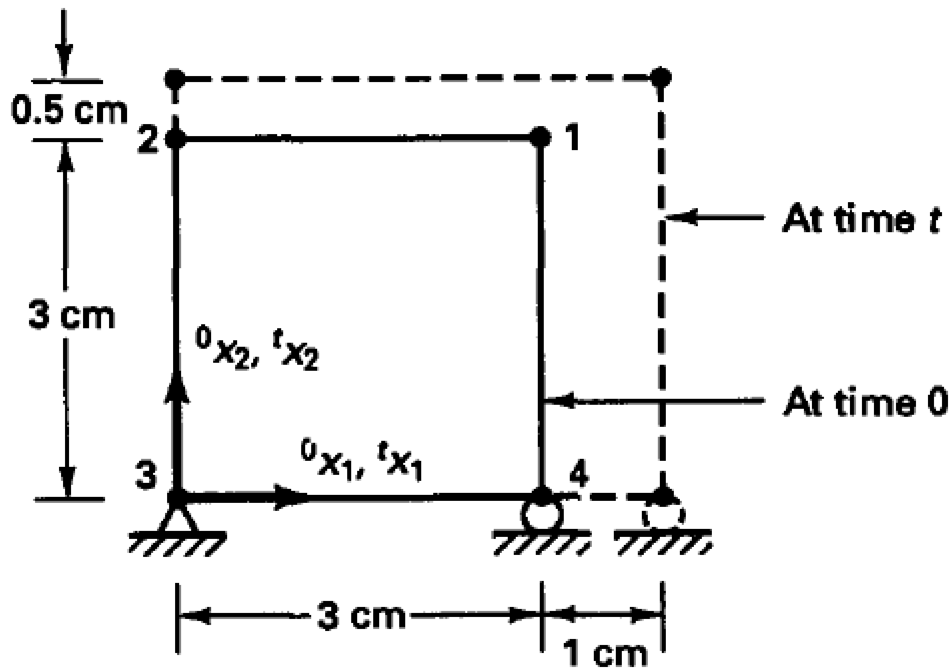
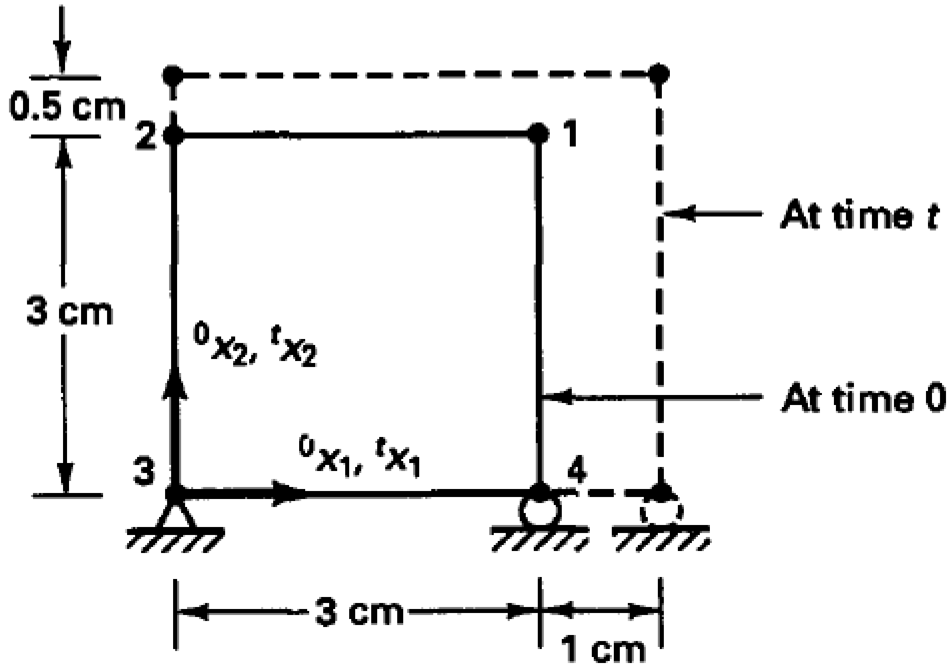
Pages 561-578

Finites Element Procedure, Klaus-Jürgen Bathe

Example 6.18

Four-node plane strain element in large displacement/large strain conditions





Quest: Establish the matrices $^t_0\mathbf{B}_{L0}$, $^t_0\mathbf{B}_{L1}$, $^t_0\mathbf{B}_{NL}$

Corresponding to TL using Table 6.5

Reminder

- TL – Total Lagrangian formulation

$$({}_0^t\mathbf{K}_L + {}_0^t\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}_0^t\mathbf{F}$$

$${}_0^t\mathbf{K}_L\hat{\mathbf{u}} = \left(\int_{{}^0V} {}_0^t\mathbf{B}_L^T\, {}_0\mathbf{C}\, {}_0^t\mathbf{B}_L\, d^0V\right)\hat{\mathbf{u}}$$

$${}_0^t\mathbf{K}_{NL}\hat{\mathbf{u}} = \left(\int_{{}^0V} {}_0^t\mathbf{B}_{NL}^T\, {}_0^t\mathbf{S}\, {}_0^t\mathbf{B}_{NL}\, d^0V\right)\hat{\mathbf{u}}$$

$${}_0^t\mathbf{F} = \int_{{}^0V} {}_0^t\mathbf{B}_L^T\, {}_0^t\hat{\mathbf{S}}\, d^0V$$

To get: $\mathbf{U}$ = vector of increments in the nodal point displacements

We need to know: ${}_0\mathbf{C}$ = incremental stress-strain material property matrices

${}_0^t\mathbf{S}$, ${}^t\hat{\mathbf{S}}$ = matrix and vector of second Piola-Kirchhoff stresses

${}_0^t\mathbf{B}_L$ = linear strain-displacement transformation matrices

${}_0^t\mathbf{B}_{NL}$ = nonlinear strain-displacement transformation matrices

Table 6.5

$${}_0^t\mathbf{B}_L = {}_0^t\mathbf{B}_{L0} + {}_0^t\mathbf{B}_{L1}$$

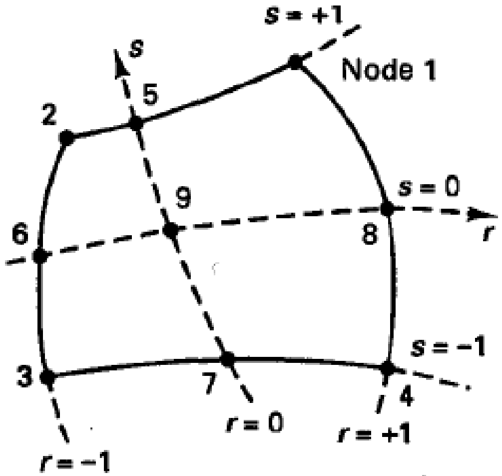
= linear strain-displacement transformation matrices

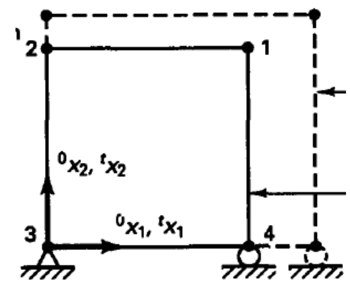
$${}_0^t\mathbf{B}_{L0} = \begin{bmatrix} {}_0h_{1,1} & 0 & {}_0h_{2,1} & 0 & {}_0h_{3,1} & 0 & \cdots & {}_0h_{N,1} & 0 \\ 0 & {}_0h_{1,2} & 0 & {}_0h_{2,2} & 0 & {}_0h_{3,2} & \cdots & 0 & {}_0h_{N,2} \\ {}_0h_{1,2} & {}_0h_{1,1} & {}_0h_{2,2} & {}_0h_{2,1} & {}_0h_{3,2} & {}_0h_{3,1} & \cdots & {}_0h_{N,2} & {}_0h_{N,1} \\ \frac{h_1}{{}^0\bar{x}_1} & 0 & \frac{h_2}{{}^0\bar{x}_1} & 0 & \frac{h_3}{{}^0\bar{x}_1} & 0 & \cdots & \frac{h_N}{{}^0\bar{x}_1} & 0 \end{bmatrix}$$

Only for 3D axisymmetric elements

where ${}_0h_{k,j} = \dfrac{\partial h_k}{\partial\, {}^0x_j}$;

