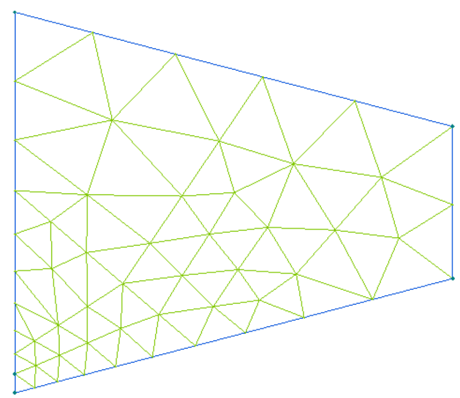
Lecture Chapter 9.
- Part 2 -



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Execice 9.12

EXAMPLE 9.12: Analyze the central difference method for its integration stability. Consider the case $\xi = 0.0$ used in (9.8) to (9.12).

We need to calculate the spectral radius of the approximation operator given in (9.70) when $\xi = 0$. The eigenvalue problem $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$ to be solved is

$$\begin{bmatrix} 2 - \omega^2 \Delta t^2 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{u} = \lambda \mathbf{u} \tag{a}$$

Task : Analyze the stability of the central difference method :

For stability, we need that the absolute values of λ_i is smaller or equal to 1
It mean that the spectral radius $\rho(A)$ of the matrix A must satisfy the condition
$\rho(A) \leq 1$and it gives the condition $\Delta t/T \leq 1/\pi$

$$\begin{bmatrix} 2 - \omega^2 \Delta t^2 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{u} = \lambda \mathbf{u}$$

Using the spectral decomposition we get : $\mathbf{A}^n = \mathbf{P}\mathbf{J}^n\mathbf{P}^{-1}$

Let's be ρ(A) the spectral radius define as : $\rho(\mathbf{A}) = \max_{i=1,2,...} |\lambda_i|$

The eigenvalues are the roots of the characteristic polynomial $p(\lambda)$ (see Section 2.5) defined as

$$p(\lambda) = (2 - \omega^2 \Delta t^2 - \lambda)(-\lambda) + 1$$

Hence,

$$\lambda_1 = \frac{2 - \omega^2 \Delta t^2}{2} + \sqrt{\frac{(2 - \omega^2 \Delta t^2)^2}{4} - 1}$$

$$\lambda_2 = \frac{2 - \omega^2 \Delta t^2}{2} - \sqrt{\frac{(2 - \omega^2 \Delta t^2)^2}{4} - 1}$$

9.5 Solution of nonlinear equations in dynamic analysis

How to obtain a solution of a nonlinear dynamic response of a finite element system ?

Plan

- 9.5 Solution of nonlinear equations in dynamic analysis
- 9.5.1 Explicit integration
- 9.5.2 Implicit integration
- 9.5.3 Solution using Mode superposition

9.5 Solution of nonlinear equations in dynamic analysis

Introduction : In order to have a solution to a nonlinear dynamic analysis, we need to used :

1. Incremental calculation (Chapter 6)
2. Iterative solutions procedure (Chapter 8.4)
3. Time integration procedure (Chapter 9.)

This chapter will explore the topics :

How these method are employ together in a nonlinear dynamic analysis

9.5.1 Explicit integration

Explicit integration consider based equilibrium at time **t.**

Most common explicit formulation in nonlinear dynamic analysis is the **central different operator**

As in linear analysis, mean the equilibrium of the finite element system is considered at time **t** in order to calculate the displacement at time **t+Δt**

$$\mathbf{M}\,{}^{t}\ddot{\mathbf{U}} = {}^{t}\mathbf{R} - {}^{t}\mathbf{F} \qquad (9.103)$$

Damping is neglected

${}^{t}F$: Nodal point force vector evaluated by 6.3 …the solution for nodal point displacement at time **t+Δt** is obtained using central difference approximation for acceleration.

