FEM2 : Finite element Method 2

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Exercice 6.14

- *Show that the second Piola-Kirchhof stress tensor is invariant under a rigid rotation of the material*.

- Second Piola-Kirchhof stress tensor

Mass density

Cauchy stress tensor

$$ {}^t_0\mathbf{S} = \frac{{}^0\rho}{{}^t\rho}\, {}^0_t\mathbf{X}\, {}^t\boldsymbol{\tau}\, {}^0_t\mathbf{X}^T \qquad \text{(a)} $$

2nd term Piola-Kirchhof stress tensor

Inverse Deformation tensor

- *If rigid body rotation is applied to material from time t to time t+Δt, the deformation gradient change to*

Deformation Matrix at t+Δt, **:** $${}^{t+\Delta t}_{0}\mathbf{X} = \mathbf{R}\,{}^{t}_{0}\mathbf{X}$$ **: Deformation Matrix at time** t

Vector of external apply force … Rotation force
Orthogonal matrix : $R^TR = RR^T = I$

End Hence :

Inverse deformation gradient at t+Δt : $${}^{0}_{t+\Delta t}\mathbf{X} = {}^{0}_{t}\mathbf{X}\mathbf{R}^{T}$$ **: Inverse deformation gradient at t** **(b)**

At t :

$$ {}^t_0\mathbf{S} = \frac{{}^0\rho}{{}^t\rho}\, {}^0_t\mathbf{X}\mathbf{R}^T \quad {}^t\boldsymbol{\tau}\mathbf{R}\; {}^0_t\mathbf{X}^T $$

At t+Δt :

$$ {}^{t+\Delta t}_{\;\;0}\mathbf{S} = \frac{{}^0\rho}{{}^t\rho}\, {}^0_t\mathbf{X}\mathbf{R}^T\; {}^{t+\Delta t}\boldsymbol{\tau}\mathbf{R}\; {}^0_t\mathbf{X}^T \qquad \text{(c)} $$

No density change

$$ {}^{\;\;\;\;\;\;0}_{t+\Delta t}\mathbf{X} = {}^0_t\mathbf{X}\mathbf{R}^T $$

During the rigid body rotation, the stress component remain constant in the rotating coordinate system.

$$ {}^{t+\Delta t}\boldsymbol{\tau} = \mathbf{R}\; {}^t\boldsymbol{\tau}\mathbf{R}^T \qquad \textbf{(d)} $$

Substituting from (d) into (c), we obtain

$$ {}^{t+\Delta t}_{0}\mathbf{S} = \frac{{}^{0}\rho}{{}^{t}\rho}\, {}^{0}_{t}\mathbf{X}^{T}\, {}^{t}\boldsymbol{\tau}\, {}^{0}_{t}\mathbf{X}^{T} = {}_{0}^{t}\mathbf{S} $$

Mean: $\mathbf{{}^{t+\Delta t}_{0}S = {}^{t}_{0}S}$

second Piola-Kirchhof stress tensor is invariant under a rigid rotation of the material

FEM2 : Finite element Method 2

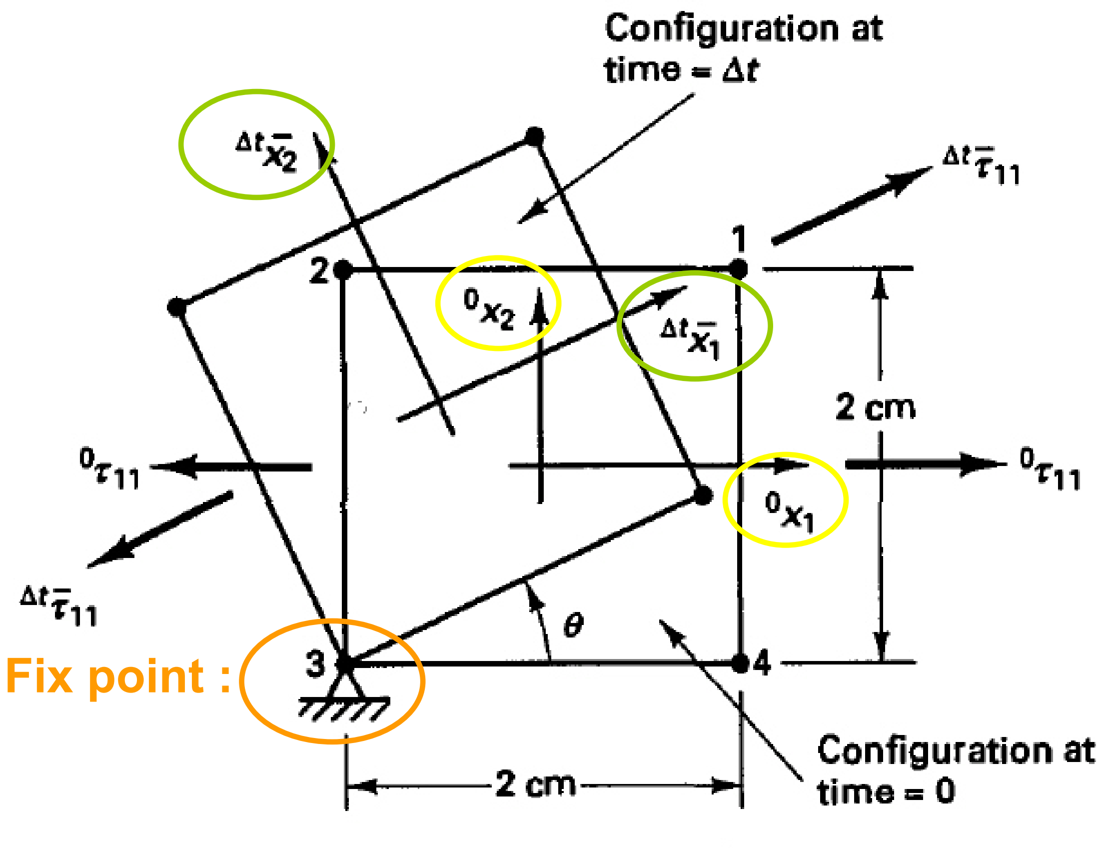
Exercice 6.15

A four element node is subject to a stress (initial stress)

$$^{0}\tau_{11}$$

The element is rotated from time 0 to time Δt as rigid body trough large angle θ and without stress change

So magnitude $^{\Delta t}\bar{\tau}_{11} = {}^{0}\tau_{11}$



- *Second Piola-Kirchhof tensor at time 0 = Cauchy stress tensor ,*

,because element deformation are 0

$$ {}_0^0\mathbf{S} = \begin{bmatrix} {}^0\tau_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(a)} $$

- *Component of the Cauchy stress tensor at time Δt expressed in the coordinates axes* ${}^0x_1$, ${}^0x_2$

Rotation tensor **Transpose Rotation tensor**

$$ {}^{\Delta t}\boldsymbol{\tau} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} {}^{\Delta t}\bar{\bar{\tau}}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \textbf{(b)} $$

↑

S :
Second
Piola-Kirchhof stress tensor

- *Relation between the Cauchy stress and the second Piola-Kirchhof stresses at time Δt :*

Deformation gradient tensor and Transposed tensor at time Δt

$$ {}^{\Delta t}\tau = \frac{{}^{\Delta t}\rho}{{}^{0}\rho}\, {}^{\Delta t}_{0}\mathbf{X}\; {}^{\Delta t}_{0}\mathbf{S}\; {}^{\Delta t}_{0}\mathbf{X}^{T} \qquad \text{(c)} $$