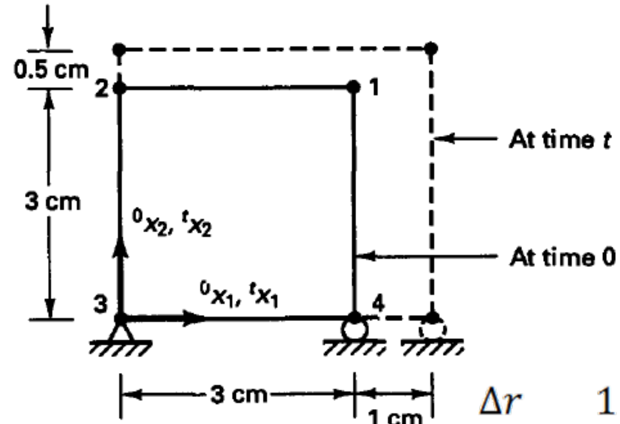
Quadrilateral Elements





Include only if node i is defined

		$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
$h_1 =$	$\frac{1}{4}(1 + r)(1 + s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1 - r)(1 + s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1 - r)(1 - s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1 + r)(1 - s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1 - r^2)(1 + s)$					$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1 - s^2)(1 - r)$					$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1 - r^2)(1 - s)$					$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1 - s^2)(1 + r)$					$-\frac{1}{2}h_9$
$h_9 =$	$(1 - r^2)(1 - s^2)$					



r=-1 r=+1

$$\frac{\Delta r}{\Delta x_1} = \frac{1-(-1)}{3cm}$$

$$h_1 = \frac{1}{4}(1+r)(1+s) \rightarrow {}_0h_{1,1} = \frac{dh_1}{dx_1} = \frac{dh_1}{dr} * \frac{dr}{dx_1} = \frac{1}{4}(1+s) * \frac{2}{3} = \frac{1}{6}(1+s)$$

$$ {}_0^t\mathbf{B}_{L0} = \frac{1}{6}\begin{bmatrix} (1+s) & 0 & -(1+s) & 0 & -(1-s) & 0 & (1-s) & 0 \\ 0 & (1+r) & 0 & (1-r) & 0 & -(1-r) & 0 & -(1+r) \\ (1+r) & (1+s) & (1-r) & -(1+s) & -(1-r) & -(1-s) & -(1+r) & (1-s) \end{bmatrix}$$

Table 6.5

$$
{}_0^t\mathbf{B}_{L1} = \begin{bmatrix}
l_{11}\,{}_0h_{1,1} & l_{21}\,{}_0h_{1,1} & l_{11}\,{}_0h_{2,1} & l_{21}\,{}_0h_{2,1} & \cdots & l_{11}\,{}_0h_{N,1} & l_{21}\,{}_0h_{N,1} \\
l_{12}\,{}_0h_{1,2} & l_{22}\,{}_0h_{1,2} & l_{12}\,{}_0h_{2,2} & l_{22}\,{}_0h_{2,2} & \cdots & l_{12}\,{}_0h_{N,2} & l_{22}\,{}_0h_{N,2} \\
(l_{11}\,{}_0h_{1,2} + l_{12}\,{}_0h_{1,1}) & (l_{21}\,{}_0h_{1,2} + l_{22}\,{}_0h_{1,1}) & (l_{11}\,{}_0h_{2,2} + l_{12}\,{}_0h_{2,1}) & (l_{21}\,{}_0h_{2,2} + l_{22}\,{}_0h_{2,1}) & \cdots & (l_{11}\,{}_0h_{N,2} + l_{12}\,{}_0h_{N,1}) & (l_{21}\,{}_0h_{N,2} + l_{22}\,{}_0h_{N,1}) \\
l_{33}\,\frac{h_1}{{}^0\bar{x}_1} & 0 & l_{33}\,\frac{h_2}{{}^0\bar{x}_1} & 0 & \cdots & l_{33}\,\frac{h_N}{{}^0\bar{x}_1} & 0
\end{bmatrix}
$$

where $l_{11} = \sum_{k=1}^{N} {}_0h_{k,1}\,{}^tu_1^k$;

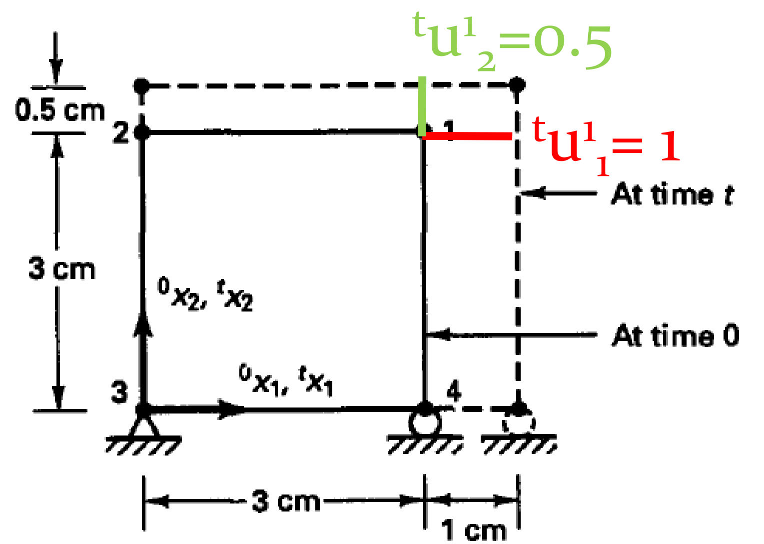
$$u_j^k = {}^{t+\Delta t}u_j^k - {}^tu_j^k;$$

$${}^tu_1^1 = 1; \qquad {}^tu_2^1 = 0.5$$

$${}^tu_1^2 = 0; \qquad {}^tu_2^2 = 0.5$$

$${}^tu_1^3 = 0; \qquad {}^tu_2^3 = 0$$

$${}^tu_1^4 = 1; \qquad {}^tu_2^4 = 0$$



$$l_{11} = \sum_{k=1}^{4} {}_0h_{k,1}\, {}^tu_1^k = \left({}_0h_{1,1} * {}^tu_1^1 + {}_0h_{4,1} * {}^tu_1^4\right) = \frac{1}{3}$$

$$h_1 = \tfrac{1}{4}(1+r)(1+s) \longrightarrow {}_0h_{1,1} = \frac{1}{6}(1+s)$$

$$h_4 = \tfrac{1}{4}(1+r)(1-s) \longrightarrow {}_0h_{4,1} = \frac{1}{6}(1-s)$$

$${}^tu_1^1 = 1; \quad {}^tu_1^2 = 0; \quad {}^tu_1^3 = 0; \quad {}^tu_1^4 = 1;$$

$${}_0^t\mathbf{B}_{L1} = \begin{bmatrix} l_{11}\ {}_0h_{1,1} & \cdots \\ l_{12}\ {}_0h_{1,2} & \cdots \\ (l_{11}\ {}_0h_{1,2} + l_{12}\ {}_0h_{1,1}) & \cdots \end{bmatrix}$$

$${}_0^t\mathbf{B}_{L1} = \frac{1}{36}\begin{bmatrix} 2(1+s) & 0 & -2(1+s) & 0 & -2(1-s) & 0 & 2(1-s) & 0 \\ 0 & (1+r) & 0 & (1-r) & 0 & -(1-r) & 0 & -(1+r) \\ 2(1+r) & (1+s) & 2(1-r) & -(1+s) & -2(1-r) & -(1-s) & -2(1+r) & (1-s) \end{bmatrix}$$

Table 6.5

$${}^{t}_{0}\mathbf{B}_{NL}$$

= nonlinear strain-displacement transformation matrices

$$
{}^{t}_{0}\mathbf{B}_{NL} = \begin{bmatrix}
{}_0h_{1,1} & 0 & {}_0h_{2,1} & 0 & {}_0h_{3,1} & 0 & \cdots & {}_0h_{N,1} & 0 \\
{}_0h_{1,2} & 0 & {}_0h_{2,2} & 0 & {}_0h_{3,2} & 0 & \cdots & {}_0h_{N,2} & 0 \\
0 & {}_0h_{1,1} & 0 & {}_0h_{2,1} & 0 & {}_0h_{3,1} & \cdots & 0 & {}_0h_{N,1} \\
0 & {}_0h_{1,2} & 0 & {}_0h_{2,2} & 0 & {}_0h_{3,2} & \cdots & 0 & {}_0h_{N,2} \\
\frac{h_1}{{}^0\bar{x}_1} & 0 & \frac{h_2}{{}^0\bar{x}_1} & 0 & \frac{h_3}{{}^0\bar{x}_1} & 0 & \cdots & \frac{h_N}{{}^0\bar{x}_1} & 0
\end{bmatrix}
$$

$$
{}^{t}_{0}\mathbf{B}_{NL} = \frac{1}{6}\begin{bmatrix}
(1+s) & 0 & -(1+s) & 0 & -(1-s) & 0 & (1-s) & 0 \\
(1+r) & 0 & (1-r) & 0 & -(1-r) & 0 & -(1+r) & 0 \\
0 & (1+s) & 0 & -(1+s) & 0 & -(1-s) & 0 & (1-s) \\
0 & (1+r) & 0 & (1-r) & 0 & -(1-r) & 0 & -(1+r)
\end{bmatrix}
$$

6.4 Displacement/Pressure Formulation for Large Deformations

(Page 561 – 567)

Why Displacement/Pressure Formulations for Large Deformations?