$$\mathbf{U}(t) = \mathbf{P}\mathbf{X}(t) \tag{9.30}$$

where $\mathbf{P}$ is an $n \times n$ square matrix and $\mathbf{X}(t)$ is a time-dependent vector of order n. The

- If we know ${}^{t-\Delta t}U$ and ${}^{t}U$, with 9.103 and 9.3 we can calculate ${}^{t+\Delta t}U$.
- The solution is just the forward marching in time.
- Advantage of the method : With M a diagonal matrix the solution of ${}^{t+\Delta t}U$ does not involve a triangular factorization of a coefficient matrix.

The central difference method. In the central difference integration scheme we use (9.3) and (9.4) to approximate the acceleration and velocity at time t, respectively. The equilibrium equation (9.62) is considered at time t; i.e., we use

$$ {}^{t}\ddot{x} + 2\xi\omega\,{}^{t}\dot{x} + \omega^2\,{}^{t}x = {}^{t}r \tag{9.65} $$

$$ {}^{t}\ddot{x} = \frac{1}{\Delta t^2}\left({}^{t-\Delta t}x - 2\,{}^{t}x + {}^{t+\Delta t}x\right) \tag{9.66} $$

$$ {}^{t}\dot{x} = \frac{1}{2\Delta t}\left(-{}^{t-\Delta t}x + {}^{t+\Delta t}x\right) \tag{9.67} $$

Substituting (9.66) and (9.67) into (9.65) and solving for ${}^{t+\Delta t}x$, we obtain

$$ {}^{t+\Delta t}x = \frac{2 - \omega^2\,\Delta t^2}{1 + \xi\omega\,\Delta t}\,{}^{t}x - \frac{1 - \xi\omega\,\Delta t}{1 + \xi\omega\,\Delta t}\,{}^{t-\Delta t}x + \frac{\Delta t^2}{1 + \xi\omega\,\Delta t}\,{}^{t}r \tag{9.68} $$

The solution (9.68) can now be written in the form (9.63); i.e., we have

$$\begin{bmatrix} {}^{t+\Delta t}x \\ {}^{t}x \end{bmatrix} = \mathbf{A} \begin{bmatrix} {}^{t}x \\ {}^{t-\Delta t}x \end{bmatrix} + \mathbf{L}\, {}^{t}r \tag{9.69}$$

where

$$\mathbf{A} = \begin{bmatrix} \dfrac{2 - \omega^2 \Delta t^2}{1 + \xi\omega \Delta t} & -\dfrac{1 - \xi\omega \Delta t}{1 + \xi\omega \Delta t} \\ 1 & 0 \end{bmatrix} \tag{9.70}$$

and

$$\mathbf{L} = \begin{bmatrix} \dfrac{\Delta t^2}{1 + \xi\omega \Delta t} \\ 0 \end{bmatrix} \tag{9.71}$$

As we pointed out in Section 9.2.1, the method is usually employed with $\xi = 0$.

- Shortcoming of the central difference method : time step size Δt, it have to be smaller than $\Delta t_{cr} = T_n/\pi$ where T_n = smaller period in the FE system.

- Result are also applicable to nonlinear analysis. For each time step the stiffness matrix change in the nonlinear analysis. It mean that the T_n is not constant and that the time step Δt need to be decreased if the systems stiffens. The time step adjustment need to satisfied $\Delta t_{cr} \leq T_n/\pi$.

- T_c : Critical time step for stability

Example :

- An analysis where all Δt is always smaller than Tc except a few successive solution steps. This time steps Δt is just larger than the T_c.
- In this case the solution will show a significant error accumulation and nor only a solution instability.
- The situation observed is different than the linear analysis where the solution is "blow up" if the time step is larger than the T_c.