Cauchy stress at *time Δt*

Second Piola-Kirchhof stress at time Δt

In this case, there is no density change at time Δt

$$^{\Delta t}\rho / {}^{0}\rho = 1.$$

Deformation gradient evaluation

Node	$^{t+\Delta t}x_1$	$^{t+\Delta t}x_2$
1	$^{\Delta t}x_1^1 = 2\cos\theta - 1 - 2\sin\theta;$	$^{\Delta t}x_2^1 = 2\sin\theta - 1 + 2\cos\theta$
2	$^{\Delta t}x_1^2 = -1 - 2\sin\theta;$	$^{\Delta t}x_2^2 = 2\cos\theta - 1$
3	$^{\Delta t}x_1^3 = -1;$	$^{\Delta t}x_2^3 = -1$
4	$^{\Delta t}x_1^4 = 2\cos\theta - 1;$	$^{\Delta t}x_2^4 = 2\sin\theta - 1$

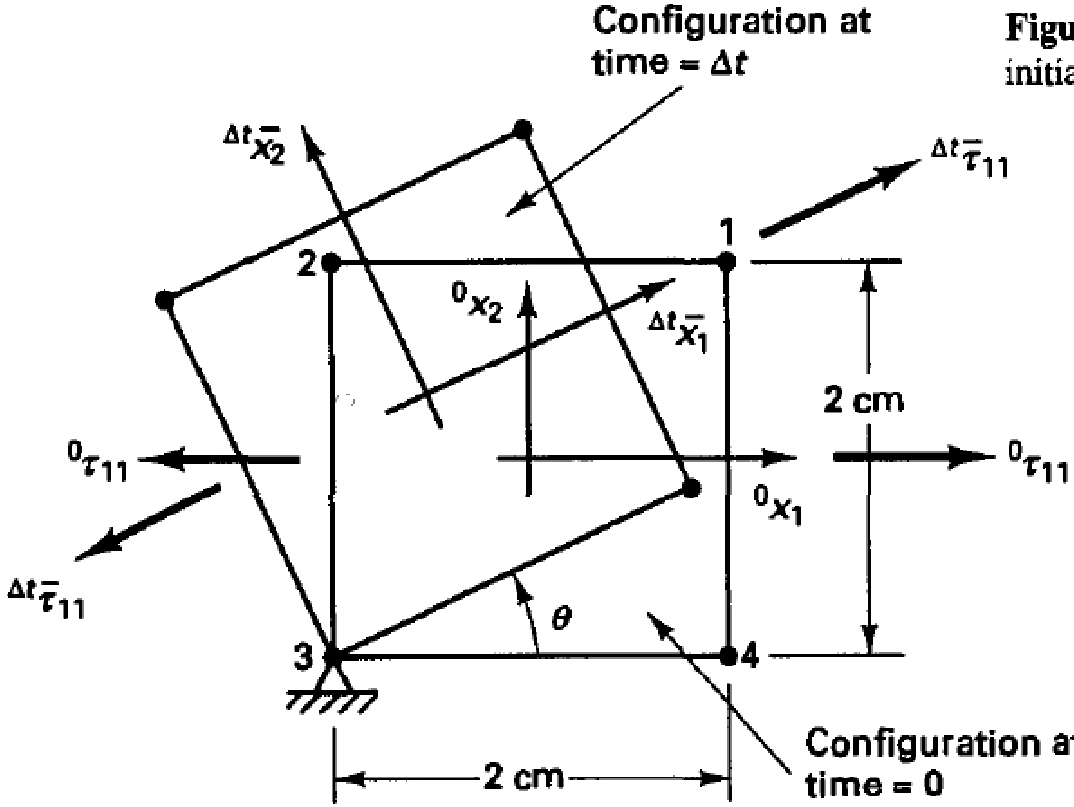
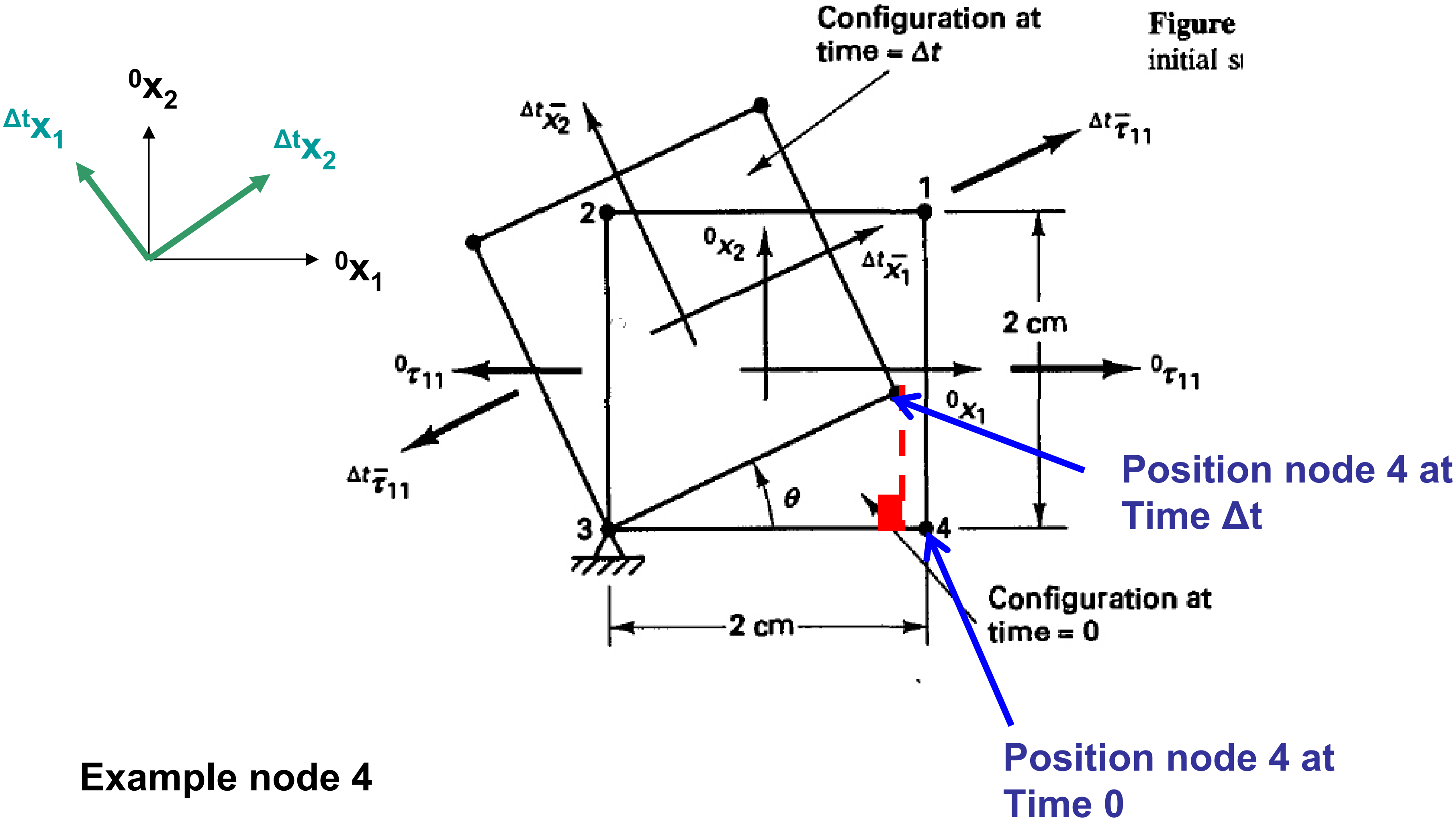
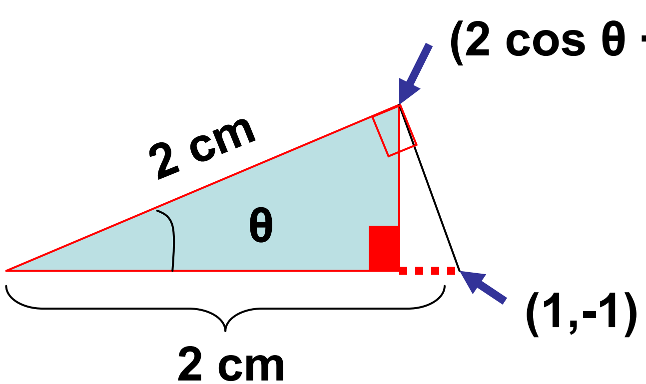
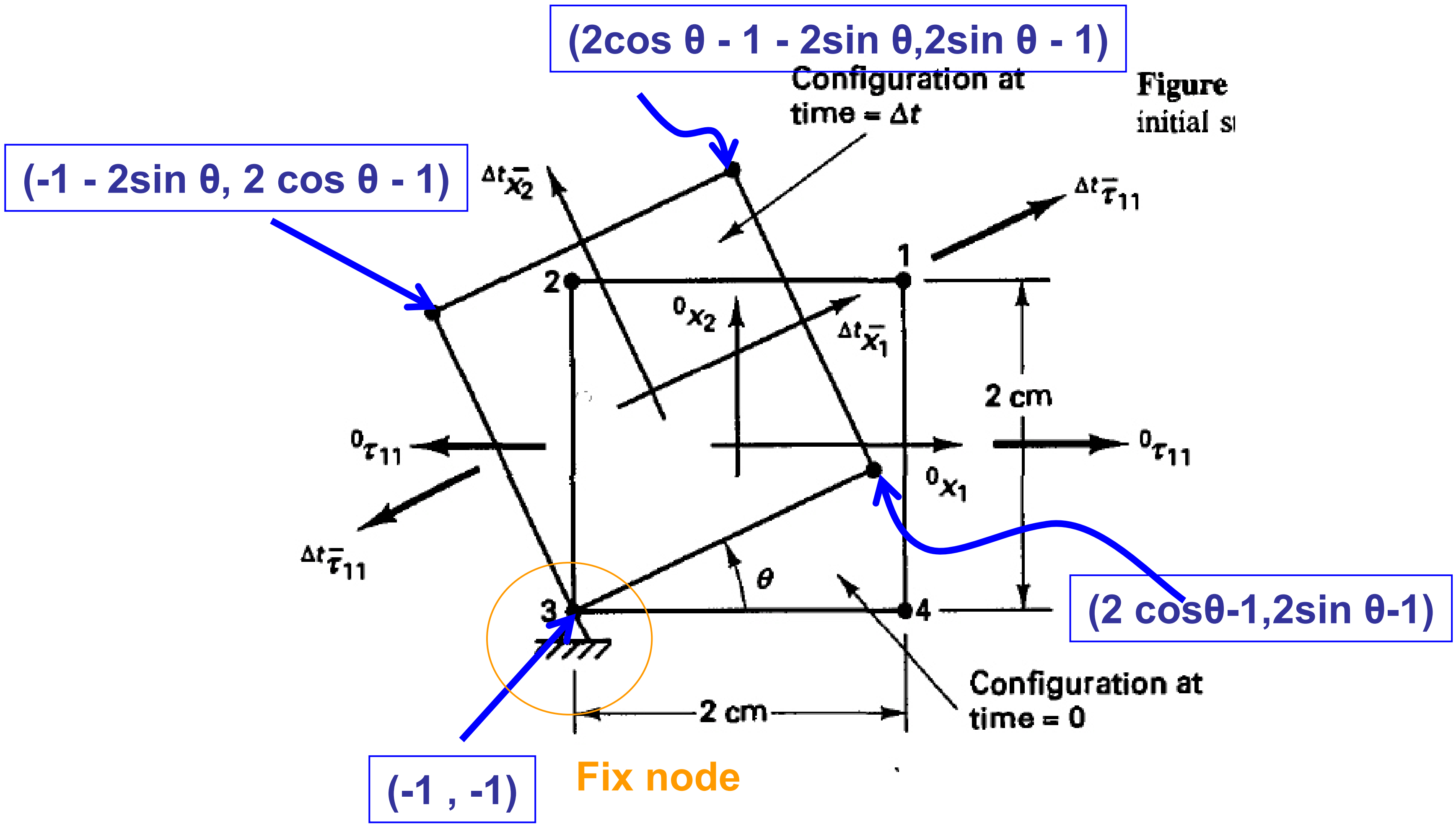


