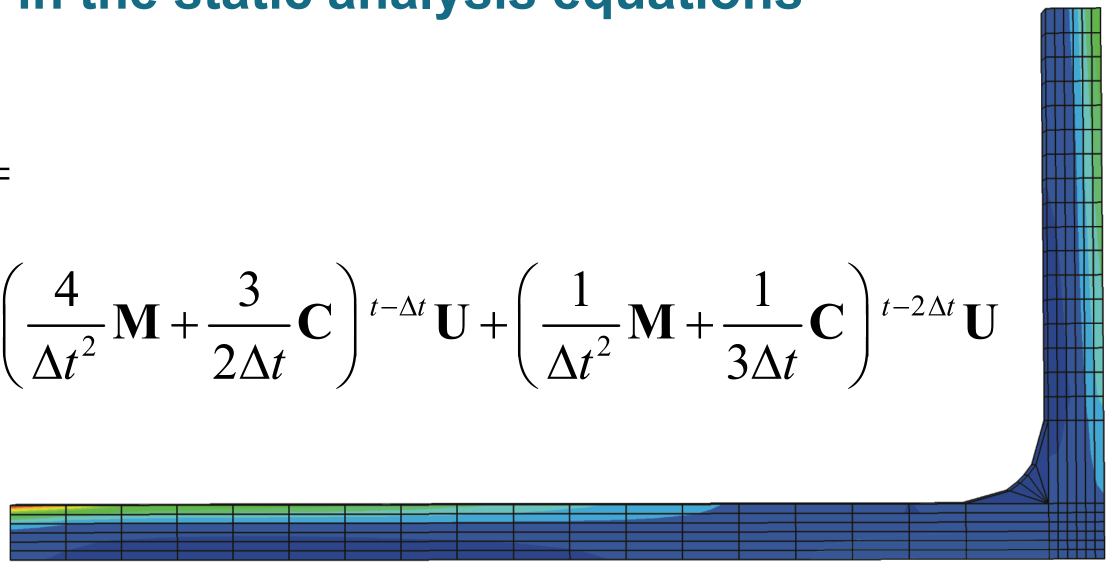
- **The Houbolt method**

  We will not consider the Houbolt in more detail – however it is noted that it is necessary to factorize the stiffness matrix (implicit method)

  Furthermore, if the mass and damping terms are neglected, the Houbolt method results in the static analysis equations

$$\left(\frac{2}{\Delta t^2}\mathbf{M}+\frac{12}{6\Delta t}\mathbf{C}+\mathbf{K}\right)^{t+\Delta t}\mathbf{U}=$$

$$^{t+\Delta t}\mathbf{R}+\left(\frac{5}{\Delta t^2}\mathbf{M}+\frac{3}{\Delta t}\mathbf{C}\right)^{t}\mathbf{U}-\left(\frac{4}{\Delta t^2}\mathbf{M}+\frac{3}{2\Delta t}\mathbf{C}\right)^{t-\Delta t}\mathbf{U}+\left(\frac{1}{\Delta t^2}\mathbf{M}+\frac{1}{3\Delta t}\mathbf{C}\right)^{t-2\Delta t}\mathbf{U}$$



Direct Integration Methods

- **The Wilson θ method**

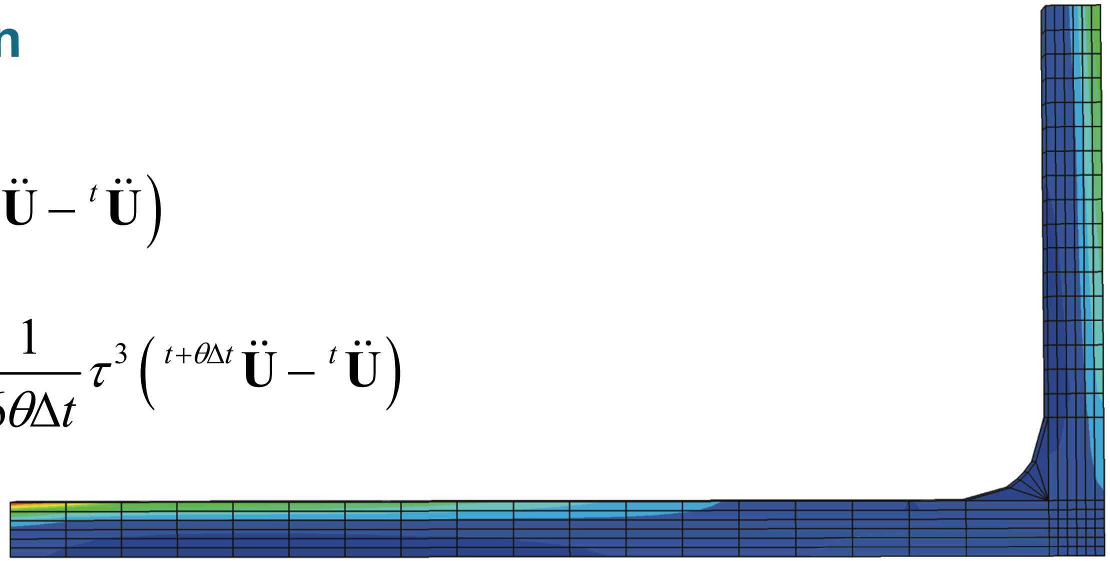
  In this method the acceleration is assumed to vary linearly from time *t* to *t* + Δ*t*

$$ {}^{t+\tau}\ddot{\mathbf{U}} = {}^{t}\ddot{\mathbf{U}} + \frac{\tau}{\theta\Delta t}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}} \right) $$