Simple one degree of freedom spring mass system

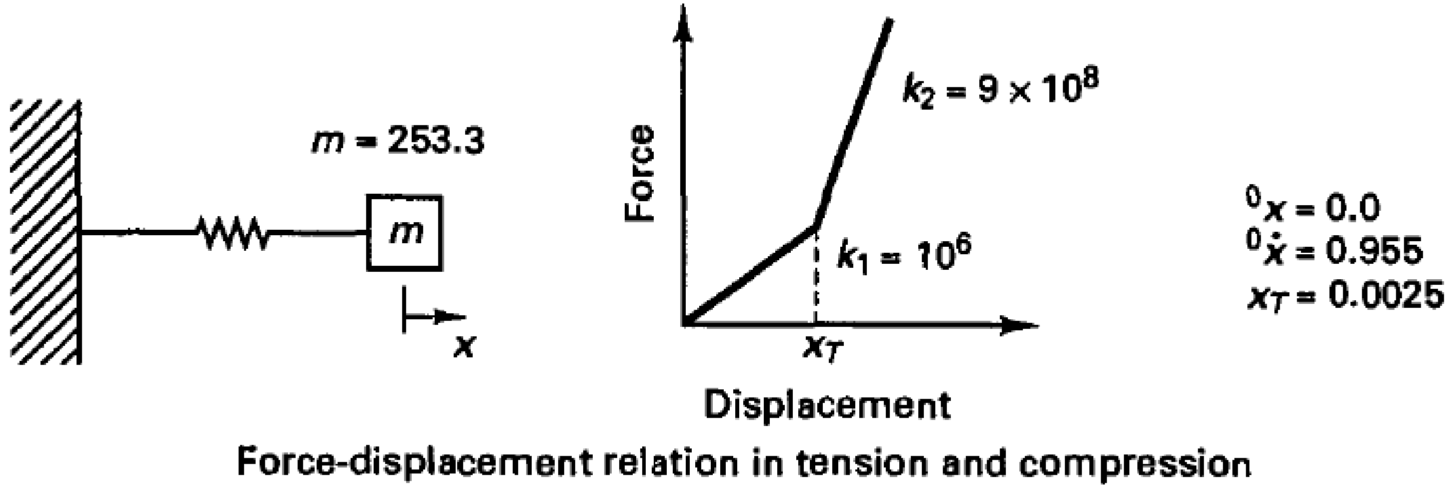
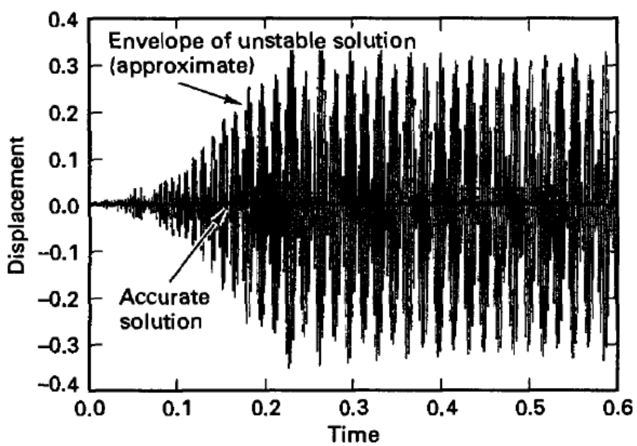


Figure 9.8 Response of bilinear elastic system as predicted using the central difference method; $\Delta t_{cr} = 0.001061027$; the accurate response with displacement $\ll 0.1$ was calculated with $\Delta t = 0.000106103$; the "unstable" response was calculated with $\Delta t = 0.00106103$.



The time step Δt is slightly > Δt_{cr} in the stiff region of the spring.

Time step correspond to a stable time step for small displacement,
Solution is partly stable and unstable

- **Important :**

The proper choice of Δt is the more important factor.

9.5.2 Implicit integration

- All the time integration schemes discuss before for linear dynamic analysis can also be employed in nonlinear dynamic response calculation. A very common technique is the trapezoidal rule which id Newman rule with $\delta = ½$ and $\alpha=1/4$ and we use this method to demonstrate the basic additional consideration involved in nonlinear analysis.