Figure
initial s



Example node 4





(2cos θ - 1 - 2sin θ,2sin θ - 1)
Configuration at time = Δt
Figure
initial s
(-1 - 2sin θ, 2 cos θ - 1)
1
2
3
4
2 cm
2 cm
θ
Configuration at time = 0
(2 cosθ-1,2sin θ-1)
(-1 , -1)
Fix node

Deformation tensor a time t : $${}^t x_i = \sum_{k=1}^{4} h_k \, {}^t x_i^k$$

Gradient deformation tensor a time t : $$\frac{\partial^t x_i}{\partial^0 x_j} = \sum_{k=1}^{4} \left(\frac{\partial h_k}{\partial^0 x_j}\right) {}^t x_i^k$$

$$h_1 = \tfrac{1}{4}(1 + {}^0x_1)(1 + {}^0x_2); \qquad h_2 = \tfrac{1}{4}(1 - {}^0x_1)(1 + {}^0x_2)$$

$$h_3 = \tfrac{1}{4}(1 - {}^0x_1)(1 - {}^0x_2); \qquad h_4 = \tfrac{1}{4}(1 + {}^0x_1)(1 - {}^0x_2)$$

$$\frac{\partial h_1}{\partial^0 x_1} = \frac{1}{4}(1 + {}^0x_2); \qquad \frac{\partial h_2}{\partial^0 x_1} = -\frac{1}{4}(1 + {}^0x_2)$$

$$\frac{\partial h_3}{\partial^0 x_1} = -\frac{1}{4}(1 - {}^0x_2); \qquad \frac{\partial h_4}{\partial^0 x_1} = \frac{1}{4}(1 - {}^0x_2)$$

$$\frac{\partial h_1}{\partial^0 x_2} = \frac{1}{4}(1 + {}^0x_1); \qquad \frac{\partial h_2}{\partial^0 x_2} = \frac{1}{4}(1 - {}^0x_1)$$

$$\frac{\partial h_3}{\partial^0 x_2} = -\frac{1}{4}(1 - {}^0x_1); \qquad \frac{\partial h_4}{\partial^0 x_2} = -\frac{1}{4}(1 + {}^0x_1)$$

$$ {}^{\Delta t}_{0}X = \begin{pmatrix} \frac{\partial^t x_1}{\partial^0 x_1} & \frac{\partial^t x_1}{\partial^0 x_2} \\ \frac{\partial^t x_2}{\partial^0 x_1} & \frac{\partial^t x_2}{\partial^0 x_2} \end{pmatrix} $$

$$ {}^{\Delta t}_{0}\mathbf{X} = \frac{1}{4} \left[\begin{array}{l|l} (1 + {}^0x_2)(2\cos\theta - 1 - 2\sin\theta) & (1 + {}^0x_1)(2\cos\theta - 1 - 2\sin\theta) \\ -(1 + {}^0x_2)(-1 - 2\sin\theta) & +(1 - {}^0x_1)(-1 - 2\sin\theta) \\ -(1 - {}^0x_2)(-1) & -(1 - {}^0x_1)(-1) \\ +(1 - {}^0x_2)(2\cos\theta - 1) & -(1 + {}^0x_1)(2\cos\theta - 1) \\ \hline (1 + {}^0x_2)(2\sin\theta - 1 + 2\cos\theta) & (1 + {}^0x_1)(2\sin\theta - 1 + 2\cos\theta) \\ -(1 + {}^0x_2)(2\cos\theta - 1) & +(1 - {}^0x_1)(2\cos\theta - 1) \\ -(1 - {}^0x_2)(-1) & -(1 - {}^0x_1)(-1) \\ +(1 - {}^0x_2)(2\sin\theta - 1) & -(1 + {}^0x_1)(2\sin\theta - 1) \end{array} \right] $$