- A pure displacement-based procedure is not effective for (almost) incompressible analysis
- Material in large deformations (like rubberlike materials or metals under large strain) behave almost imcompressible
- Therefore total and updated Lagrangian formulations should be extended to incompressible analysis

Total Lagrangian Formulation pure displacement-based

Equation of motion (based on the principle of virtual displacements):

$$\int_{^0V} {}^{t+\Delta t}_{0}S_{ij}\,\delta\,{}^{t+\Delta t}_{0}\epsilon_{ij}\,d^0V = {}^{t+\Delta t}\mathcal{R}$$

We define the incremental potential*:

$$d\,{}^t_0\overline{W} = {}^t_0\bar{S}_{ij}\,d\,{}^t_0\epsilon_{ij}$$

Now the principle of virtual displacement can be rewritten as:

$$\delta\left(\int_{^0V} {}^t_0\overline{W}\,d^0V\right) = {}^t\mathcal{R}$$

*The overbar denotes that that only displacement fields where considered (not pressure fields)

Displacement/Pressure Total Lagrangian Formulation

$$\delta\left(\int_{^0V} {}_0^tW \, d^0V\right) = {}^t\mathcal{R}$$

$${}_0^tW = {}_0^t\overline{W} + {}_0^tQ$$

${}_0^tW$: Total Potential

${}_0^t\overline{W}$: Potential of the displacements

${}_0^tQ$: Potential of the pressure

Potential Q

$$ {}_0^tQ = -\frac{1}{2\kappa}({}^t\bar{p} - {}^t\tilde{p})^2 $$

${}^t\bar{p}$: Pressure of the displacement field

${}^t\tilde{p}$: Total element pressure at time t

κ: bulk modulus

$$ {}^tu_i = \sum_{k=1}^{N} h_k\, {}^tu_i^k; \qquad {}^t\tilde{p} = \sum_{i=1}^{q} g_i\, {}^t\hat{p}_i $$

$$ {}_0^tW = {}_0^t\overline{W} - \frac{1}{2\kappa}({}^t\bar{p} - {}^t\tilde{p})^2 $$

Bulk Modulus κ

$$ {}_{0}^{t}Q = -\frac{1}{2\kappa}({}^{t}\bar{p} - {}^{t}\tilde{p})^2 $$

- The bulk modulus (Kompressionsmodul) is defined as the pressure increase needed to cause a given relative decrease in volume

$$ \kappa = -V\frac{\partial P}{\partial V} $$

Linearization

$$\begin{pmatrix} {}^t\mathbf{KUU} & {}^t\mathbf{KUP} \\ {}^t\mathbf{KPU} & {}^t\mathbf{KPP} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} {}^{t+\Delta t}\mathbf{R} \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} {}^t\mathbf{FU} \\ {}^t\mathbf{FP} \end{pmatrix}$$

Vectors FU, FP contains these Elements

$${}^tFU_i = \frac{\partial}{\partial^t \hat{u}_i}\left(\int_{{}^0V} {}_0^tW \, d^0V\right) = \int_{{}^0V} {}_0^tS_{ki} \frac{\partial {}_0^t\epsilon_{ki}}{\partial^t \hat{u}_i} \, d^0V$$

$${}^tFP_i = \frac{\partial}{\partial^t \hat{p}_i}\left(\int_{{}^0V} {}_0^tW \, d^0V\right) = \int_{{}^0V} \frac{1}{\kappa}({}^t\overline{p} - {}^t\tilde{p}) \frac{\partial^t \tilde{p}}{\partial^t \hat{p}_i} \, d^0V$$

↑

$${}_0^tQ = -\frac{1}{2\kappa}({}^t\overline{p} - {}^t\tilde{p})^2$$

Linearization

$$\begin{pmatrix} {}^t\mathbf{KUU} & {}^t\mathbf{KUP} \\ {}^t\mathbf{KPU} & {}^t\mathbf{KPP} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} {}^{t+\Delta t}\mathbf{R} \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} {}^t\mathbf{FU} \\ {}^t\mathbf{FP} \end{pmatrix}$$

Matrices KUU, KUP, KPP, KPU contains these Elements

$${}^tKUU_{ij} = \frac{\partial^t FU_i}{\partial^t \hat{u}_j} = \int_{{}^0V} {}_0CUU_{klrs} \frac{\partial {}_0^t\epsilon_{kl}}{\partial^t \hat{u}_i} \frac{\partial {}_0^t\epsilon_{rs}}{\partial^t \hat{u}_j} \, d^0V + \int_{{}^0V} {}_0^tS_{kl} \frac{\partial^2 \, {}_0^t\epsilon_{kl}}{\partial^t \hat{u}_i \, \partial^t \hat{u}_j} \, d^0V$$

$${}^tKUP_{ij} = \frac{\partial^t FU_i}{\partial^t \hat{p}_j} = \frac{\partial^t FP_j}{\partial^t \hat{u}_i} = {}^tKPU_{ji} = \int_{{}^0V} {}_0CUP_{kl} \frac{\partial {}_0^t\epsilon_{kl}}{\partial^t \hat{u}_i} \frac{\partial^t \tilde{p}}{\partial^t \hat{p}_j} \, d^0V$$

$${}^tKPP_{ij} = \frac{\partial^t FP_i}{\partial^t \hat{p}_j} = \int_{{}^0V} -\frac{1}{\kappa} \frac{\partial^t \tilde{p}}{\partial^t \hat{p}_i} \frac{\partial^t \tilde{p}}{\partial^t \hat{p}_j} \, d^0V$$

$$ {}_0^tS_{kl} = {}_0^t\bar{S}_{kl} - \frac{1}{\kappa}({}^t\bar{p} - {}^t\tilde{p})\frac{\partial{}^t\bar{p}}{\partial{}_0^t\epsilon_{kl}} $$

$$ {}_0CUU_{klrs} = {}_0\bar{C}_{klrs} - \frac{1}{\kappa}\frac{\partial{}^t\bar{p}}{\partial{}_0^t\epsilon_{kl}}\frac{\partial{}^t\bar{p}}{\partial{}_0^t\epsilon_{rs}} - \frac{1}{\kappa}({}^t\bar{p} - {}^t\tilde{p})\frac{\partial^2\,{}^t\bar{p}}{\partial{}_0^t\epsilon_{kl}\,\partial{}_0^t\epsilon_{rs}} $$

$$ {}_0CUP_{kl} = \frac{1}{\kappa}\frac{\partial{}^t\bar{p}}{\partial{}_0^t\epsilon_{kl}} $$

$$ \frac{\partial{}^t\tilde{p}}{\partial{}^t\hat{p}_i} = g_i $$

$$ \frac{\partial{}_0^t\epsilon_{kl}}{\partial{}^tu_n^L} = \frac{1}{2}({}_0^tx_{n,k}\;{}_0h_{L,l} + {}_0^tx_{n,l}\;{}_0h_{L,k}) $$

$$ \frac{\partial^2\,{}_0^t\epsilon_{kl}}{\partial{}^tu_n^L\,\partial{}^tu_m^M} = \frac{1}{2}({}_0h_{L,k}\;{}_0h_{M,l} + {}_0h_{L,l}\;{}_0h_{M,k})\delta_{nm} $$

Displacement/Pressure Updated Lagrangian Formulation

Total Lagrange: $$\delta \int_{{}^0V} \left({}_0^t\overline{W} + {}_0^tQ \right) d^0V = {}^t\mathcal{R}$$

$$\downarrow$$

Updated Lagrange: $$\delta \int_{{}^\tau V} ({}_\tau^t\overline{W} + {}_\tau^tQ)\, d^\tau V = {}^t\mathcal{R}$$

$${}_\tau^t\overline{W}\, d^\tau V = {}_0^t\overline{W}\, d^0V \qquad {}_\tau^tQ\, d^\tau V = {}_0^tQ\, d^0V$$

Innovation in UL formulation

$${}_{T}^{t}Q \, d^{T}V = {}_{0}^{t}Q \, d^{0}V$$

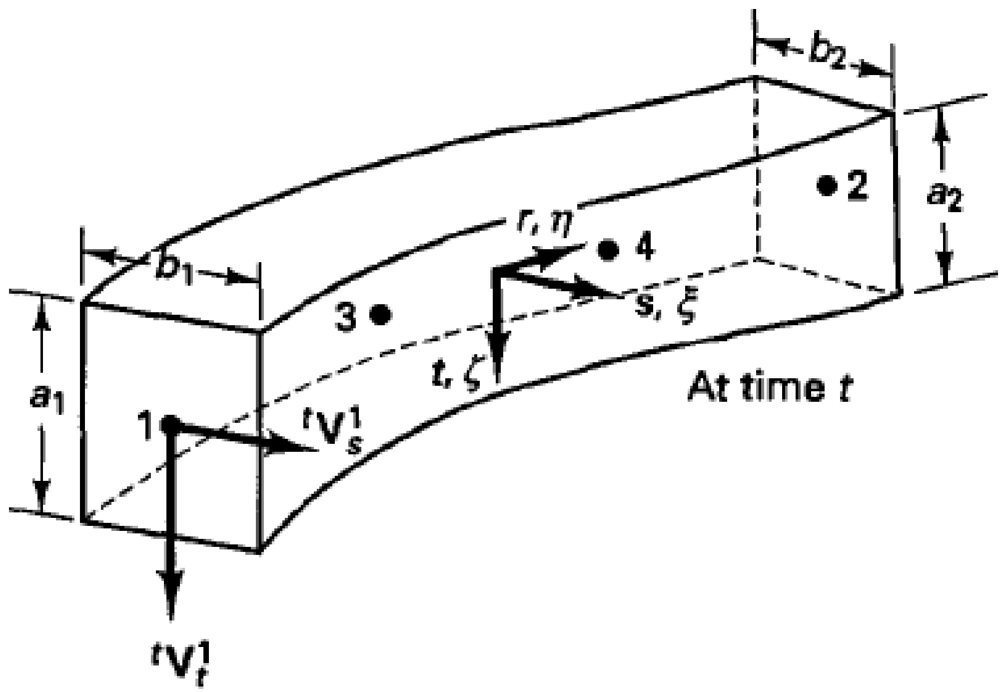
$${}_{0}^{t}Q = -\frac{1}{2\kappa}({}^{t}\bar{p} - {}^{t}\tilde{p})^{2}$$