  By integration we obtain

$$ {}^{t+\tau}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + {}^{t}\ddot{\mathbf{U}}\tau + \frac{\tau^2}{2\theta\Delta t}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}} \right) $$

$$ {}^{t+\tau}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\tau + \frac{1}{2}{}^{t}\ddot{\mathbf{U}}\tau^2 + \frac{1}{6\theta\Delta t}\tau^3\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}} \right) $$



Direct Integration Methods

- **The Wilson θ method**

Setting $\tau = \theta\,\Delta t$ we get

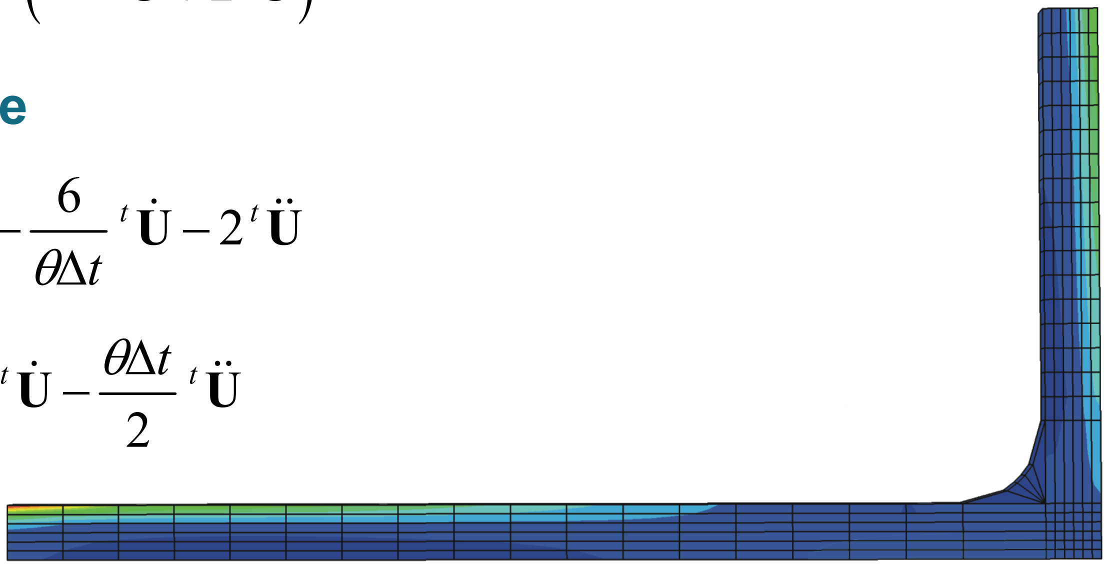
$$^{t+\theta\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \frac{\theta\Delta t}{2}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} + {}^{t}\ddot{\mathbf{U}} \right)$$

$$^{t+\theta\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\,\theta\Delta t + \frac{1}{6}\left(\theta\Delta t\right)^2\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} + 2\,{}^{t}\ddot{\mathbf{U}} \right)$$

from which we can solve

$$^{t+\theta\Delta t}\ddot{\mathbf{U}} = \frac{6}{\left(\theta\Delta t\right)^2}\left({}^{t+\theta\Delta t}\mathbf{U} - {}^{t}\mathbf{U} \right) - \frac{6}{\theta\Delta t}\,{}^{t}\dot{\mathbf{U}} - 2\,{}^{t}\ddot{\mathbf{U}}$$

$$^{t+\theta\Delta t}\dot{\mathbf{U}} = \frac{3}{\theta\Delta t}\left({}^{t+\theta\Delta t}\mathbf{U} - {}^{t}\mathbf{U} \right) - 2\,{}^{t}\dot{\mathbf{U}} - \frac{\theta\Delta t}{2}\,{}^{t}\ddot{\mathbf{U}}$$



Direct Integration Methods

$$^{t+\theta\Delta t}\ddot{\mathbf{U}} = \frac{6}{(\theta\Delta t)^2}\left(^{t+\theta\Delta t}\mathbf{U} - {}^{t}\mathbf{U}\right) - \frac{6}{\theta\Delta t}\,{}^{t}\dot{\mathbf{U}} - 2\,{}^{t}\ddot{\mathbf{U}}$$