- As linear analysis, using implicit time integration, we consider equilibrium at time $t+ \Delta t$.

- Using the Newton-Raphson integration, we get the governing equilibrium equation

$$\mathbf{M}\ {}^{t+\Delta t}\ddot{\mathbf{U}}^{(k)} + {}^{t}\mathbf{K}\ \Delta\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} \tag{9.104}$$

$${}^{t+\Delta t}\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{U}^{(k-1)} + \Delta\mathbf{U}^{(k)} \tag{9.105}$$

- Using trapezoidal rule of integration we get :

$${}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \frac{\Delta t}{2}({}^{t}\dot{\mathbf{U}} + {}^{t+\Delta t}\dot{\mathbf{U}}) \tag{9.106}$$

$${}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \frac{\Delta t}{2}({}^{t}\ddot{\mathbf{U}} + {}^{t+\Delta t}\ddot{\mathbf{U}}) \tag{9.107}$$

Using the relations in (9.105) to (9.107), we thus obtain

$$ {}^{t+\Delta t}\ddot{\mathbf{U}}^{(k)} = \frac{4}{\Delta t^2}\left({}^{t+\Delta t}\mathbf{U}^{(k-1)} - {}^{t}\mathbf{U} + \Delta\mathbf{U}^{(k)}\right) - \frac{4}{\Delta t}\,{}^{t}\dot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}} \tag{9.108} $$

and substituting into (9.104), we have

$$ {}^{t}\hat{\mathbf{K}}\,\Delta\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} - \mathbf{M}\left(\frac{4}{\Delta t^2}\left({}^{t+\Delta t}\mathbf{U}^{(k-1)} - {}^{t}\mathbf{U}\right) - \frac{4}{\Delta t}\,{}^{t}\dot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}\right) \tag{9.109} $$

where

$$ {}^{t}\hat{\mathbf{K}} = {}^{t}\mathbf{K} + \frac{4}{\Delta t^2}\mathbf{M} \tag{9.110} $$

- We can notice : Iterative equation in dynamic analysis using implicit integration is the same form as equation consider in static linear analysis except that both coefficient matrix and the nodal point force vector contain contribution from the inertia of the system.

- So, all iterative strategies discuss for static analysis are directly applicable to 9.109

$$^{t}\hat{\mathbf{K}}\,\Delta\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} - \mathbf{M}\left(\frac{4}{\Delta t^2}({}^{t+\Delta t}\mathbf{U}^{(k-1)} - {}^{t}\mathbf{U}) - \frac{4}{\Delta t}\,{}^{t}\dot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}\right) \qquad (9.109)$$

- Inertia of the system renders “more smooth ” response, convergence expect to be more rapid than static analysis and convergence behavior can be increase by decreasing Δt.

- The numerical reason to this lies in the contribution of mass matrix to the coefficient matrix. It increase and become dominant as time steps decrease.

- Interesting : First solutions of nonlinear finite element response, equilibrium were not performed by step-to step incremental analysis but simply solved for k=1 and incremental displacement $\Delta U^{(1)}$ was accepted as an accurate approximation to the actual displacement increment from time t to time t+ Δt.
- It was recognized that iteration can be upmost importance since any error admitted in the incremental solution at a particularly time directly affect in a path-dependant manner the solution at a subsequent time.
- Any nonlinear dynamic response is highly path dependant, the nonlinear analysis dynamic response requires iteration at each time step more stringently than static analysis.
- See example 9.9