$${}_{T}^{t}Q = -\frac{1}{2\kappa^{*}}({}^{t}\bar{p} - {}^{t}\tilde{p})^{2}$$

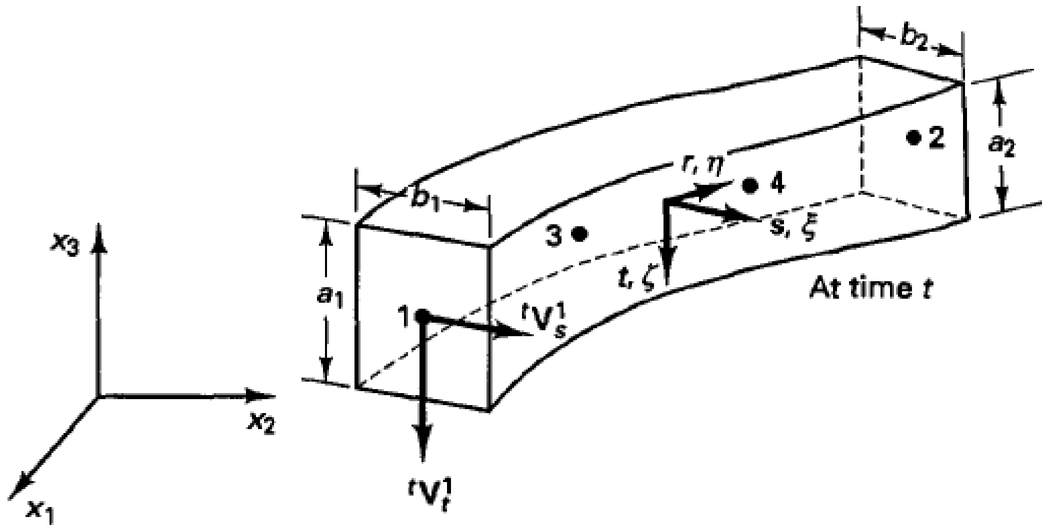
$$\kappa^{*} = \kappa \, \frac{d^{T}V}{d^{0}V} = \kappa \det {}_{0}^{T}\mathbf{X}$$

6.5.1 Structural Elements:
Beam and Axisymmetric Shell Elements

(Page 568 – 572)



Dimensions of the Beam



$$ {}^t x_i = \sum_{k=1}^{q} h_k \, {}^t x_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^t V_{ti}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^t V_{si}^k $$

The vectors V_s and V_t define the Orientation of the cross-section of the beam: they are normal to the axis of the beam and to each other
The values a and b define the size of the cross-section of the beam

Displacement ${}^t u$ / Incremental Displacement u

$${}^t u_i = {}^t x_i - {}^0 x_i$$

$${}^t x_i = \sum_{k=1}^{q} h_k \, {}^t x_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^t V_{ti}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^t V_{si}^k$$

$${}^t u_i = \sum_{k=1}^{q} h_k \, {}^t u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k ({}^t V_{ti}^k - {}^0 V_{ti}^k) + \frac{s}{2} \sum_{k=1}^{q} b_k h_k ({}^t V_{si}^k - {}^0 V_{si}^k)$$

$$u_i = {}^{t+\Delta t} x_i - {}^t x_i$$

$$V_{ti}^k = {}^{t+\Delta t} V_{ti}^k - {}^t V_{ti}^k$$

$$V_{si}^k = {}^{t+\Delta t} V_{si}^k - {}^t V_{si}^k$$

$$u_i = \sum_{k=1}^{q} h_k u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{ti}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k V_{si}^k$$

Vector Vt and Vs

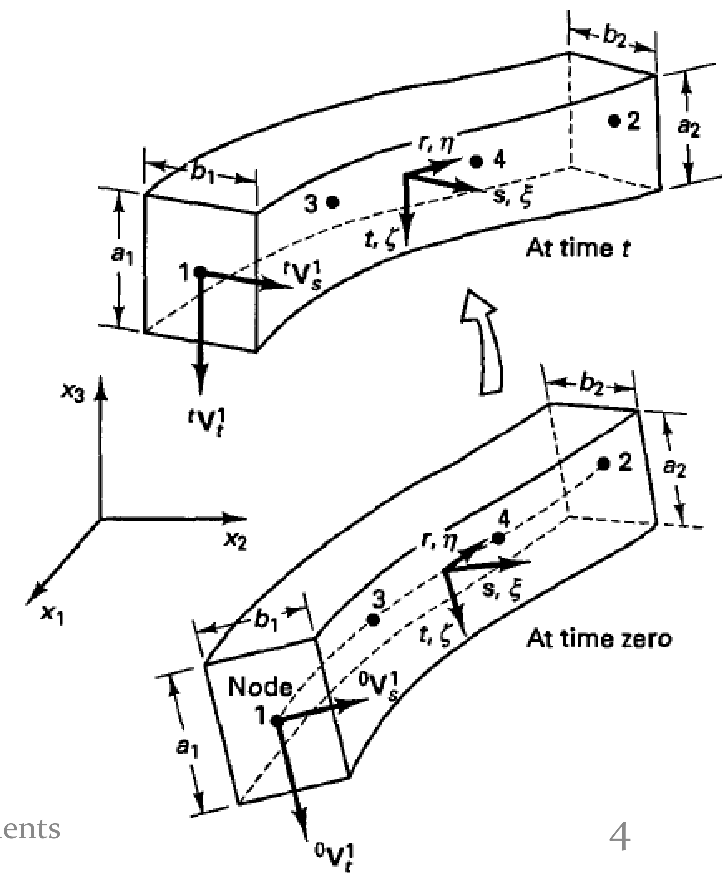
FEM I: $\mathbf{V}_t^k = \boldsymbol{\theta}_k \times {}^0\mathbf{V}_t^k = e_{x1} * (\theta_{x2}^k * {}^0V_{t,x3}^k - \theta_{x3}^k * {}^0V_{t,x2}^k) + e_{x2} \dots$

$$\mathbf{V}_s^k = \boldsymbol{\theta}_k \times {}^0\mathbf{V}_s^k$$

$$\boldsymbol{\theta}_k \times {}^0\mathbf{V}_t^k = \det \begin{bmatrix} e_x & e_y & e_z \\ \theta_x^k & \theta_y^k & \theta_z^k \\ V_{t,x}^k & V_{t,y}^k & V_{t,z}^k \end{bmatrix}$$

$$\boldsymbol{\theta}_k = \begin{bmatrix} \theta_{x_1}^k \\ \theta_{x_2}^k \\ \theta_{x_3}^k \end{bmatrix}$$

$\boldsymbol{\theta}_k$ is a vector listing the nodal point rotations at nodal point k



$$V_{ti}^k = {}^{t+\Delta t}V_{ti}^k - {}^tV_{ti}^k$$

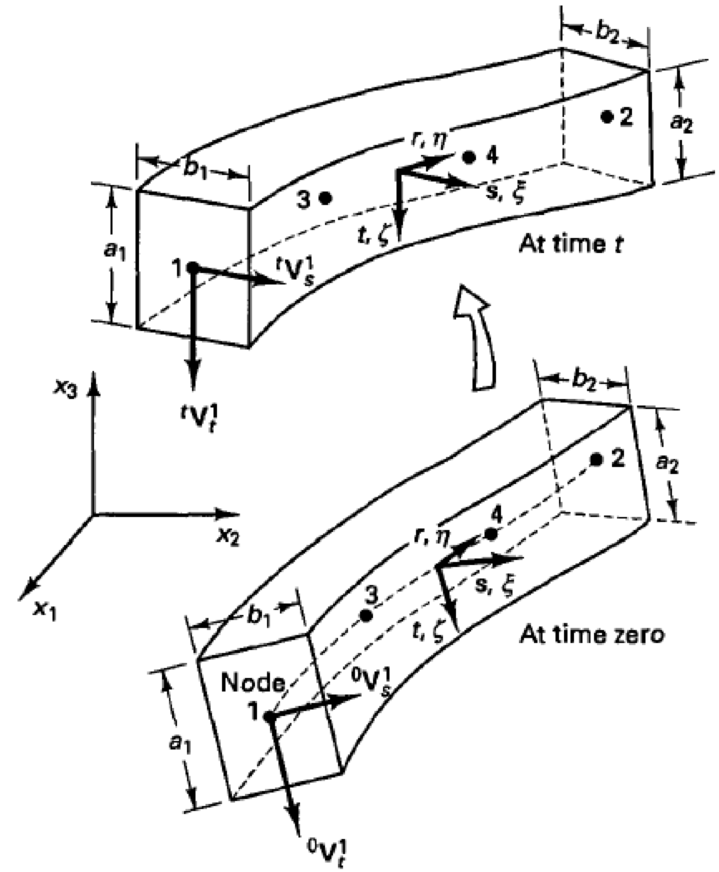
$$V_{si}^k = {}^{t+\Delta t}V_{si}^k - {}^tV_{si}^k$$