Example :

Hence,

$$\frac{\partial^t x_1}{\partial^0 x_1} = \frac{1}{4}[(1 + {}^0x_2)(2) - (1 + {}^0x_2)(-1) - (1 - {}^0x_2)(-1) + (1 - {}^0x_2)(1)]$$

$$= \frac{1}{4}(5 + {}^0x_2)$$

and

$$\frac{\partial^t x_1}{\partial^0 x_2} = \frac{1}{4}(1 + {}^0x_1); \qquad \frac{\partial^t x_2}{\partial^0 x_1} = \frac{1}{8}(1 + {}^0x_2)$$

$$\frac{\partial^t x_2}{\partial^0 x_2} = \frac{1}{8}(9 + {}^0x_1)$$

so that the deformation gradient is

$$ {}_0^t\mathbf{X} = \frac{1}{4}\begin{bmatrix} (5 + {}^0x_2) & (1 + {}^0x_1) \\ \frac{1}{2}(1 + {}^0x_2) & \frac{1}{2}(9 + {}^0x_1) \end{bmatrix}$$

$$
{}_{0}^{\Delta t}\mathbf{X} = \frac{1}{4}\left[\begin{array}{l|l}
(1 + {}^{0}x_2)(2\cos\theta - 1 - 2\sin\theta) & (1 + {}^{0}x_1)(2\cos\theta - 1 - 2\sin\theta) \\
-(1 + {}^{0}x_2)(-1 - 2\sin\theta) & +(1 - {}^{0}x_1)(-1 - 2\sin\theta) \\
-(1 - {}^{0}x_2)(-1) & -(1 - {}^{0}x_1)(-1) \\
+(1 - {}^{0}x_2)(2\cos\theta - 1) & -(1 + {}^{0}x_1)(2\cos\theta - 1) \\
\hline
(1 + {}^{0}x_2)(2\sin\theta - 1 + 2\cos\theta) & (1 + {}^{0}x_1)(2\sin\theta - 1 + 2\cos\theta) \\
-(1 + {}^{0}x_2)(2\cos\theta - 1) & +(1 - {}^{0}x_1)(2\cos\theta - 1) \\
-(1 - {}^{0}x_2)(-1) & -(1 - {}^{0}x_1)(-1) \\
+(1 - {}^{0}x_2)(2\sin\theta - 1) & -(1 + {}^{0}x_1)(2\sin\theta - 1)
\end{array}\right]
$$

Compute

$$
{}_{0}^{\Delta t}\mathbf{X} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{(d)}
$$

Substituting

$$ {}^{\Delta t}\boldsymbol{\tau} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} {}^{\Delta t}\bar{\tau}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{(b)} $$

and

$$ {}^{\Delta t}_{0}\mathbf{X} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{(d)} $$

in

$$ {}^{\Delta t}\boldsymbol{\tau} = \frac{{}^{\Delta t}\rho}{{}^{0}\rho}\, {}^{\Delta t}_{0}\mathbf{X}\, {}^{\Delta t}_{0}\mathbf{S}\, {}^{\Delta t}_{0}\mathbf{X}^{T} $$

$$ {}^{\Delta t}_{0}S = \begin{bmatrix} {}^{\Delta t}\bar{\tau}_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{(d)} $$

Because $^{\Delta t}\bar{\tau}_{11} = {}^{0}\tau_{11}$, $^{0}_{0}S = \begin{bmatrix} {}^{0}\tau_{11} & 0 \\ 0 & 0 \end{bmatrix}$ and $^{\Delta t}_{0}S = \begin{bmatrix} {}^{\Delta t}\bar{\tau}_{11} & 0 \\ 0 & 0 \end{bmatrix}$

Show that component of the second Piola-Kirchhoh stress tensor did not change During th erigid body relation. There is no change because in this case the Deformation gradient corresponds to a rotation matrix :

Rotation tensor

Transpose Rotation tensor

$$^{\Delta t}\boldsymbol{\tau} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} {}^{\Delta t}\bar{\tau}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

6.3 Displacement-Based isoparametric continuum finite elements

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Chap.6 Finite Element Analysis in Solid and Structural Mechanics

6.3 Introduction

- From previous section (Chap.5) :

Developed linearized principle of virtual displacements in continuum form. Only variable is displacement u.

If only nodal point displacement as degree of freedom, finite element matrix is a full linearization of the virtual displacement at time t.

The derivation will show that if other displacement degree of freedom like rotation or stress mixed, the linearization is more efficiently achieved by direct Taylor expansion.

6.3.1. Linearization of the principle of virtual work with respect to finite element variable

- Prininciple of virtual displacement In the total Lagrangian formulation'

$$\int_{^0V} {}^{t+\Delta t}_{\ 0}S_{ij}\, \delta\, {}^{t+\Delta t}_{\ 0}\epsilon_{ij}\, d^0V = {}^{t+\Delta t}\mathcal{R} \qquad (6.89)$$

Strain tensor

- We linearize the expression and assume ${}^{t+\Delta t}\mathcal{R}$ independent of the deformation

$${}^{t+\Delta t}_{\ 0}S_{ij}\, \delta\, {}^{t+\Delta t}_{\ 0}\epsilon_{ij} \doteq {}^t_0S_{ij}\, \delta\, {}^t_0\epsilon_{ij} + \frac{\partial}{\partial\, {}^t a_k}\left({}^t_0S_{ij}\, \delta\, {}^t_0\epsilon_{ij}\right) da_k \qquad (6.90)$$

where da_k is a differential increment in ${}^t a_k$. We note that

$$\delta\, {}^t_0\epsilon_{ij} = \frac{\partial\, {}^t_0\epsilon_{ij}}{\partial\, {}^t a_l}\, \delta a_l \qquad (6.91)$$

- The second term

$$\frac{\partial}{\partial^t a_k}({}_0^t S_{ij}\ \delta_0^t\epsilon_{ij})\ da_k \quad = {}_0C_{ijrs}\frac{\partial_0^t\epsilon_{rs}}{\partial^t a_k}\frac{\partial_0^t\epsilon_{ij}}{\partial^t a_l}\delta a_l\ da_k + {}_0^t S_{ij}\ \frac{\partial^2\ {}_0^t\epsilon_{ij}}{\partial^t a_k\ \partial^t a_l}\delta a_l\ da_k \tag{6.92}$$