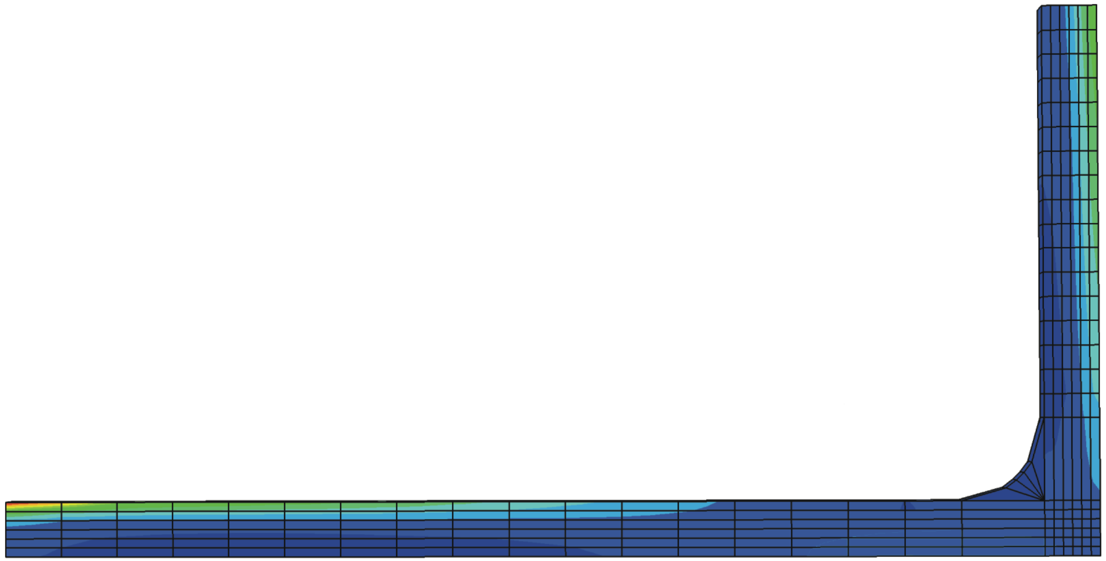
$$^{t+\theta\Delta t}\dot{\mathbf{U}} = \frac{3}{\theta\Delta t}\left(^{t+\theta\Delta t}\mathbf{U} - {}^{t}\mathbf{U}\right) - 2\,{}^{t}\dot{\mathbf{U}} - \frac{\theta\Delta t}{2}\,{}^{t}\ddot{\mathbf{U}}$$

- **The Wilson θ method**

We now solve for the displacements, velocities and accelerations by inserting into the dynamic equilibrium equation

$$\mathbf{M}\,{}^{t+\theta\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\theta\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\theta\Delta t}\mathbf{U} = {}^{t+\theta\Delta t}\overline{\mathbf{R}}$$

$$^{t+\theta\Delta t}\overline{\mathbf{R}} = {}^{t}\mathbf{R} + \theta({}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{R})$$



Direct Integration Methods

- **The Wilson θ method : Solution procedure:**

A: Initial calculations

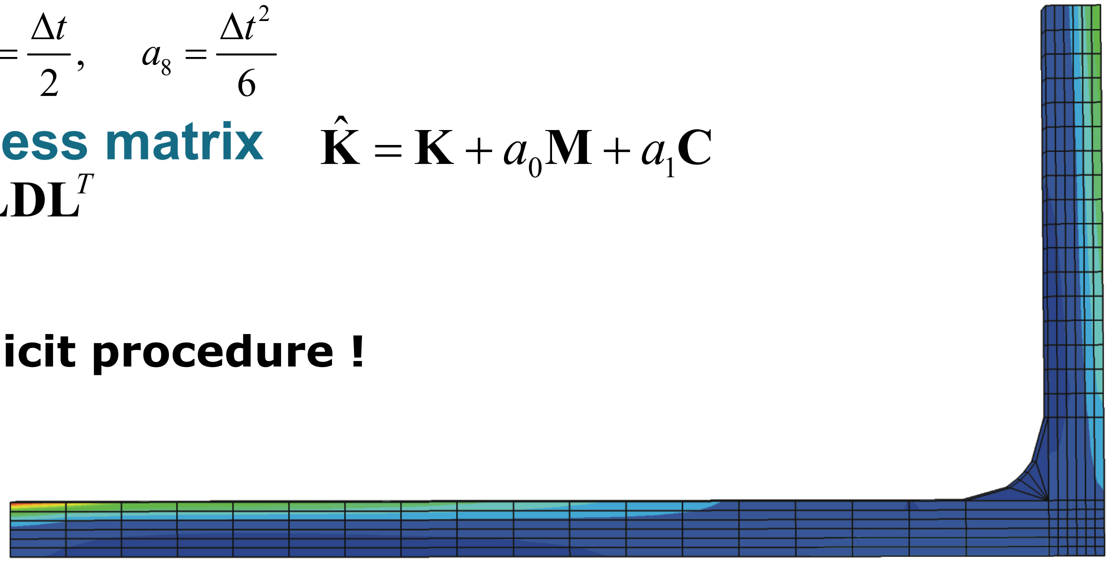
1) **Form stiffness matrix, mass matrix and damping matrix**
2) **Initialize** $^{0}\mathbf{U},\ ^{0}\dot{\mathbf{U}}$ **and** $^{0}\ddot{\mathbf{U}}$
3) **Select time step Δt and calculate integration constants θ = 1.4**

$$a_0 = \frac{6}{(\theta\Delta t)^2}, \quad a_1 = \frac{3}{\theta\Delta t}, \quad a_2 = 2a_1, \quad a_3 = \frac{\theta\Delta t}{2}, \quad a_4 = \frac{a_0}{\theta}$$

$$a_5 = \frac{-a_2}{\theta}, \quad a_6 = 1 - \frac{3}{\theta}, \quad a_7 = \frac{\Delta t}{2}, \quad a_8 = \frac{\Delta t^2}{6}$$

4) **Form effective stiffness matrix** $\hat{\mathbf{K}} = \mathbf{K} + a_0\mathbf{M} + a_1\mathbf{C}$
5) **Triangularize** $\hat{\mathbf{K}} = \mathbf{LDL}^T$

Implicit procedure !