Problem: Large Rotation

-> therefore second-order has to be taken into account
(higher order terms are neglected)

$$\mathbf{V}_t^k = \boldsymbol{\theta}_k \times {}^t\mathbf{V}_t^k + \frac{1}{2}\boldsymbol{\theta}_k \times (\boldsymbol{\theta}_k \times {}^t\mathbf{V}_t^k)$$

$$\mathbf{V}_s^k = \boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k + \frac{1}{2}\boldsymbol{\theta}_k \times (\boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k)$$



Result

Now the Equation can be solved

$$ {}^{t}\mathbf{K} \begin{bmatrix} \vdots \\ \mathbf{u}_k \\ \boldsymbol{\theta}_k \\ \vdots \end{bmatrix} = {}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F} $$

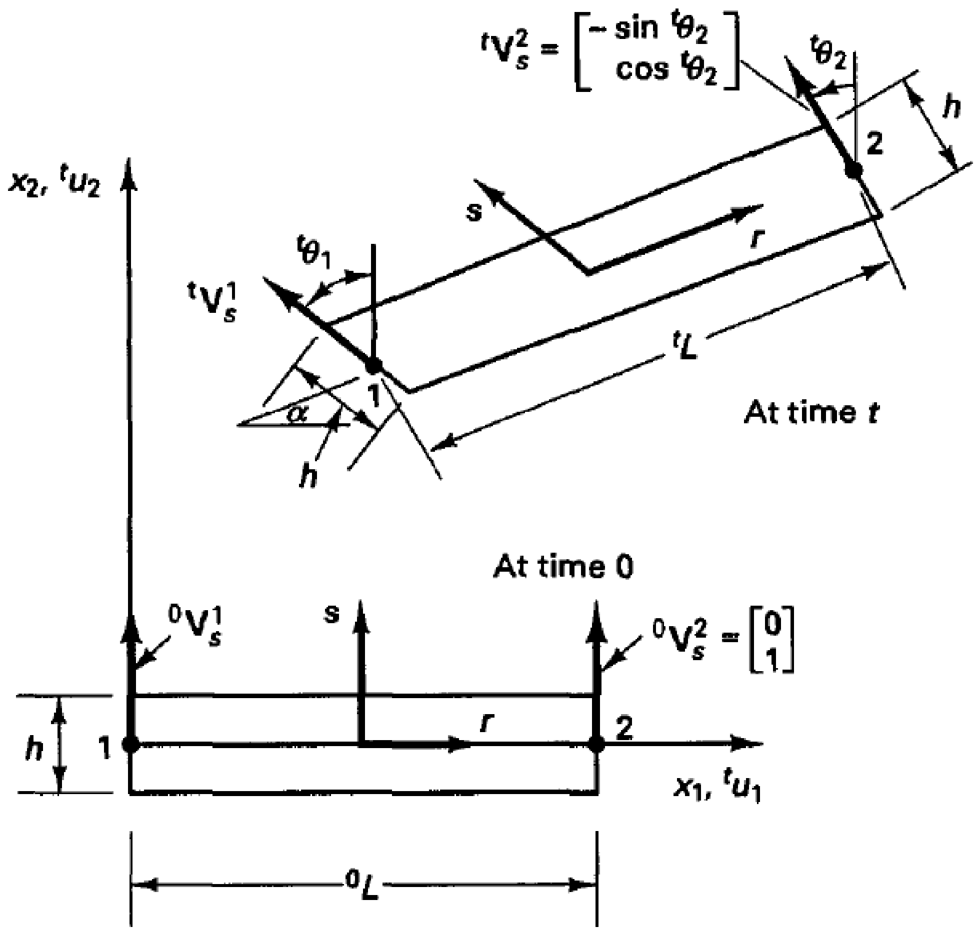
$$ {}^{t+\Delta t}\mathbf{u}_k = {}^{t}\mathbf{u}_k + \mathbf{u}_k $$

$$ {}^{t+\Delta t}\mathbf{V}_t^k = {}^{t}\mathbf{V}_t^k + \int_{\boldsymbol{\theta}_k} d\boldsymbol{\theta}_k \times {}^{\tau}\mathbf{V}_t^k $$

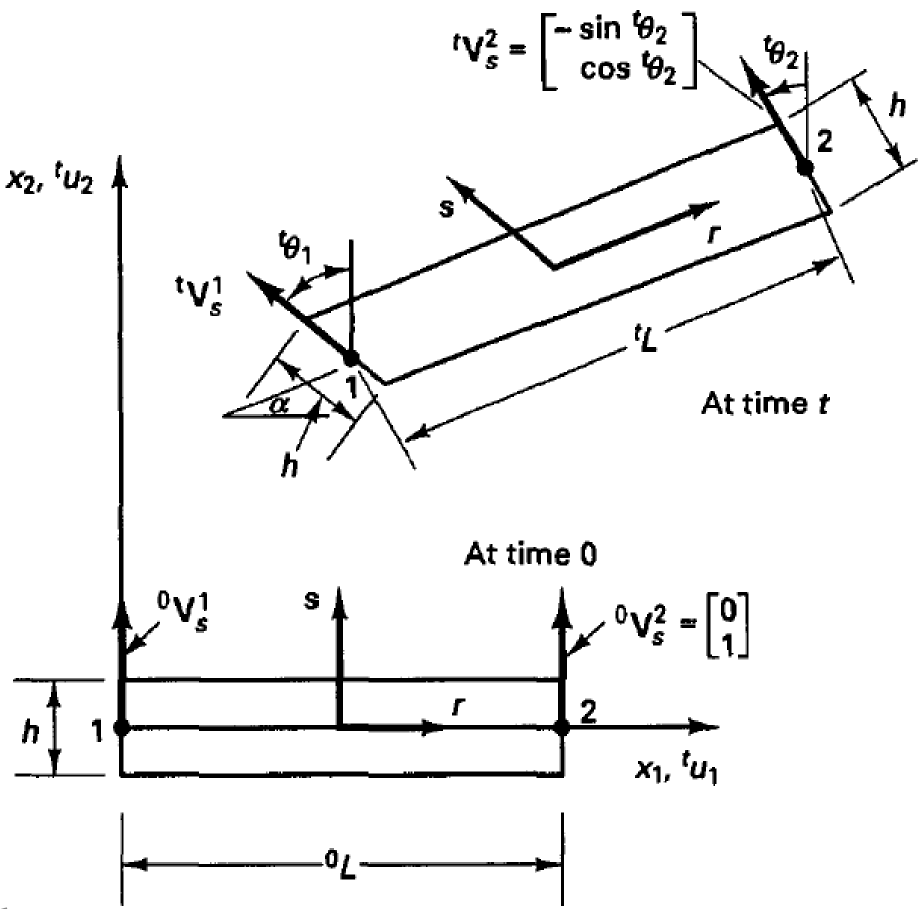
$$ {}^{t+\Delta t}\mathbf{V}_s^k = {}^{t}\mathbf{V}_s^k + \int_{\boldsymbol{\theta}_k} d\boldsymbol{\theta}_k \times {}^{\tau}\mathbf{V}_s^k $$

Example 6.20

Two Node Beam Element in Large Displacement and Rotation



Evaluate the coordinate and displacement interpolations and derivates that are required for the calculation of the strain-displacement matrices of the UL and TL formulations



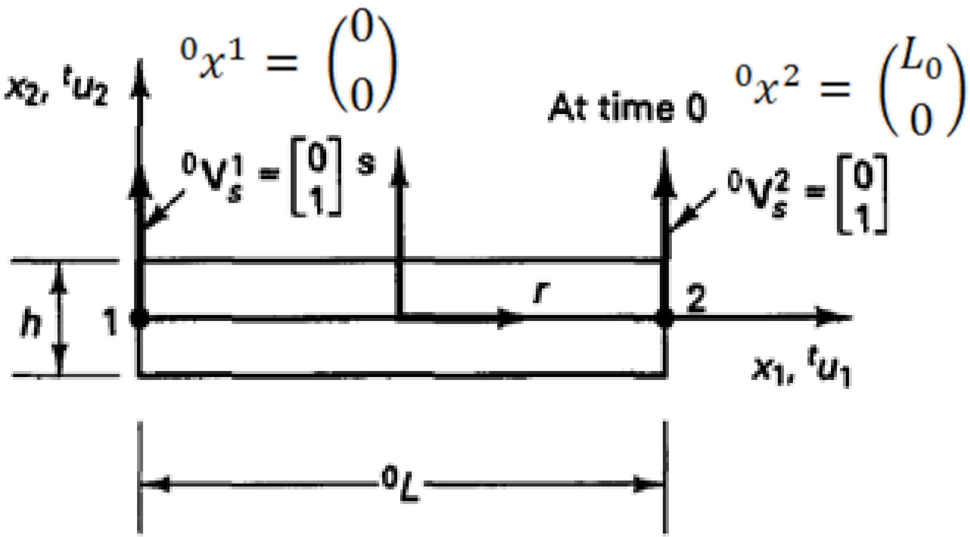
First we have to calculate:

${}^{t}x_1 \quad {}^{t}x_2$

${}^{0}x_1 \quad {}^{0}x_2$

q = 2

$$ {}^t x_i = \sum_{k=1}^{q} h_k \, {}^t x_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^t V_{ti}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^t V_{si}^k $$