- And next :

$$\left\{\int_{^0V} {}_0C_{ijrs}\frac{\partial_0^t\epsilon_{rs}}{\partial^t a_k}\frac{\partial_0^t\epsilon_{ij}}{\partial^t a_l}\ d^0V + \int_{^0V} {}_0^t S_{ij}\frac{\partial^2\ {}_0^t\epsilon_{ij}}{\partial^t a_k\ \partial^t a_l}\ d^0V\right\} da_k\ \delta a_l = {}^{t+\Delta t}\mathcal{R}_l - \left(\int_{^0V} {}_0^t S_{ij}\ \frac{\partial_0^t\epsilon_{ij}}{\partial^t a_l}\ d^0V\right)\delta a_l \tag{6.96}$$

- For isoparametric displacement-based continuum elements with nodal displacement degrees of freedom, both expressions can be directly and easily be employed to obtain the same finite element equations. For rotational element (6.96) is more direct

6.3.2 General Matrix Equations of Displacement-Based Continuum Elements

- We consider in more detail the matrices of isoparametric continuum finite element with displacement degrees freedom only.
- Basic step in the derivation of the governing finite element equations are the same as those used in linear analysis.
- By invoking the linearized principle of virtual displacement for each of the nodal points displacement in turn the governing finite element equation are obtained.
- As in linear analysis we need to consider a single element of a specific type in the derivation because equation of equilibrium are an assemblage of elements directly constructed from using direct stiffness procedure.

- By substituting the element coordinate and displacement interpolation into the equation we did in linear analysis, we obtain for a single element or for an assemblage of elements :

In material-nonlinear-only analysis

static analysis:

Displacement vector

strain

$$ {}^t\mathbf{K}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^t\mathbf{F} \tag{6.97} $$

External forces

dynamic analysis, implicit time integration:

$$ \mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + {}^t\mathbf{K}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^t\mathbf{F} \tag{6.98} $$

dynamic analysis, explicit time integration:

$$ \mathbf{M}\,{}^t\ddot{\mathbf{U}} = {}^t\mathbf{R} - {}^t\mathbf{F} \tag{6.99} $$

using the TL formulation:
static analysis:

$$ ({}_0^t\mathbf{K}_L + {}_0^t\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}_0^t\mathbf{F} \tag{6.100} $$

dynamic analysis, implicit time integration:

$$\mathbf{M}\ {}^{t+\Delta t}\ddot{\mathbf{U}} + ({}^{t}_{0}\mathbf{K}_L + {}^{t}_{0}\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}_{0}\mathbf{F} \tag{6.101}$$

dynamic analysis, explicit time integration:

$$\mathbf{M}\ {}^{t}\ddot{\mathbf{U}} = {}^{t}\mathbf{R} - {}^{t}_{0}\mathbf{F} \tag{6.102}$$

and using the UL formulation:
static analysis:

$$({}^{t}_{t}\mathbf{K}_L + {}^{t}_{t}\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}_{t}\mathbf{F} \tag{6.103}$$

dynamic analysis, implicit time integration:

$$\mathbf{M}\ {}^{t+\Delta t}\ddot{\mathbf{U}} + ({}^{t}_{t}\mathbf{K}_L + {}^{t}_{t}\mathbf{K}_{NL})\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}_{t}\mathbf{F} \tag{6.104}$$

dynamic analysis, explicit time integration:

$$\mathbf{M}\ {}^{t}\ddot{\mathbf{U}} = {}^{t}\mathbf{R} - {}^{t}_{t}\mathbf{F} \tag{6.105}$$

where $\mathbf{M}$ = time-independent mass matrix
$^{t}\mathbf{K}$ = linear strain incremental stiffness matrix, not including the initial displacement effect
${}^{t}_{0}\mathbf{K}_L$, ${}^{t}_{t}\mathbf{K}_L$ = linear strain incremental stiffness matrices
${}^{t}_{0}\mathbf{K}_{NL}$, ${}^{t}_{t}\mathbf{K}_{NL}$ = nonlinear strain (geometric or initial stress) incremental stiffness matrices
${}^{t+\Delta t}\mathbf{R}$ = vector of externally applied nodal point loads at time $t + \Delta t$; this vector is also used at time t in explicit time integration
${}^{t}\mathbf{F}$, ${}^{t}_{0}\mathbf{F}$, ${}^{t}_{t}\mathbf{F}$ = vectors of nodal point forces equivalent to the element stresses at time t
$\mathbf{U}$ = vector of increments in the nodal point displacements
${}^{t}\ddot{\mathbf{U}}$, ${}^{t+\Delta t}\ddot{\mathbf{U}}$ = vectors of nodal point accelerations at times t and $t + \Delta t$

TABLE 6.4 *Finite element matrices*

Analysis type	Integral	Matrix evaluation
In all analyses	$\int_{^0V} {}^0\rho\, {}^{t+\Delta t}\ddot{u}_i\, \delta u_i\, d^0V$	$\mathbf{M}\, {}^{t+\Delta t}\ddot{\hat{\mathbf{u}}} = \left(\int_{^0V} {}^0\rho\, \mathbf{H}^T\mathbf{H}\, d^0V\right) {}^{t+\Delta t}\ddot{\hat{\mathbf{u}}}$
	${}^{t+\Delta t}\mathcal{R} = \int_{^0S_f} {}^{t+\Delta t}_{\ 0}f_i^S\, \delta u_i^S\, d^0S + \int_{^0V} {}^{t+\Delta t}_{\ 0}f_i^B\, \delta u_i\, d^0V$	${}^{t+\Delta t}\mathbf{R} = \int_{^0S_f} \mathbf{H}^{S^T}\, {}^{t+\Delta t}_{\ 0}\mathbf{f}^S\, d^0S + \int_{^0V} \mathbf{H}^T\, {}^{t+\Delta t}_{\ 0}\mathbf{f}^B\, d^0V$
Materially-nonlinear-only	$\int_V C_{ijrs}\, e_{rs}\, \delta e_{ij}\, dV$	${}^t\mathbf{K}\hat{\mathbf{u}} = \left(\int_V \mathbf{B}_L^T\mathbf{C}\mathbf{B}_L\, dV\right)\hat{\mathbf{u}}$
	$\int_V {}^t\sigma_{ij}\, \delta e_{ij}\, dV$	${}^t\mathbf{F} = \int_V \mathbf{B}_L^T\, {}^t\hat{\boldsymbol{\Sigma}}\, dV$
Total Lagrangian formulation	$\int_{^0V} {}_0C_{ijrs}\, {}_0e_{rs}\, \delta_0e_{ij}\, d^0V$	${}^t_0\mathbf{K}_L\hat{\mathbf{u}} = \left(\int_{^0V} {}^t_0\mathbf{B}_L^T\, {}_0\mathbf{C}\, {}^t_0\mathbf{B}_L\, d^0V\right)\hat{\mathbf{u}}$
	$\int_{^0V} {}^t_0S_{ij}\, \delta_0\eta_{ij}\, d^0V$	${}^t_0\mathbf{K}_{NL}\hat{\mathbf{u}} = \left(\int_{^0V} {}^t_0\mathbf{B}_{NL}^T\, {}^t_0\mathbf{S}\, {}^t_0\mathbf{B}_{NL}\, d^0V\right)\hat{\mathbf{u}}$
	$\int_{^0V} {}^t_0S_{ij}\, \delta_0e_{ij}\, d^0V$	${}^t_0\mathbf{F} = \int_{^0V} {}^t_0\mathbf{B}_L^T\, {}^t_0\hat{\mathbf{S}}\, d^0V$
Updated Lagrangian formulation	$\int_{^tV} {}_tC_{ijrs}\, {}_te_{rs}\, \delta_te_{ij}\, d^tV$	${}^t_t\mathbf{K}_L\hat{\mathbf{u}} \left(\int_{^tV} {}^t_t\mathbf{B}_L^T\, {}_t\mathbf{C}\, {}^t_t\mathbf{B}_L\, d^tV\right)\hat{\mathbf{u}}$
	$\int_{^tV} {}^t\tau_{ij}\, \delta_t\eta_{ij}\, d^tV$	${}^t_t\mathbf{K}_{NL}\hat{\mathbf{u}} = \left(\int_{^tV} {}^t_t\mathbf{B}_{NL}^T\, {}^t\boldsymbol{\tau}\, {}^t_t\mathbf{B}_{NL}\, d^tV\right)\hat{\mathbf{u}}$
	$\int_{^tV} {}^t\tau_{ij}\, \delta_te_{ij}\, d^tV$	${}^t_t\mathbf{F} = \int_{^tV} {}^t_t\mathbf{B}_L^T\, {}^t\hat{\boldsymbol{\tau}}\, d^tV$