Direct Integration Methods

- **The Wilson θ method : Solution procedure:**

B: For each time step

1) Calculate effective loads at time *t*+Δ*t*:

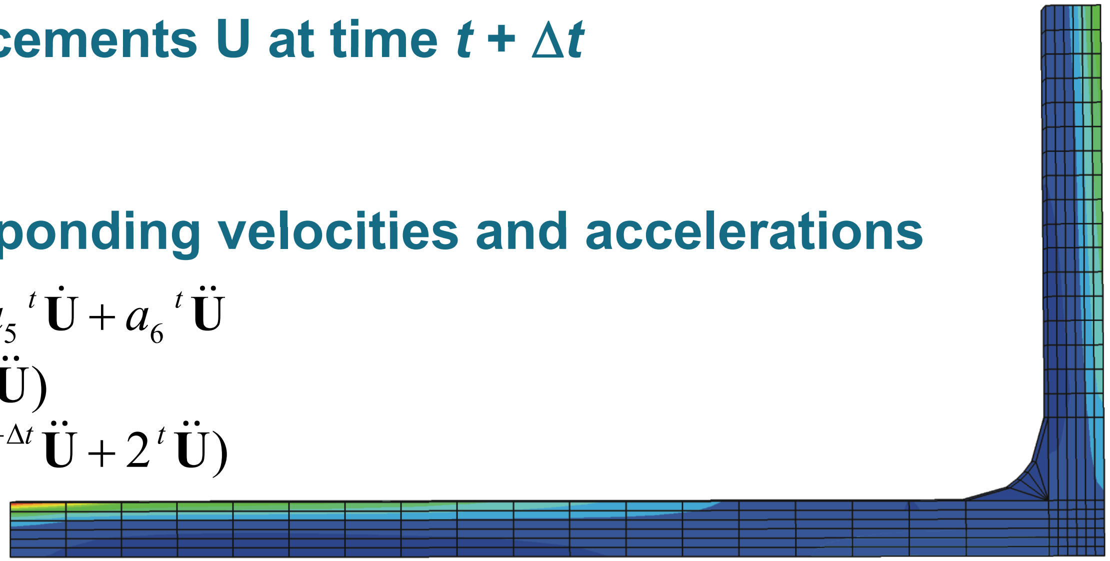
$$ {}^{t+\theta\Delta t}\hat{\mathbf{R}} = {}^{t}\mathbf{R} + \theta({}^{t+\theta\Delta t}\mathbf{R} - {}^{t}\mathbf{R}) + \mathbf{M}(a_0\,{}^{t}\mathbf{U} + a_2\,{}^{t}\dot{\mathbf{U}} + 2\,{}^{t}\ddot{\mathbf{U}}) $$
$$ + \mathbf{C}(a_1\,{}^{t}\mathbf{U} + 2\,{}^{t}\dot{\mathbf{U}} + a_3\,{}^{t}\ddot{\mathbf{U}}) $$

2) Solve for the displacements U at time *t* + Δ*t*

$$ \mathbf{L}\mathbf{D}\mathbf{L}^{T}\,{}^{t+\theta\Delta t}\mathbf{U} = {}^{t+\theta\Delta t}\hat{\mathbf{R}} $$

3) Solve for the corresponding velocities and accelerations

$$ {}^{t+\Delta t}\ddot{\mathbf{U}} = a_4({}^{t+\theta\Delta t}\mathbf{U} - {}^{t}\mathbf{U}) + a_5\,{}^{t}\dot{\mathbf{U}} + a_6\,{}^{t}\ddot{\mathbf{U}} $$
$$ {}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + a_7({}^{t+\Delta t}\ddot{\mathbf{U}} + {}^{t}\ddot{\mathbf{U}}) $$
$$ {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \Delta t\,{}^{t}\dot{\mathbf{U}} + a_8({}^{t+\Delta t}\ddot{\mathbf{U}} + 2\,{}^{t}\ddot{\mathbf{U}}) $$



Direct Integration Methods

- **The Newmark method**

This method may be seen as an extension of the Wilson θ method

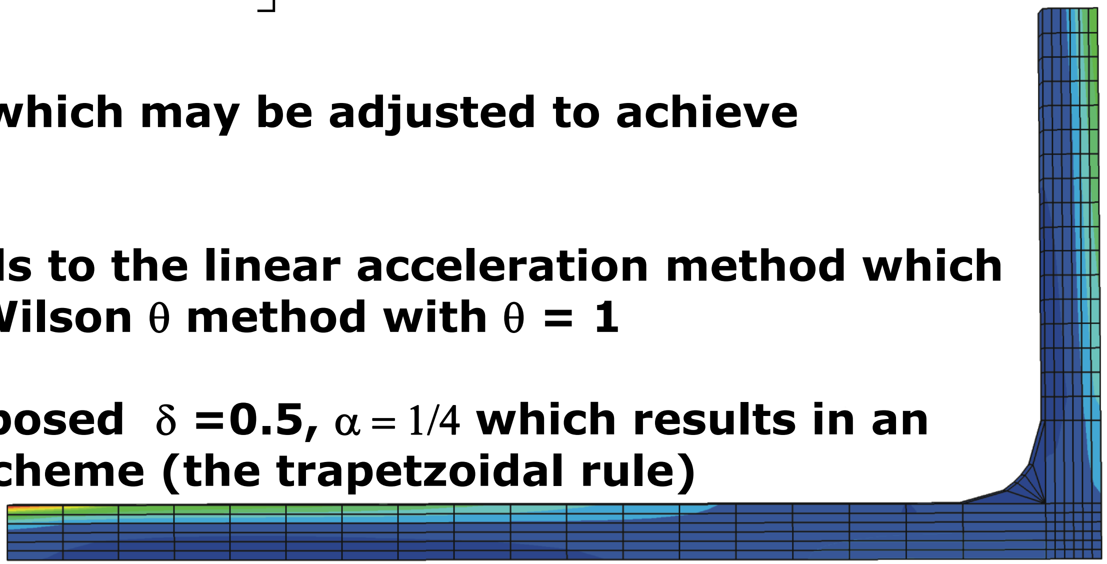
$$ {}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \left[(1-\delta)\,{}^{t}\ddot{\mathbf{U}} + \delta\,{}^{t+\Delta t}\ddot{\mathbf{U}} \right] \Delta t $$