Two-Node Element:

$$ h_1 = \frac{1}{2}(1 - r) $$

$$ h_2 = \frac{1}{2}(1 + r) $$

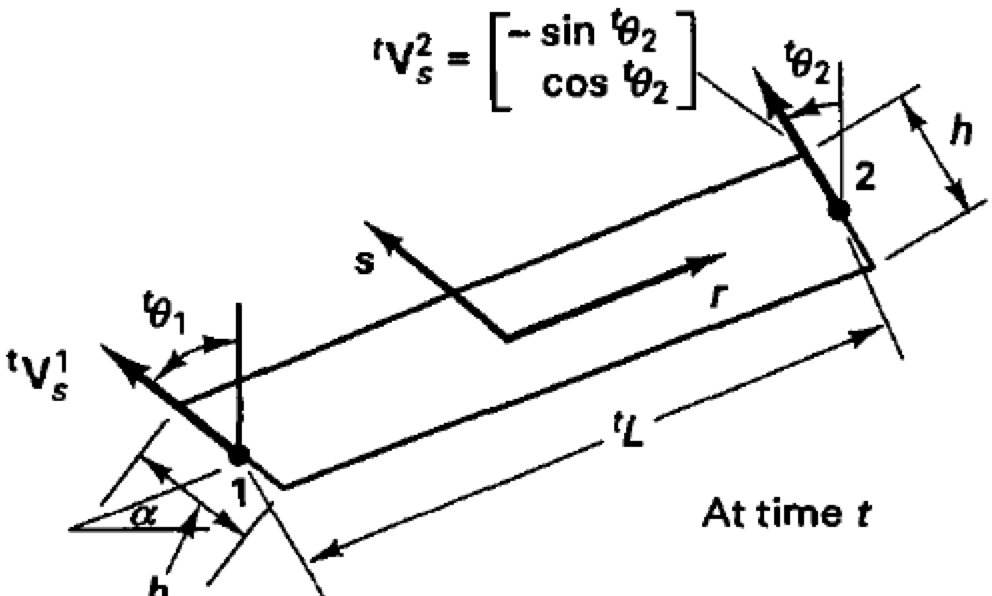
=0 =0 =0

$$ {}^0 x_1 = h_1 \, {}^0 x_1^1 + h_2 \, {}^0 x_1^2 + \frac{S}{2} b_1 h_1 \, {}^0 V_{s1}^1 + \frac{S}{2} b_2 h_2 \, {}^0 V_{s1}^2 $$

$$ {}^0 x_1 = \left(\frac{1 + r}{2} \right) {}^0 L \qquad {}^0 x_2 = \frac{sh}{2} $$

Coordinates

$$ {}^t x_i = \sum_{k=1}^{q} h_k \, {}^t x_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^t V_{ti}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k \, {}^t V_{si}^k $$

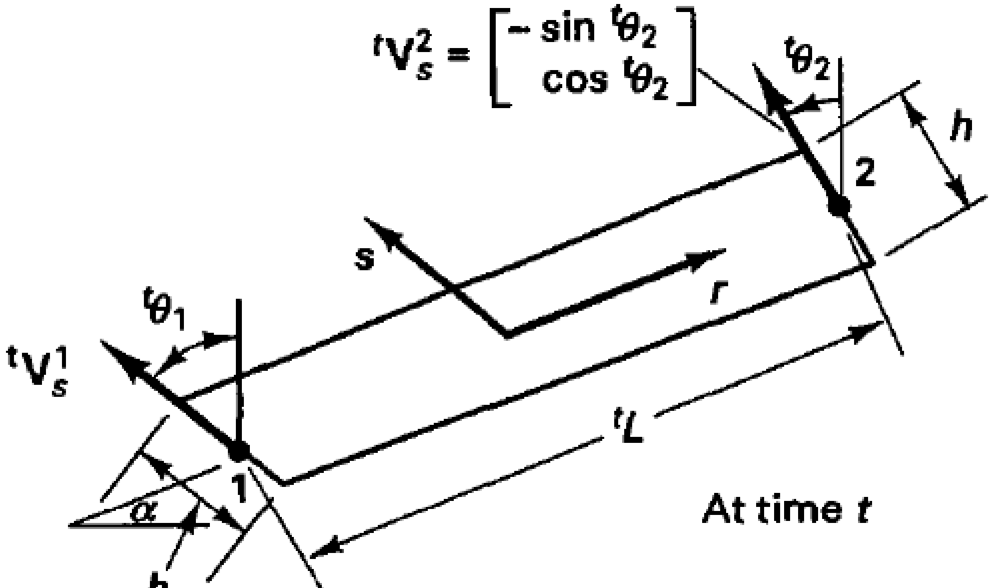


$$ {}^t x_1 = \left(\frac{1-r}{2}\right) {}^t x_1^1 + \left(\frac{1+r}{2}\right) {}^t x_1^2 - \frac{sh}{2}\left(\frac{1-r}{2}\right) \sin {}^t\theta_1 - \frac{sh}{2}\left(\frac{1+r}{2}\right) \sin {}^t\theta_2 $$

$$ {}^t x_2 = \left(\frac{1-r}{2}\right) {}^t x_2^1 + \left(\frac{1+r}{2}\right) {}^t x_2^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right) \cos {}^t\theta_1 + \frac{sh}{2}\left(\frac{1+r}{2}\right) \cos {}^t\theta_2 $$

Displacement

$$ {}^t u_i = \sum_{k=1}^{q} h_k \, {}^t u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k ({}^t V_{ti}^k - {}^0 V_{ti}^k) + \frac{s}{2} \sum_{k=1}^{q} b_k h_k ({}^t V_{si}^k - {}^0 V_{si}^k) $$



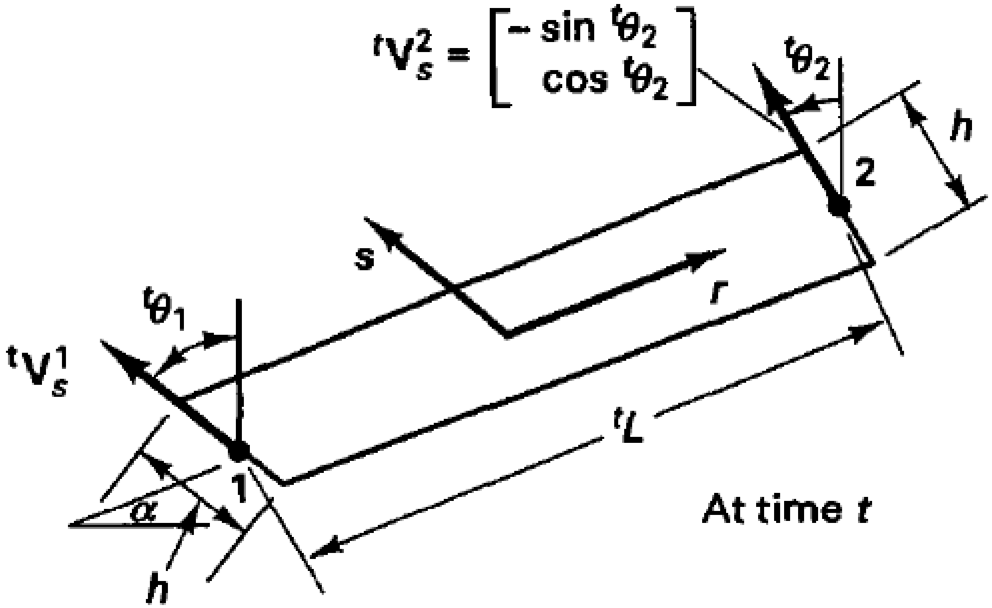
$$ {}^t u_i = {}^t x_i - {}^0 x_i $$

$$ {}^t u_1 = \left(\frac{{}^t x_1^1 + {}^t x_1^2 - {}^0 L}{2} \right) + \left(\frac{{}^t x_1^2 - {}^t x_1^1 - {}^0 L}{2} \right) r - \frac{sh}{2} \left[\left(\frac{1-r}{2} \right) \sin {}^t\theta_1 + \left(\frac{1+r}{2} \right) \sin {}^t\theta_2 \right] $$

$$ {}^t u_2 = \left(\frac{{}^t x_2^1 + {}^t x_2^2}{2} \right) + \left(\frac{{}^t x_2^2 - {}^t x_2^1}{2} \right) r + \frac{sh}{2} \left[\left(\frac{1-r}{2} \right) \cos {}^t\theta_1 + \left(\frac{1+r}{2} \right) \cos {}^t\theta_2 - 1 \right] $$

Incremental Displacement

$$u_i = \sum_{k=1}^{q} h_k u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{ti}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k V_{si}^k$$