6.3.3. Truss and Cable Element

- Truss element : structural element capable of transmitting stresses only in the direction normal to the cross-sectional area
- We consider a truss element that has arbitrary orientation in space

- Element described by two to four nodes as Fig.6.3. It subjected to large displacements an large strains.

- Global coordinates of a nodal points

At time 0 : ${}^{0}x_1^{k}$, ${}^{0}x_2^{k,}$ ${}^{0}x_3^{k,}$ ${}^{0}x_4^{k}$

At time t : ${}^{t}x_1^{k}$, ${}^{t}x_2^{k,}$ ${}^{t}x_3^{k,}$ ${}^{t}x_4^{k}$

Where k=1 … N with N = nodes numbers ($2 \le N \le 4$)

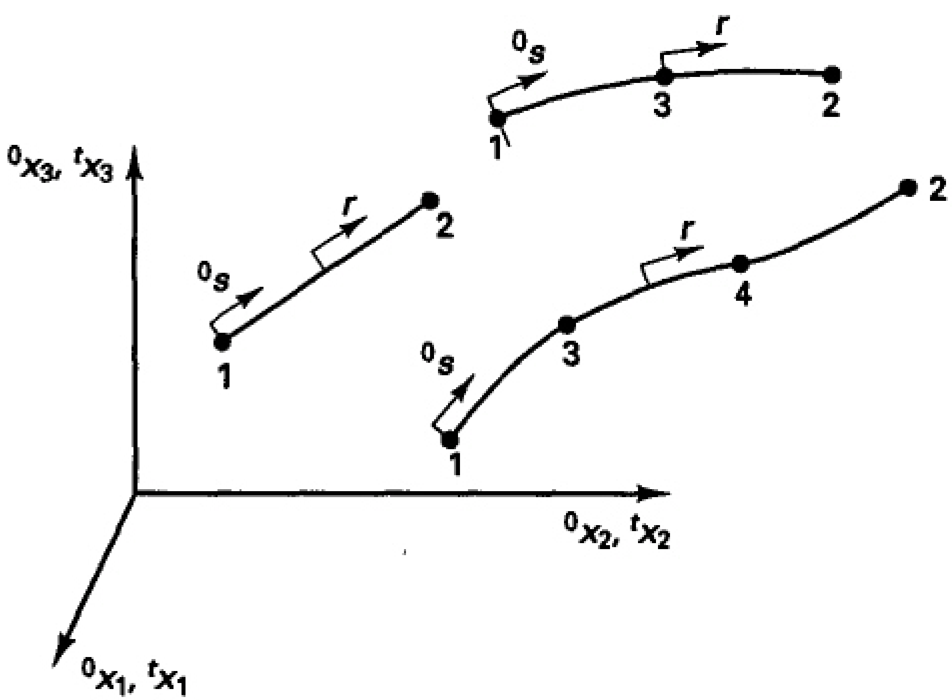


Figure 6.3 Two- to four-node truss element

The nodal point coordinate assume to determinate the spatial configuration of the truss a time 0 and t using :

$$ {}^{0}x_1(r) = \sum_{k=1}^{N} h_k \, {}^{0}x_1^k; \qquad {}^{0}x_2(r) = \sum_{k=1}^{N} h_k \, {}^{0}x_2^k; \qquad {}^{0}x_3(r) = \sum_{k=1}^{N} h_k \, {}^{0}x_3^k \tag{6.106} $$

$$ {}^{t}x_1(r) = \sum_{k=1}^{N} h_k \, {}^{t}x_1^k; \qquad {}^{t}x_2(r) = \sum_{k=1}^{N} h_k \, {}^{t}x_2^k; \qquad {}^{t}x_3(r) = \sum_{k=1}^{N} h_k \, {}^{t}x_3^k \tag{6.107} $$

Where the Interpolation functions $h_k(r)$ are defined

$$ {}^{t}u_i(r) = \sum_{k=1}^{N} h_k \, {}^{t}u_i^k \tag{6.108} $$

$$ u_i(r) = \sum_{k=1}^{N} h_k u_i^k, \qquad i = 1, 2, 3 \tag{6.109} $$

- Since for truss element the only stress is the normal stress on its cross-sectional area, we consider the corresponding longitudinal strain. We have the TL formulation :

$$ {}_0^t\tilde{\epsilon}_{11} = \frac{d\,{}^0x_i}{d\,{}^0s}\frac{d\,{}^tu_i}{d\,{}^0s} + \frac{1}{2}\frac{d\,{}^tu_i}{d\,{}^0s}\frac{d\,{}^tu_i}{d\,{}^0s} \qquad (6.110) $$

- Where ${}^0s(r)$ is the arc length at time 0 of the material point ${}^0x_1(r), {}^0x_2(r), {}^0x_3(r)$ given by :

$$ {}^0s(r) = \sum_{k=1}^{N} h_k \, {}^0s_k \tag{6.111} $$

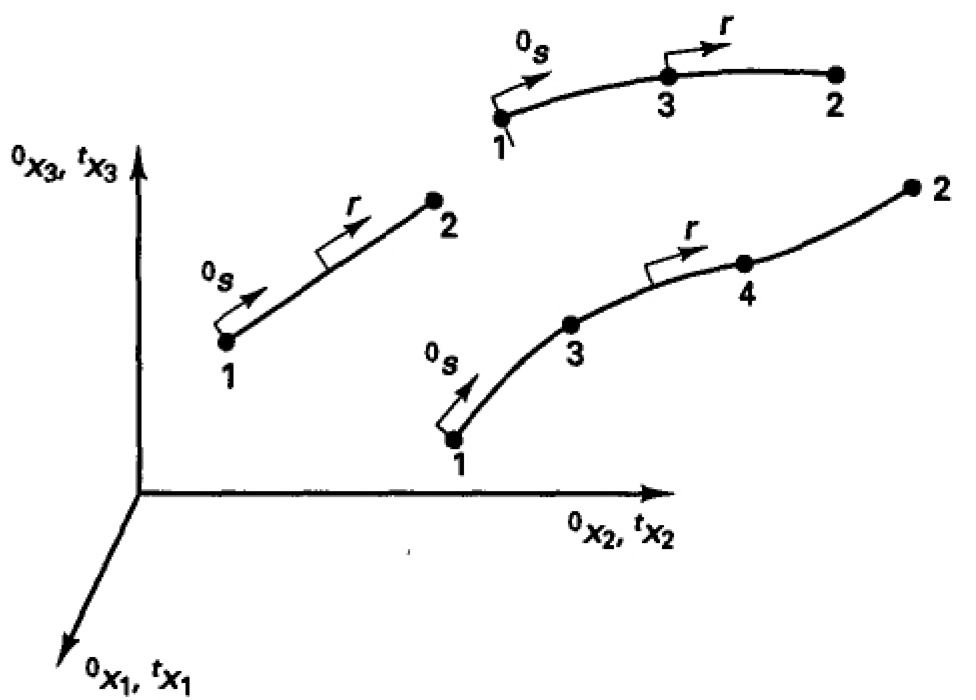


Figure 6.3 Two- to four-node truss element

The increment in the strain component ${}_0^t\varepsilon_{11}$ is denoted ${}_0\tilde{\varepsilon}_{11}$