$$ {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\Delta t + \left[\left(\frac{1}{2}-\alpha\right){}^{t}\ddot{\mathbf{U}} + \alpha\,{}^{t+\Delta t}\ddot{\mathbf{U}} \right] \Delta t^2 $$

δ and α are parameters which may be adjusted to achieve accuracy and stability

δ =0.5, $\alpha = 1/6$ corresponds to the linear acceleration method which also correspond to the Wilson θ method with θ = 1

Newmark originally proposed δ =0.5, $\alpha = 1/4$ which results in an unconditionally stable scheme (the trapetzoidal rule)



Direct Integration Methods

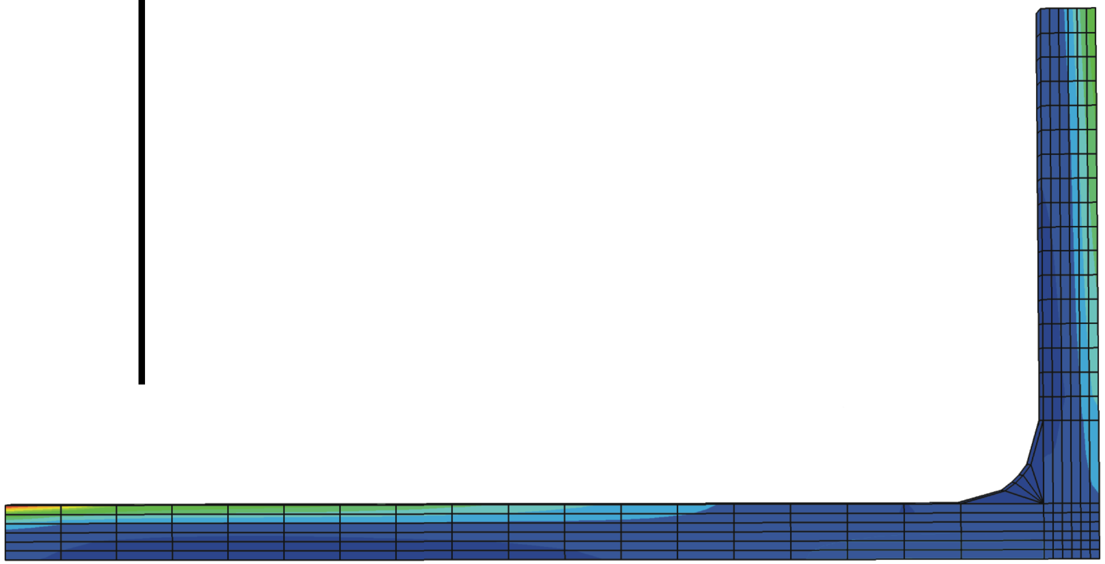
- **The Newmark method**

  We now solve for the displacements, velocities and accelerations by inserting into the dynamic equilibrium equation

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\Delta t}\mathbf{U} = {}^{t+\Delta t}\mathbf{R}$$

$${}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \left[(1-\delta)\,{}^{t}\ddot{\mathbf{U}} + \delta\,{}^{t+\Delta t}\ddot{\mathbf{U}}\right]\Delta t$$

$${}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\Delta t + \left[(\frac{1}{2}-\alpha)\,{}^{t}\ddot{\mathbf{U}} + \alpha\,{}^{t+\Delta t}\ddot{\mathbf{U}}\right]\Delta t^2$$



Direct Integration Methods

- **The Newmark method : Solution procedure:**

A: Initial calculations

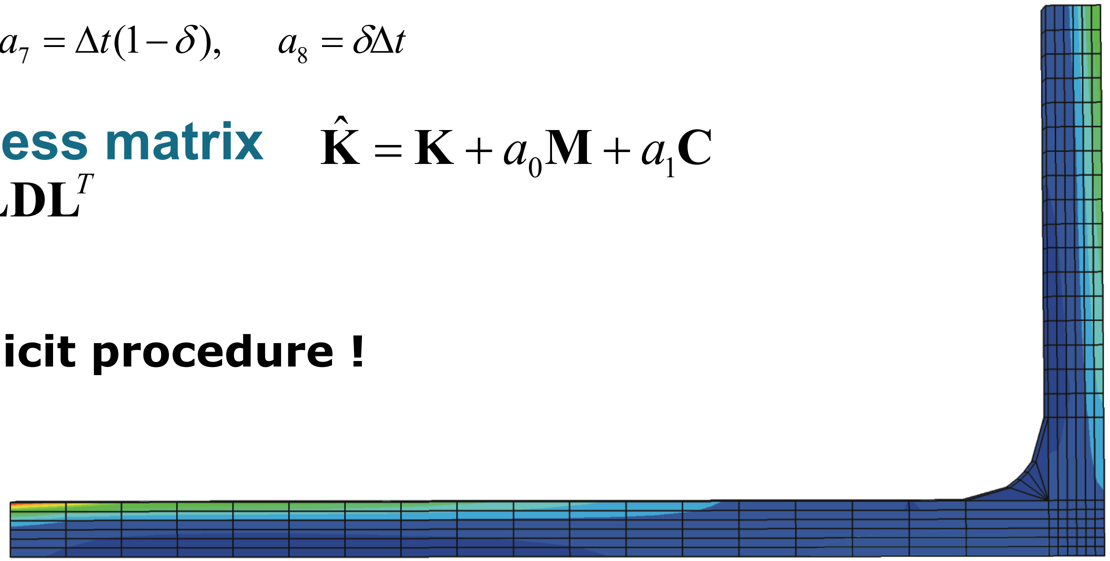
1) Form stiffness matrix, mass matrix and damping matrix
2) Initialize ${}^{0}\mathbf{U},\ {}^{0}\dot{\mathbf{U}}$ and ${}^{0}\ddot{\mathbf{U}}$
3) Select time step Δt and parameters α and δ