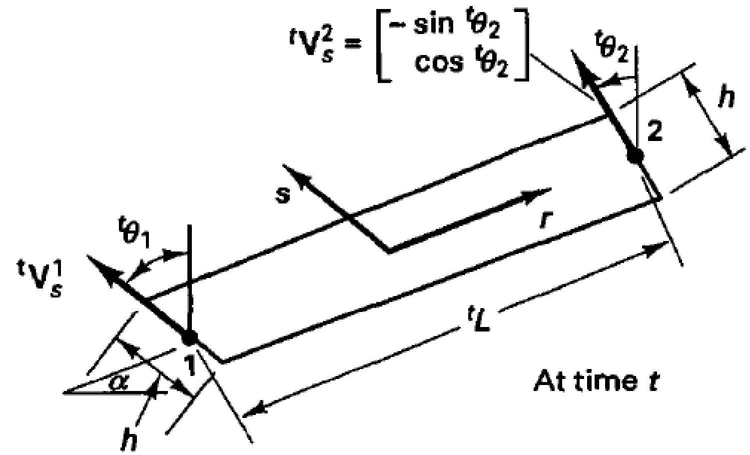


second-order terms

$$\mathbf{V}_t^k = \boldsymbol{\theta}_k \times {}^t\mathbf{V}_t^k + \frac{1}{2}\boldsymbol{\theta}_k \times (\boldsymbol{\theta}_k \times {}^t\mathbf{V}_t^k)$$

$$\mathbf{V}_s^k = \boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k + \frac{1}{2}\boldsymbol{\theta}_k \times (\boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k)$$

$$\mathbf{V}_s^k = \boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k + \frac{1}{2}\boldsymbol{\theta}_k \times (\boldsymbol{\theta}_k \times {}^t\mathbf{V}_s^k)$$



$$\theta_k \times {}^tV_s^k = det\begin{bmatrix} e_{x1} & e_{x2} & e_{x3} \\ 0 & 0 & \theta_k \\ -sin({}^t\theta_k) & \cos({}^t\theta_k) & 0 \end{bmatrix} = \begin{pmatrix} -\theta_k \cos({}^t\theta_k) \\ -\theta_k sin({}^t\theta_k) \\ 0 \end{pmatrix}$$

$$\theta_k \times (\theta_k \times {}^tV_s^k) = det\begin{bmatrix} e_{x1} & e_{x2} & e_{x3} \\ 0 & 0 & \theta_k \\ -\theta_k \cos({}^t\theta_k) & -\theta_k sin({}^t\theta_k) & 0 \end{bmatrix} = \begin{pmatrix} {\theta_k}^2 sin({}^t\theta_k) \\ -{\theta_k}^2 cos({}^t\theta_k) \\ 0 \end{pmatrix}$$

$$V_s^k = \begin{pmatrix} -\theta_k \cos({}^t\theta_k) + \frac{1}{2}{\theta_k}^2 sin({}^t\theta_k) \\ -\theta_k sin({}^t\theta_k) - \frac{1}{2}{\theta_k}^2 cos({}^t\theta_k) \\ 0 \end{pmatrix}$$

$$u_i = \sum_{k=1}^{q} h_k u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{ti}^k + \frac{s}{2} \sum_{k=1}^{q} b_k h_k V_{si}^k$$

$$V_s^2 = \begin{pmatrix} -\theta_k \cos({}^t\theta_k) + \frac{1}{2}\theta_k{}^2 sin\,({}^t\theta_k) \\ -\theta_k sin({}^t\theta_k) - \frac{1}{2}\theta_k{}^2 cos({}^t\theta_k) \\ 0 \end{pmatrix}$$

$$u_1 = \frac{1-r}{2} u_1^1 + \frac{1+r}{2} u_1^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right)\left[(-\cos {}^t\theta_1)\theta_1 + \frac{1}{2} \sin {}^t\theta_1 (\theta_1)^2\right]$$

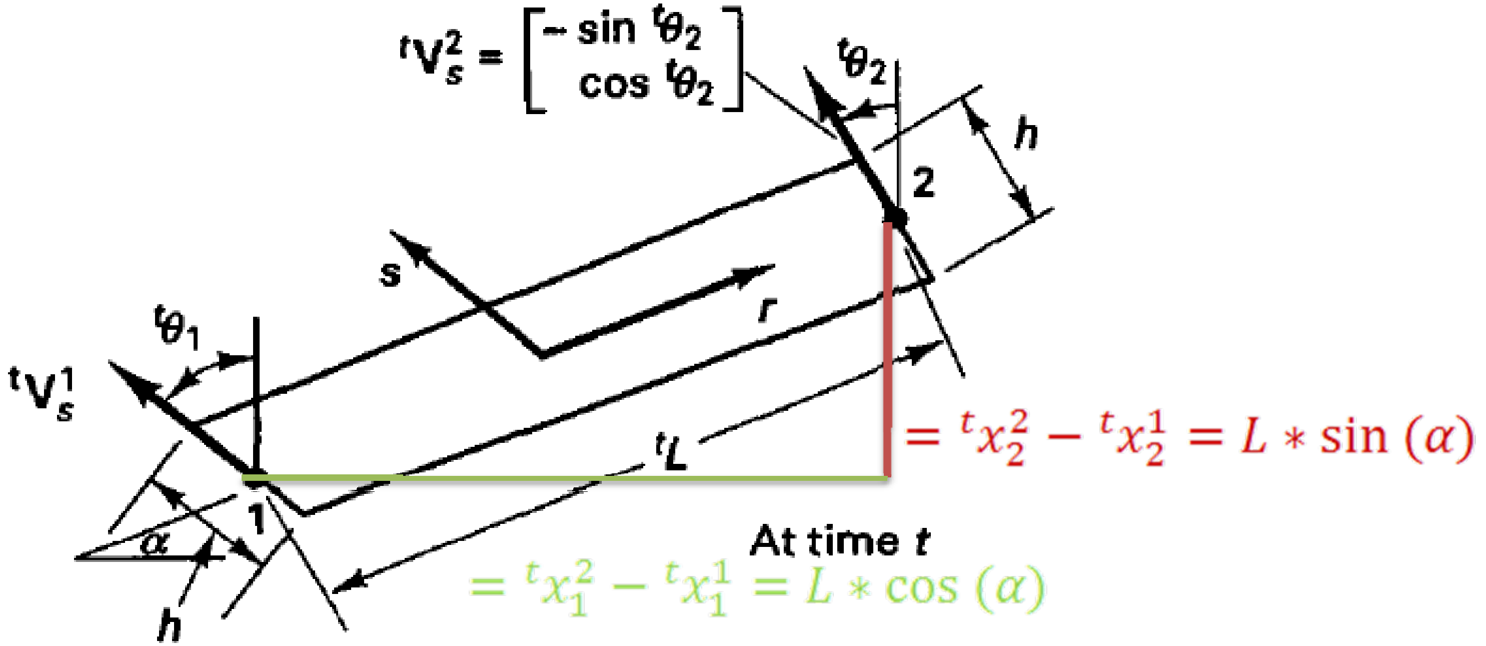
$$+ \frac{sh}{2}\left(\frac{1+r}{2}\right)\left[(-\cos {}^t\theta_2)\theta_2 + \frac{1}{2} \sin {}^t\theta_2 (\theta_2)^2\right]$$

$$u_2 = \frac{1-r}{2} u_2^1 + \frac{1+r}{2} u_2^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right)\left[(-\sin {}^t\theta_1)\underline{\theta}_1 - \frac{1}{2}\cos {}^t\theta_1(\underline{\theta}_1)^2\right]$$

$$+ \frac{sh}{2}\left(\frac{1+r}{2}\right)\left[(-\sin {}^t\theta_2)\underline{\theta}_2 - \frac{1}{2}\cos {}^t\theta_2(\underline{\theta}_2)^2\right]$$

Derivatives

$$^tx_1 = \left(\frac{1-r}{2}\right) {}^tx_1^1 + \left(\frac{1+r}{2}\right) {}^tx_1^2 - \frac{sh}{2}\left(\frac{1-r}{2}\right) \sin {}^t\theta_1 - \frac{sh}{2}\left(\frac{1+r}{2}\right) \sin {}^t\theta_2$$



$$\frac{\partial^t x_1}{\partial r} = \frac{L \cos \alpha}{2} - \frac{sh}{4}(\sin {}^t\theta_2 - \sin {}^t\theta_1)$$

Derivatives

$$ {}^t x_1 = \left(\frac{1-r}{2}\right) {}^t x_1^1 + \left(\frac{1+r}{2}\right) {}^t x_1^2 - \frac{sh}{2}\left(\frac{1-r}{2}\right) \sin {}^t\theta_1 - \frac{sh}{2}\left(\frac{1+r}{2}\right) \sin {}^t\theta_2 $$

$$ {}^t x_2 = \left(\frac{1-r}{2}\right) {}^t x_2^1 + \left(\frac{1+r}{2}\right) {}^t x_2^2 + \frac{sh}{2}\left(\frac{1-r}{2}\right) \cos {}^t\theta_1 + \frac{sh}{2}\left(\frac{1+r}{2}\right) \cos {}^t\theta_2 $$

$$ \frac{\partial {}^t x_1}{\partial r} = \frac{L \cos \alpha}{2} - \frac{sh}{4}(\sin {}^t\theta_2 - \sin {}^t\theta_1) $$