$$ {}_0\tilde{\varepsilon}_{11} = {}_0\tilde{e}_{11} + {}_0\tilde{\eta}_{11} $$

Strains :

$$ {}_0\tilde{e}_{11} = \frac{d\,{}^0x_i}{d\,{}^0s}\frac{du_i}{d\,{}^0s} + \frac{d\,{}^tu_i}{d\,{}^0s}\frac{du_i}{d\,{}^0s} \tag{6.112} $$

$$ {}_0\tilde{\eta}_{11} = \frac{1}{2}\frac{du_i}{d\,{}^0s}\frac{du_i}{d\,{}^0s} \tag{6.113} $$

For the strain-displacement matrices we define

$$ {}^0\hat{\mathbf{x}}^T = [{}^0x_1^1 \quad {}^0x_2^1 \quad {}^0x_3^1 \quad \cdots \quad {}^0x_1^N \quad {}^0x_2^N \quad {}^0x_3^N] \tag{6.114} $$

$$ {}^t\hat{\mathbf{u}}^T = [{}^tu_1^1 \quad {}^tu_2^1 \quad {}^tu_3^1 \quad \cdots \quad {}^tu_1^N \quad {}^tu_2^N \quad {}^tu_3^N] $$

$$ \hat{\mathbf{u}}^T = [u_1^1 \quad u_2^1 \quad u_3^1 \quad \cdots \quad u_1^N \quad u_2^N \quad u_3^N] \tag{6.115} $$

$$ \mathbf{H} = \left[h_1\mathbf{I}_3 \;\vdots\; \cdots \;\vdots\; h_N\mathbf{I}_3\right]; \qquad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.116} $$

Volume-displacement interpolation matrix

And hence

$$ {}_0^t\mathbf{B}_L = ({}^0J^{-1})^2\,({}^0\hat{\mathbf{x}}^T\mathbf{H}_{,r}^T\mathbf{H}_{,r} + {}^t\hat{\mathbf{u}}^T\mathbf{H}_{,r}^T\mathbf{H}_{,r}) \tag{6.117} $$

$$ {}_0^t\mathbf{B}_{NL} = {}^0J^{-1}\mathbf{H}_{,r} \tag{6.118} $$

With $J^{-1} = dr/d^0s$

With $J^{-1} = dr/d^0s$

The only nonzero component is ${}_0^t\tilde{S}_{11}$, which we assume to be given as a function of Green-Lagrange strain ${}_0^t\tilde{\epsilon}_{11}$ at time t. The tangent stress-strain relationship Is therefore.

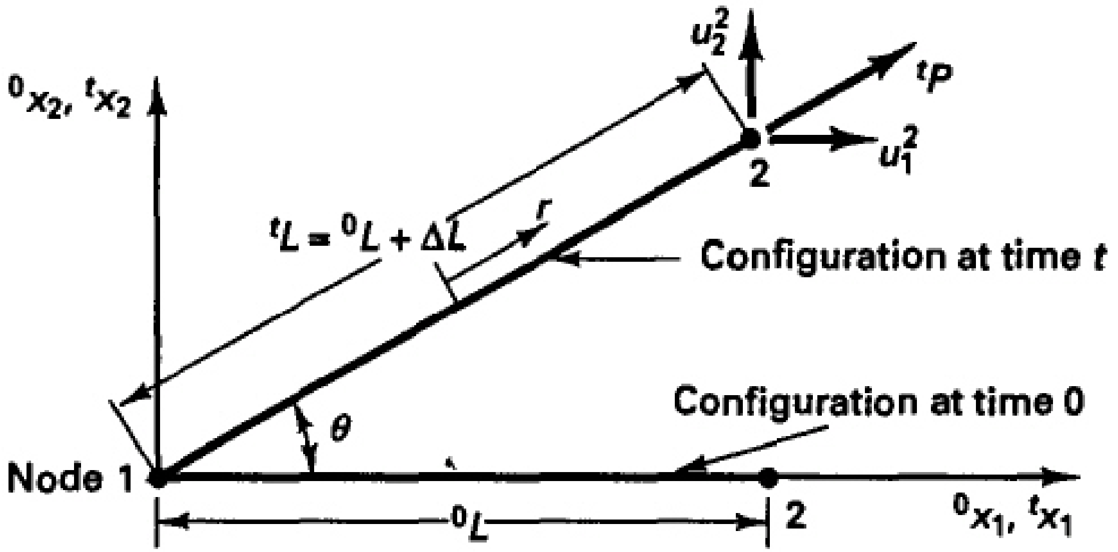
Incremental stress-strain material property matrix : $\quad {}_0\tilde{C}_{1111} = \dfrac{\partial\, {}_0^t\tilde{S}_{11}}{\partial\, {}_0^t\tilde{\epsilon}_{11}}$

The above relation can be employed to develop the UL formulation and the Materially-nonlinear formulation.