$$a_0 = \frac{1}{\alpha \Delta t^2}, \quad a_1 = \frac{\delta}{\alpha \Delta t}, \quad a_2 = \frac{1}{\alpha \Delta t}, \quad a_3 = \frac{1}{2\alpha} - 1, \quad a_4 = \frac{\delta}{2\alpha} - 1$$

$$a_5 = \frac{\delta}{\alpha} - 1, \quad a_6 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right), \quad a_7 = \Delta t(1-\delta), \quad a_8 = \delta \Delta t$$

4) Form effective stiffness matrix $\hat{\mathbf{K}} = \mathbf{K} + a_0\mathbf{M} + a_1\mathbf{C}$
5) Triangularize $\hat{\mathbf{K}} = \mathbf{LDL}^T$

Implicit procedure !



Direct Integration Methods

- **The Newmark method : Solution procedure:**

B: For each time step

1) Calculate effective loads at time *t*:

$$^{t+\Delta t}\hat{\mathbf{R}} = {}^{t+\Delta t}\mathbf{R} + \mathbf{M}(a_0\,{}^{t}\mathbf{U} + a_2\,{}^{t}\dot{\mathbf{U}} + a_3\,{}^{t}\ddot{\mathbf{U}})$$

$$+\mathbf{C}(a_1\,{}^{t}\mathbf{U} + a_4\,{}^{t}\dot{\mathbf{U}} + a_5\,{}^{t}\ddot{\mathbf{U}})$$

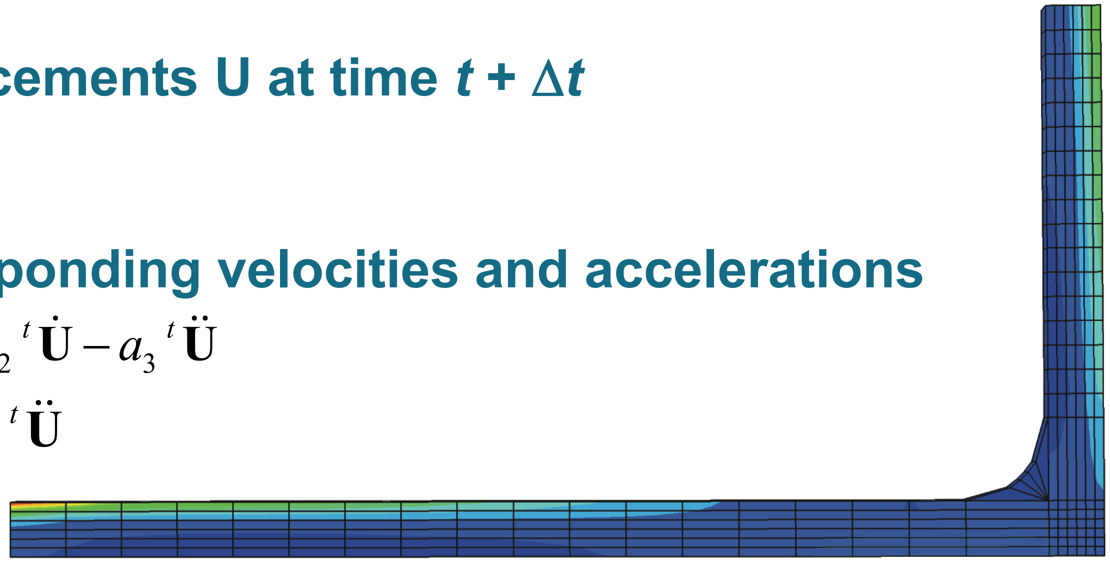
2) Solve for the displacements U at time *t* + Δ*t*

$$\mathbf{LDL}^{T}\,{}^{t+\Delta t}\mathbf{U} = {}^{t+\Delta t}\hat{\mathbf{R}}$$

3) Solve for the corresponding velocities and accelerations

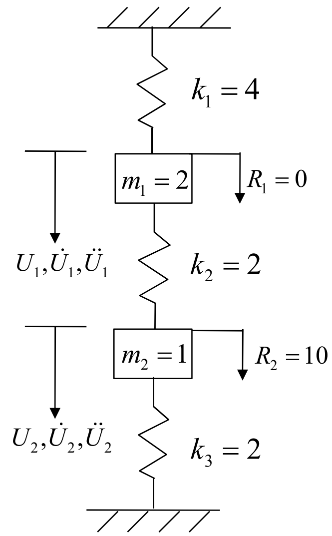
$$^{t+\Delta t}\ddot{\mathbf{U}} = a_0({}^{t+\Delta t}\mathbf{U} - {}^{t}\mathbf{U}) - a_2\,{}^{t}\dot{\mathbf{U}} - a_3\,{}^{t}\ddot{\mathbf{U}}$$