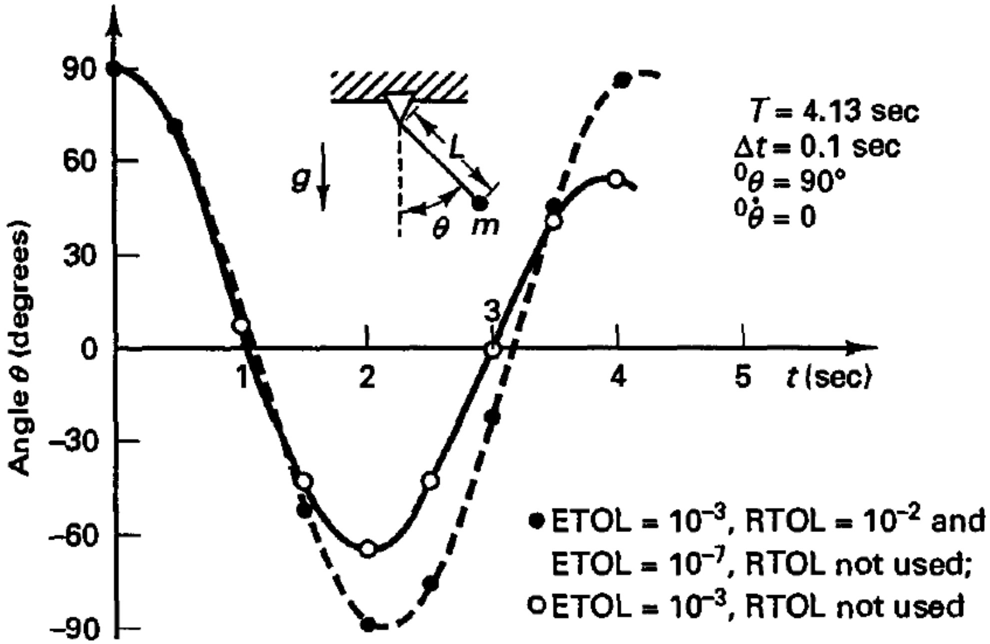


Figure 9.9 Analysis of simple pendulum using trapezoidal rule, RNORM = mg

- Result obtain in the simple analysis of a pendulum idealized as a truss element with concentrated mass at its free end.
- Pendulum was released from horizontal position and response calculated for one period oscillation.

The convergence criteria used include inertia, it mean that the converge is reached when these conditions are satisfied :

$$\frac{\left\| {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(i-1)} - \mathbf{M}\, {}^{t+\Delta t}\ddot{\mathbf{U}}^{(i-1)} \right\|_2}{\text{RNORM}} \leq \text{RTOL} \tag{9.111}$$

$$\frac{\Delta\mathbf{U}^{(i)T}\left({}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(i-1)} - \mathbf{M}\, {}^{t+\Delta t}\ddot{\mathbf{U}}^{(i-1)}\right)}{\Delta\mathbf{U}^{(1)T}\left({}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F} - \mathbf{M}\, {}^{t}\ddot{\mathbf{U}}\right)} \leq \text{ETOL} \tag{9.112}$$

- RTOL : Tolerance force
- ETOL: Energy tolerance
- Fig 9.9 demonstrated the importance of iteration doing with a sufficiently tight convergence tolerance.
- Energy is lost if the convergence tolerance

Conclusion : implicit time integration

- In summary, for a nonlinear dynamic analysis using implicit time integration, the analyst should employ :

- An operator that is unconditionally stable in linear analysis (Good choice is the trapezoid rule)
- Equilibrium iteration with tight enough convergence tolerance
- Select a time step size based on criteria a
- Convergence in the equilibrium iteration must be achieved

9.5.3 Solution using Mode superposition

- In linear analysis :
  - Essence of superposition is a transformation from element nodal point degree of freedom to the generalized degrees of freedom of the vibration mode shapes.
  - The dynamic equilibrium equations in the basis of the mode shape vectors decouple, mode superposition can be very effective in linear analysis if only some vibration modes are excited by the loading.
  - The same principle can be applicable in the nonlinear analysis. However, in this case the vibration mode shapes and frequencies change and to transform the coefficient of the matrix