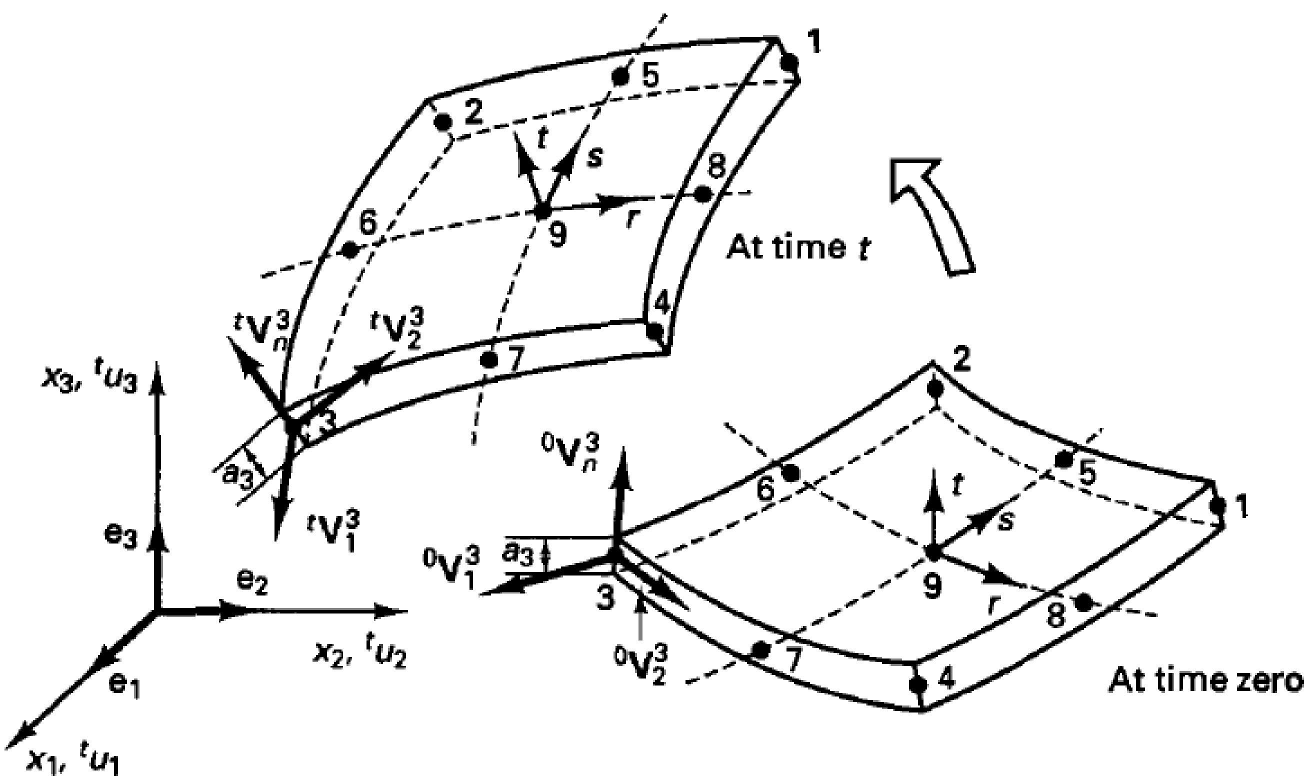
$$ \frac{\partial {}^t x_1}{\partial s} = \left(-\frac{h}{2}\right)\left[\left(\frac{1-r}{2}\right) \sin {}^t\theta_1 + \left(\frac{1+r}{2}\right) \sin {}^t\theta_2\right] $$

$$ \frac{\partial {}^t x_2}{\partial r} = \frac{L \sin \alpha}{2} + \frac{sh}{4}(\cos {}^t\theta_2 - \cos {}^t\theta_1) $$

$$ \frac{\partial {}^t x_2}{\partial s} = \frac{h}{2}\left[\left(\frac{1-r}{2}\right) \cos {}^t\theta_1 + \left(\frac{1+r}{2}\right) \cos {}^t\theta_2\right] $$

6.5.2 Structural Elements:
Structural Elements: Plate and General Shell Elements

(Page 575 – 578)



We now extend the theory for beams for general shell elements

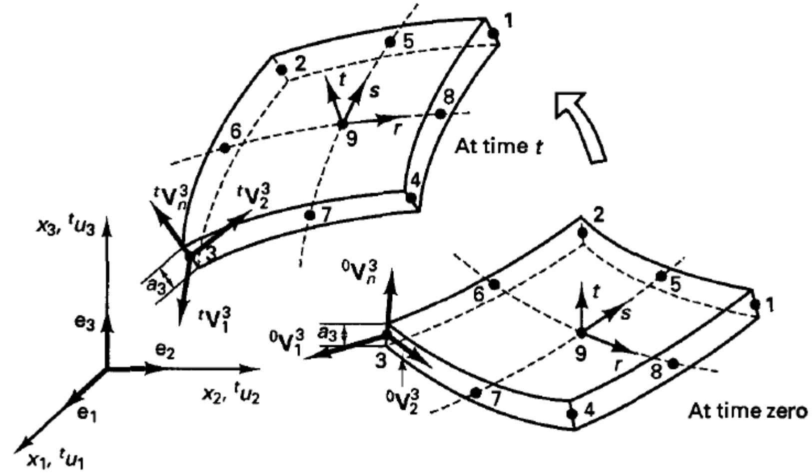
$$ {}^{t}x_i = \sum_{k=1}^{q} h_k\, {}^{t}x_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k\, {}^{t}V_{ni}^k $$

$$ \Downarrow $$

$$ {}^{t}u_i = {}^{t}x_i - {}^{0}x_i $$

$$ \Downarrow $$

$$ {}^{t}u_i = \sum_{k=1}^{q} h_k\, {}^{t}u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \left({}^{t}V_{ni}^k - {}^{0}V_{ni}^k \right) $$



$$ u_i = {}^{t+\Delta t}x_i - {}^{t}x_i \qquad V_{ni}^k = {}^{t+\Delta t}V_{ni}^k - {}^{t}V_{ni}^k $$

$$ \Downarrow \qquad \Downarrow $$

$$ u_i = \sum_{k=1}^{q} h_k u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{ni}^k $$

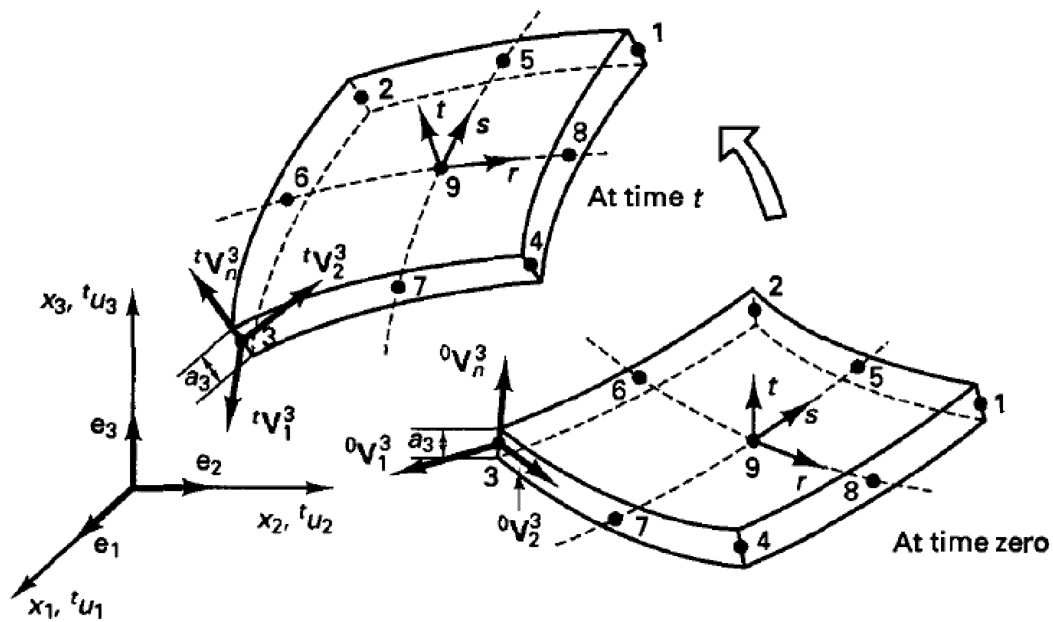
Rotation

$${}^0\mathbf{V}_1^k = \frac{\mathbf{e}_2 \times {}^0\mathbf{V}_n^k}{\| \mathbf{e}_2 \times {}^0\mathbf{V}_n^k \|_2}$$

$${}^0\mathbf{V}_2^k = {}^0\mathbf{V}_n^k \times {}^0\mathbf{V}_1^k$$

${}^t\mathbf{V}_1^k$ and ${}^t\mathbf{V}_2^k$ are defined at time 0

Normal Vector



α and β are the Rotation of the Vector ${}^t\mathbf{V}_n^k$ about ${}^t\mathbf{V}_1^k$ and ${}^t\mathbf{V}_2^k$

$$\mathbf{V}_n^k = -{}^t\mathbf{V}_2^k\, \alpha_k + {}^t\mathbf{V}_1^k\, \beta_k - \tfrac{1}{2}(\alpha_k^2 + \beta_k^2)\, {}^t\mathbf{V}_n^k$$

$$ {}^{t+\Delta t}\mathbf{V}_n^k = {}^{t}\mathbf{V}_n^k + \int_{\alpha_k,\beta_k} -{}^{\tau}\mathbf{V}_2^k \, d\alpha_k + {}^{\tau}\mathbf{V}_1^k \, d\beta_k $$

$$ u_i = \sum_{k=1}^{q} h_k u_i^k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \left[-\, {}^{t}V_{2i}^k \, \alpha_k + {}^{t}V_{1i}^k \, \beta_k - \frac{1}{2}(\alpha_k^2 + \beta_k^2) \, {}^{t}V_{ni}^k \right] $$

Note: the procedure for computing the displacements, etc. is now the same as always.
But as seen in FEM I there can be problems with shear-locking.
Therefore Mixed Interpolation should be considered.