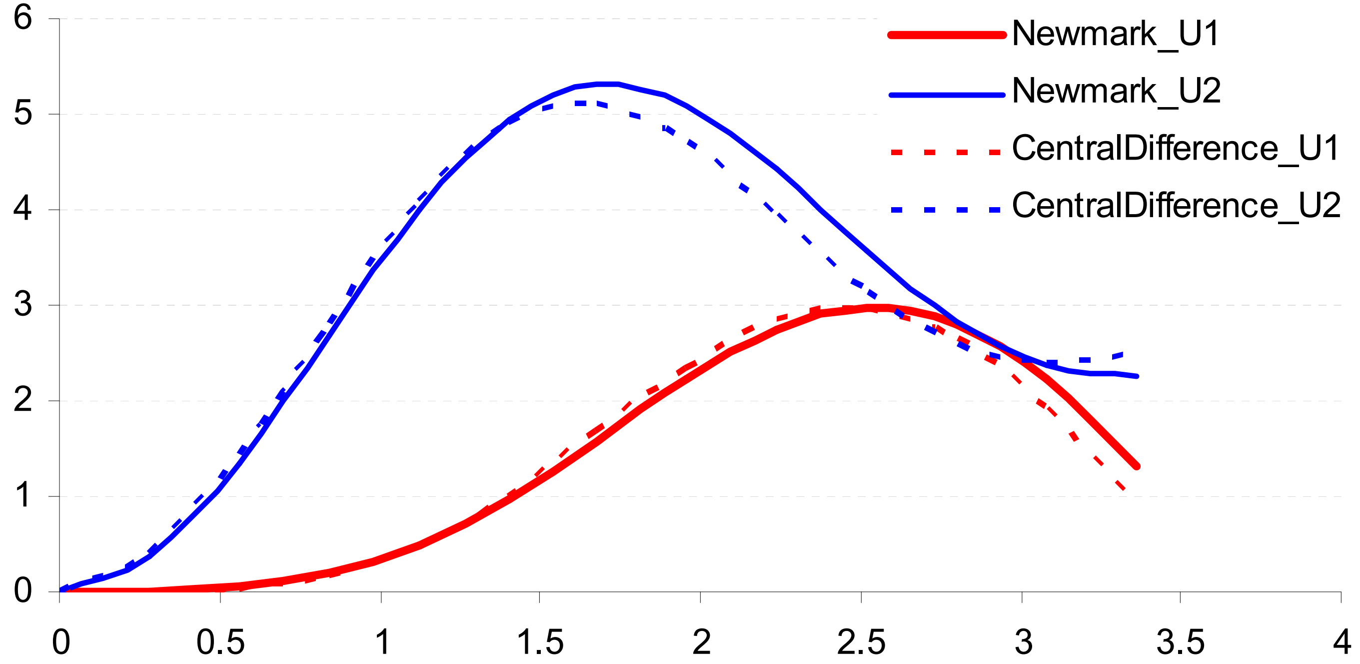
$$^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + a_7\,{}^{t+\Delta t}\ddot{\mathbf{U}} + a_6\,{}^{t}\ddot{\mathbf{U}}$$