9.5.3 Solution using Mode superposition

$$\mathbf{M}\ {}^{t+\Delta t}\ddot{\mathbf{U}}^{(k)} + {}^{\tau}\mathbf{K}\ \Delta\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} \qquad k = 1, 2, \ldots \tag{9.113}$$

where ${}^{\tau}\mathbf{K}$ is the stiffness matrix corresponding to the configuration at some previous time τ. In the mode superposition analysis we now use

$$^{t+\Delta t}\mathbf{U} = \sum_{i=r}^{s} \boldsymbol{\phi}_i\ {}^{t+\Delta t}x_i \tag{9.114}$$

where ${}^{t+\Delta t}x_i$ is the ith generalized modal displacement at time $t + \Delta t$ and

$${}^{\tau}\mathbf{K}\boldsymbol{\phi}_i = \omega_i^2\mathbf{M}\boldsymbol{\phi}_i; \qquad i = r, \ldots, s \tag{9.115}$$

$$^{t+\Delta t}\mathbf{U} = \sum_{i=r}^{s} \boldsymbol{\phi}_i\ {}^{t+\Delta t}x_i \tag{9.114}$$

$${}^{\tau}\mathbf{K}\boldsymbol{\phi}_i = \omega_i^2\mathbf{M}\boldsymbol{\phi}_i; \qquad i = r, \ldots, s \tag{9.115}$$

- The complete mode superposition analysis of nonlinear dynamic response is generally effective only when the solution can be obtained without updating the stiffness matrix too frequency. In this case , the governing finite element equilibrium equations for the solution of the response at time t+Δt is :

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}}^{(k)} + {}^{\tau}\mathbf{K}\,\Delta\mathbf{U}^{(k)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} \qquad k = 1, 2, \ldots \tag{9.113}$$

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$$ {}^{\tau}\mathbf{K}\boldsymbol{\phi}_i = \omega_i^2 \mathbf{M}\boldsymbol{\phi}_i; \qquad i = r, \ldots, s \tag{9.115}$$

that is, ω_i, $\boldsymbol{\phi}_i$ are free-vibration frequencies (radians/second) and mode shape vectors of the system at time τ. Using (9.114) in the usual way, the equations in (9.113) are transformed to

$$^{t+\Delta t}\ddot{\mathbf{X}}^{(k)} + \boldsymbol{\Omega}^2\,\Delta\mathbf{X}^{(k)} = \boldsymbol{\Phi}^T(^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)}) \qquad k = 1, 2, \ldots \tag{9.116}$$

where

$$\boldsymbol{\Omega}^2 = \begin{bmatrix} \omega_r^2 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \omega_s^2 \end{bmatrix}; \qquad \boldsymbol{\Phi} = [\boldsymbol{\phi}_r, \ldots, \boldsymbol{\phi}_s]; \qquad ^{t+\Delta t}\mathbf{X} = \begin{bmatrix} ^{t+\Delta t}x_r \\ \vdots \\ ^{t+\Delta t}x_s \end{bmatrix} \tag{9.117}$$

- The relations in (9.116)are the equilibrium equations at time t+Δt in the generalized modal displacement of time τ:
- The corresponding mass matrix is an identity matrix
- Ω^2 : the stiffness matrix
- The external load vector is $\Phi^T\ {}^{t+\Delta t}R$
- The force vector corresponding to the element stress at the end iteration (k-1) is $\Phi^T\ {}^{t+\Delta t}F^{(k-1)}$
- The solution of 9.116 can be obtained using for example the trapezoidal rule of iteration (see 9.5.2)

Conclusion mode superposition

- In general, the use of mode superposition in non linear dynamic analysis can be effective if only a relatively few mode shapes need to be considered in the analysis.

- Such conditions may be encountered for example in the analysis of earthquake response and vibration excitation and it is in this area that the technique has been employed.