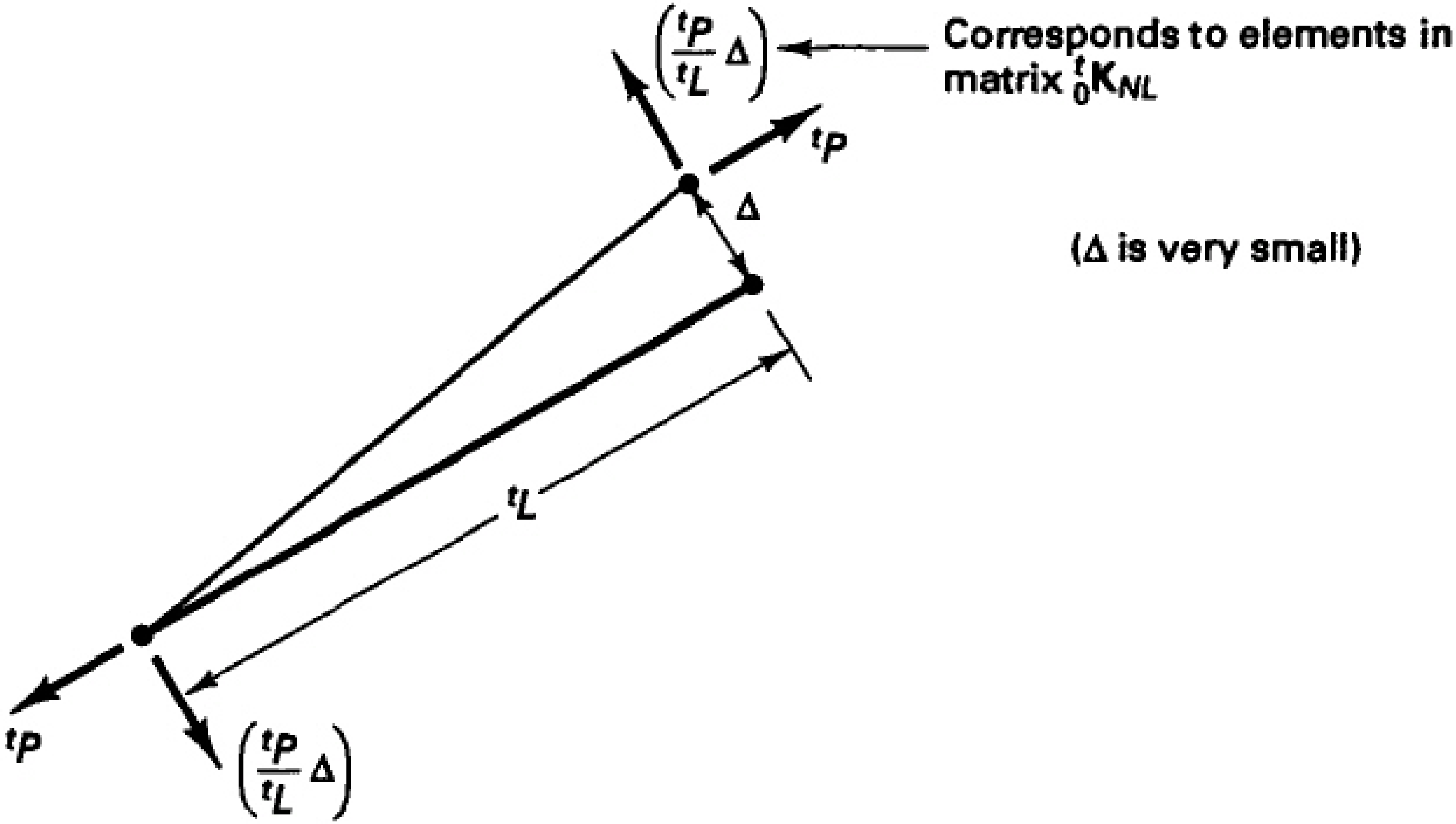
Ex 6.16

- *For the 2 nodes element :*

Develop the tangent stiffness matrix and force vector at time t. Consider large displacement and large strain conditions.



(a) Two-node element



(b) Moment equilibrium of element

Figure E6.16 Formulation of two-node truss element

Using TL formulation we express ${}_0e_{11}$ and ${}_0\eta_{11}$. The truss element Undergoes displacement only in the ${}^0x_1, {}^0x_2$ plane

$$ {}_0e_{11} = \frac{\partial u_1}{\partial {}^0x_1} + \frac{\partial {}^t u_1}{\partial {}^0x_1}\frac{\partial u_1}{\partial {}^0x_1} + \frac{\partial {}^t u_2}{\partial {}^0x_1}\frac{\partial u_2}{\partial {}^0x_1} $$

$$ {}_0\eta_{11} = \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial {}^0x_1}\right)^2 + \left(\frac{\partial u_2}{\partial {}^0x_1}\right)^2\right] $$

By geometry and using ${}^t u_i = \sum_{k=1}^{2} h_k \, {}^t u_i^k$

We compute

$$ {}^t u_1^1 = 0,\ {}^t u_2^1 = 0,\ {}^t u_1^2 = ({}^0L + \Delta L)\cos\theta - {}^0L,\ {}^t u_2^2 = ({}^0L + \Delta L)\sin\theta. $$

And obtain

$$ \frac{\partial {}^t u_1}{\partial {}^0x_1} = \frac{({}^0L + \Delta L)\cos\theta}{{}^0L} - 1; \qquad \frac{\partial {}^t u_2}{\partial {}^0x_1} = \frac{({}^0L + \Delta L)\sin\theta}{{}^0L} $$

- We replace the result into

$$ {}_0e_{ij} = \tfrac{1}{2}({}_0u_{i,j} + {}_0u_{j,i} + \underbrace{{}_0^tu_{k,i}\,{}_0u_{k,j} + {}_0u_{k,i}\,{}_0^tu_{k,j}}_{\text{Initial displacement effect}}); \qquad {}_0\eta_{ij} = \tfrac{1}{2}{}_0u_{k,i}\,{}_0u_{k,j} $$

$$ {}_0e_{11} = \frac{\partial u_1}{\partial^0 x_1} + \frac{\partial^t u_1}{\partial^0 x_1}\frac{\partial u_1}{\partial^0 x_1} + \frac{\partial^t u_2}{\partial^0 x_1}\frac{\partial u_2}{\partial^0 x_1} $$

$$ {}_0e_{11} = \frac{1}{{}^0L}\left\{[-1 \quad 0 \quad 1 \quad 0] + \left(\frac{{}^0L + \Delta L}{{}^0L}\cos\theta - 1\right)[-1 \quad 0 \quad 1 \quad 0] + \left(\frac{{}^0L + \Delta L}{{}^0L}\sin\theta\right)[0 \quad -1 \quad 0 \quad 1]\right\}\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix} $$

$$ = \frac{{}^0L + \Delta L}{({}^0L)^2}[-\cos\theta \quad -\sin\theta \quad \cos\theta \quad \sin\theta]\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix} $$

- And hence

$$ {}_0^t\mathbf{B}_L = \frac{{}^0L + \Delta L}{({}^0L)^2}[-\cos\theta \quad -\sin\theta \quad \cos\theta \quad \sin\theta] $$

- In the total Lagrangian formulation we assume that ${}_0^tS_{11}$ is given in term of ${}_0^t\varepsilon_{11}$

$$ {}_0C_{1111} = \frac{\partial_0^t S_{11}}{\partial_0^t \epsilon_{11}} $$

If we use ${}^t_0S_{11} = E\ {}^t_0\varepsilon_{11}$, we have ${}_0C_{1111} = E$

- The tangent matrix and force vector are

$$ {}_0^t\mathbf{K} = {}_0C_{1111}\frac{({}^0L + \Delta L)^2}{({}^0L)^3}\,{}^0A \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ & & \cos^2\theta & \sin\theta\cos\theta \\ \text{Symmetric} & & & \sin^2\theta \end{bmatrix} $$

$$ + \frac{{}^tP}{{}^0L + \Delta L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \text{(a)} $$

Linear strain stiffness matrix

$$ {}_0^t\mathbf{F} = {}^tP\begin{bmatrix} -\cos\theta \\ -\sin\theta \\ \cos\theta \\ \sin\theta \end{bmatrix} $$

Non linear strain
Stiffness matrix,independent from θ

- Where ${}^{t}P$ is the current force carried in the truss element. Here we have used, with the Cauchy stress equal to ${}^{t}P/A$

$$
{}_0^tS_{11} = \frac{{}^0\rho}{{}^t\rho}\left(\frac{{}^0L}{{}^0L + \Delta L}\right)^2 \frac{{}^tP}{{}^tA}; \qquad {}_0^t\epsilon_{11} = \frac{\Delta L}{{}^0L} + \frac{1}{2}\left(\frac{\Delta L}{{}^0L}\right)^2
$$

$$
{}^0\rho\, {}^0L\, {}^0A = {}^t\rho({}^0L + \Delta L)\, {}^tA; \qquad {}_0^tS_{11} = \frac{{}^0L}{{}^0L + \Delta L} \frac{{}^tP}{{}^0A} \tag{b}
$$

$$
{}^tP = {}_0^tS_{11}\, {}^0A \frac{{}^0L + \Delta L}{{}^0L}
$$