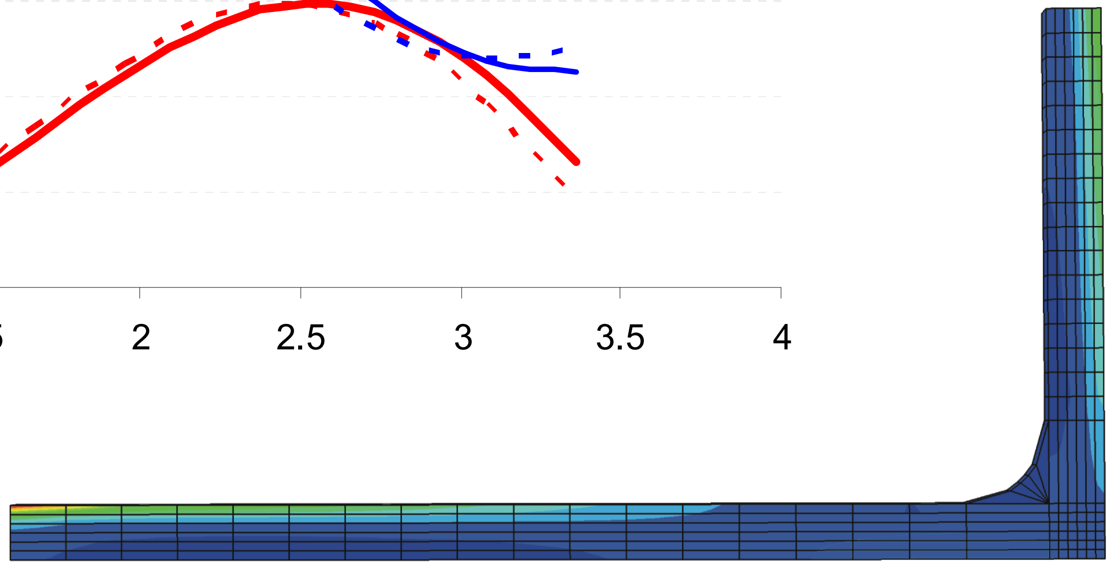


Direct Integration Methods

- **The Newmark method : Example:**

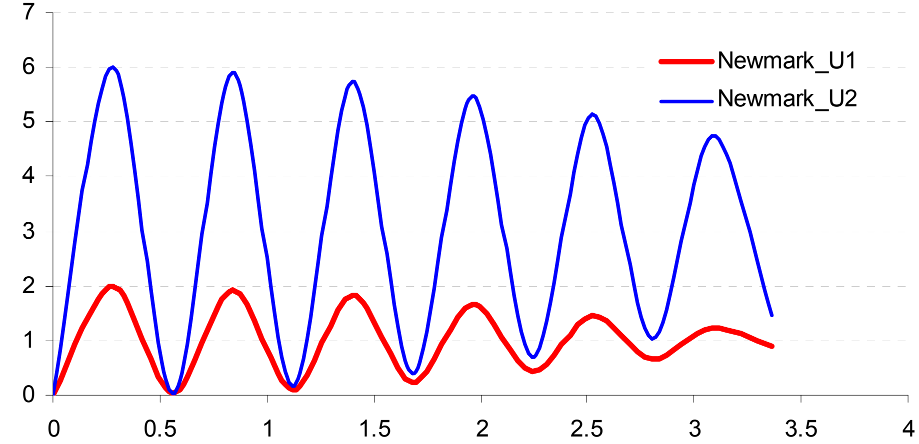


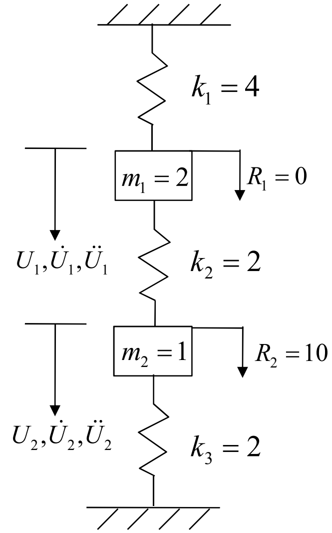




Direct Integration Methods

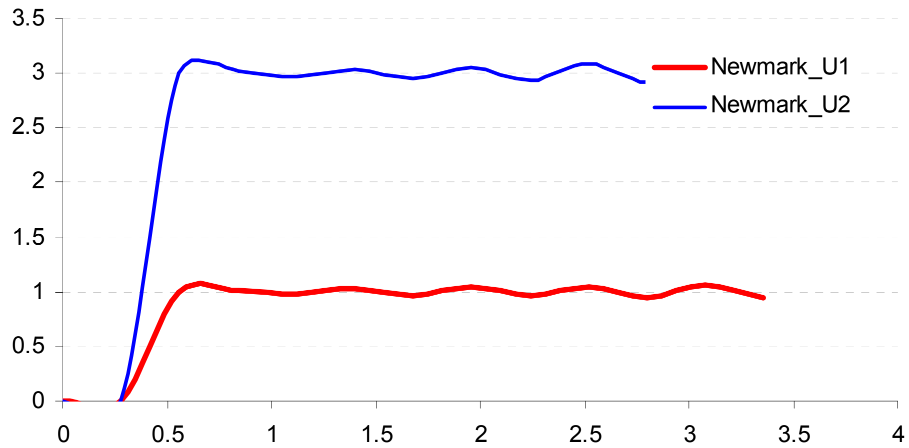
- **The Newmark method : Example:**





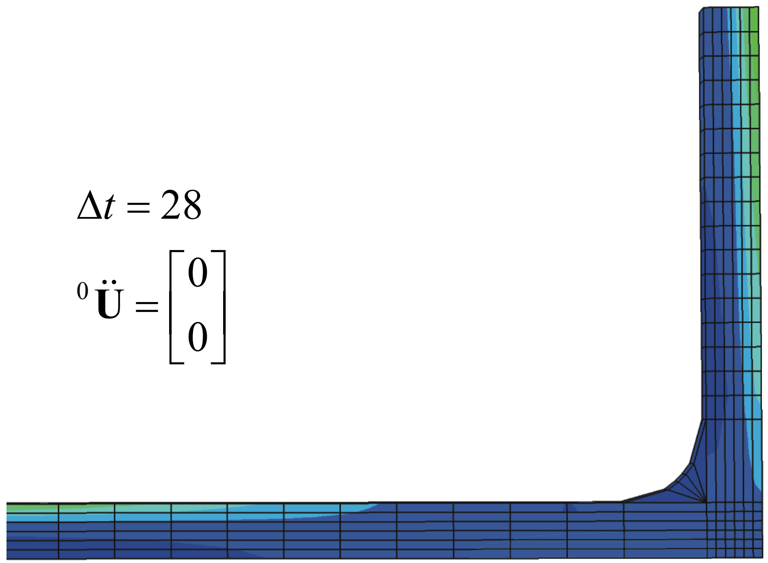
$$\Delta t = 28$$

$$^{0}\ddot{\mathbf{U}} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$



$$\Delta t = 28$$

$$^{0}\ddot{\mathbf{U}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Direct Integration Methods

- **Coupling of integration operators**

  For some problems it may be an advantage to combine the different types of integration schemes – e.g. if a structure is subjected to dynamic load effect from hydrodynamic loading then the analysis of the hydrodynamic forces may be assessed using an explicit scheme and the structural response by using an implicit scheme.

  The best choice of strategy will depend on the problem in regard stability and accuracy!